La-Net

MTH310

MATHEMATICAL METHODS II

Suggested Outline for Series 1: Sturm–Liouville Problems

- **Part 1.1: Introduction to Sturm–Liouville Problems**
  - Definition and historical background
  - General form of Sturm–Liouville equations
  - Importance in mathematical physics
- **Part 1.2: Properties of Sturm–Liouville Systems**
  - Self-adjoint operators
  - Eigenvalues and eigenfunctions
  - Orthogonality of solutions
- **Part 1.3: Applications of Sturm–Liouville Problems**
  - Boundary value problems
  - Vibrating string and heat conduction examples
  - Connection to Fourier series
- **Part 1.4: Solution Techniques**
  - Separation of variables
  - Normalisation of eigenfunctions
  - Worked examples

Introduction

In many areas of applied mathematics and physics, we encounter problems that involve finding solutions to differential equations subject to specific boundary conditions. These are known as **Sturm–Liouville problems**, and they form a cornerstone of mathematical methods used in engineering, physics, and applied sciences. They are particularly important because they lead to sets of functions that are orthogonal, which in turn provide powerful tools for expanding and solving more complex problems such as those involving partial differential equations. By studying this Series, you will see how Sturm–Liouville theory underpins much of the material that follows in the course.

The aim of this Series is to introduce you to the concept of Sturm–Liouville problems, explain their properties, and demonstrate how they are applied to practical boundary value problems. You will also learn solution techniques that will prepare you for later topics such as Fourier series and eigenfunction expansions.
We shall commence this Series by defining Sturm–Liouville problems and exploring their historical background and general form. Next, we will examine the properties of Sturm–Liouville systems, including self-adjoint operators, eigenvalues, and orthogonality. Following that, we will consider applications to boundary value problems, with examples drawn from vibrating strings and heat conduction. Finally, we will wrap up by studying solution techniques such as separation of variables and normalisation of eigenfunctions, supported by worked examples.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 1.1** define Sturm–Liouville problems and explain their general form and historical background,
- **SLO 1.2** analyse the properties of Sturm–Liouville systems, including self-adjoint operators, eigenvalues, and orthogonality of solutions,
- **SLO 1.3** apply Sturm–Liouville theory to boundary value problems in physics and engineering, such as vibrating strings and heat conduction, and
- **SLO 1.4** demonstrate solution techniques for Sturm–Liouville problems, including separation of variables and normalisation of eigenfunctions, supported by worked examples.

1.1 Introduction to Sturm–Liouville Problems

Sturm–Liouville problems are a special class of **boundary value problems** where the solution involves finding eigenvalues and eigenfunctions of a differential operator. A **boundary value problem** is a type of differential equation that requires the solution to satisfy certain conditions at the boundaries of the domain. These problems are fundamental in mathematical physics because they provide orthogonal sets of functions that can be used to expand solutions of more complex equations.

Fill in the blank: A boundary value problem requires the solution to satisfy conditions at the __________ of the domain.

1.1.1 Definition and historical background

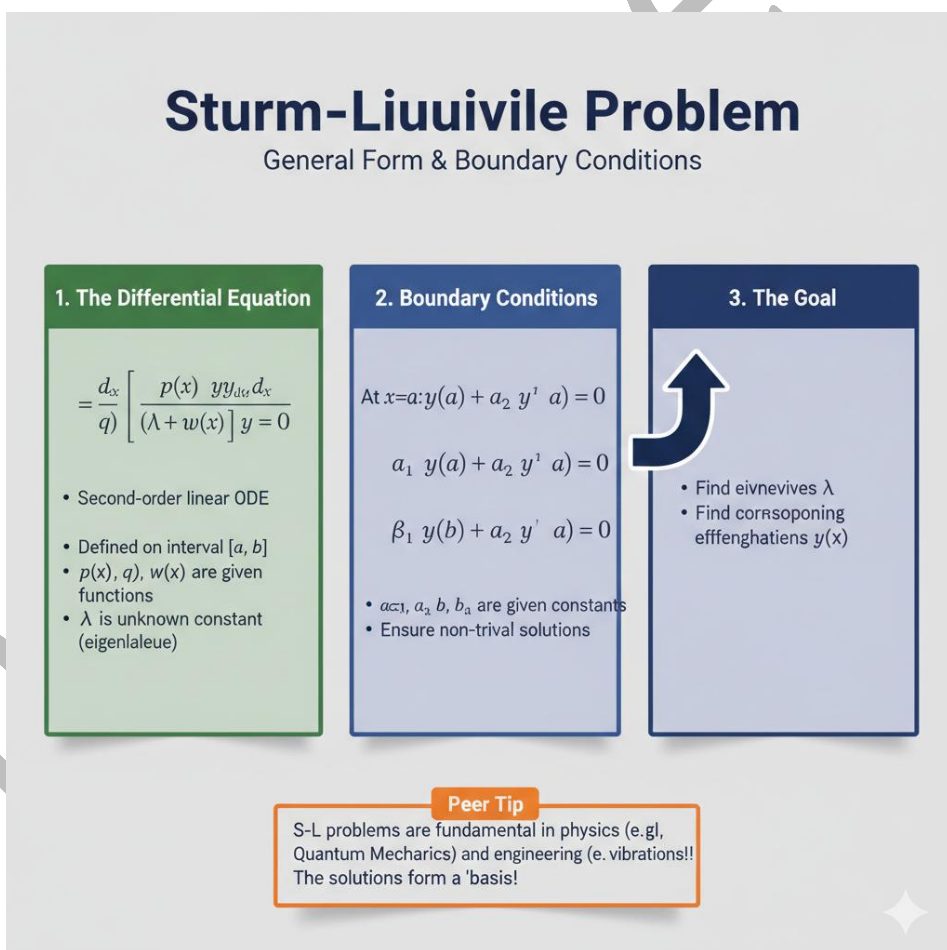
A **Sturm–Liouville problem** is a second-order linear differential equation of the form:

$$ddxp(x)dydx+\lambda w(x)-q(x)y=0$$

Here, p(x), q(x), and w(x)are given functions, is an eigenvalue, and y(x)is the eigenfunction. The problem is accompanied by boundary conditions at the endpoints of the interval.

Historically, these problems were studied by Jacques Charles François Sturm and Joseph Liouville in the 19th century. Their work laid the foundation for modern spectral theory and applications in physics.

Fill in the blank: In a Sturm–Liouville problem, the unknown function y(x)is called the __________.



General form of a Sturm–Liouville problem.

1.1.2 General form of Sturm–Liouville equations

The general Sturm–Liouville equation involves:

- A **differential operator**: an operator acting on functions, involving derivatives.
- A **weight function** w(x): a positive function that defines the inner product space.
- **Eigenvalues** : constants that arise from solving the equation.
- **Eigenfunctions** y(x): functions corresponding to each eigenvalue.

These equations are important because they generate orthogonal sets of eigenfunctions with respect to the weight function. This property is crucial for expansions such as Fourier series.

Fill in the blank: The function w(x)in a Sturm–Liouville problem is called the __________ function.

Key components of a Sturm–Liouville equation

- Differential operator
- Weight function
- Eigenvalues
- Eigenfunctions

1. Fundamental Mathematical Parts

Component	Mathematical Term	Role in the System
$p(x)$	**Coefficient Function**	Must be continuously differentiable and usually $p(x) > 0$. It often represents physical properties like stiffness or thermal conductivity.
$q(x)$	**Potential Function**	A continuous function that represents external forces or potential energy within the system.
$w(x)$	**Weight (or Density) Function**	Must be continuous and $w(x) > 0$. It defines the "inner product" and ensures the orthogonality of the solutions.
λ	**Eigenvalue**	A constant (often unknown) that represents specific allowed states, such as harmonics, energy levels, or vibration frequencies.
$y(x)$	**Eigenfunction**	The non-trivial solution ($y \neq 0$) that satisfies both the equation and the boundary conditions.

1.1.3 Importance in mathematical physics

Sturm–Liouville problems are widely used in physics and engineering because they provide a systematic way to solve partial differential equations. For example:

- In **vibrating string problems**, eigenfunctions represent modes of vibration.
- In **heat conduction**, they describe temperature distributions.
- In **quantum mechanics**, they appear in the Schrödinger equation.

Fill in the blank: In quantum mechanics, Sturm–Liouville problems appear in the __________ equation.

Applications of Sturm–Liouville problems in physics

Discipline	**Example**	**Notes**
Mechanics	Vibrating string	Modes of vibration
Heat conduction	Heat equation	Temperature distribution
Quantum mechanics	Schrödinger equation	Energy eigenstates

Pilot questions 1.1

1. A boundary value problem requires the solution to satisfy conditions at the: A. Boundaries of the domain B. Centre of the domain C. Infinity D. None of the above (SLO 1.1)
2. In a Sturm–Liouville problem, the unknown function $y(x)$ is called the: A. Eigenfunction B. Eigenvalue C. Weight function D. None of the above (SLO 1.1)
3. The function $w(x)$ in a Sturm–Liouville problem is called the: A. Weight function B. Eigenfunction C. Differential operator D. None of the above (SLO 1.1)
4. Sturm–Liouville problems were studied by: A. Sturm and Liouville B. Newton and Leibniz C. Fourier and Laplace D. None of the above (SLO 1.1)
5. In quantum mechanics, Sturm–Liouville problems appear in the: A. Schrödinger equation B. Heat equation C. Wave equation D. None of the above (SLO 1.1)

1.2 Properties of Sturm–Liouville Systems

Sturm–Liouville systems possess several important properties that make them extremely useful in solving boundary value problems and in constructing expansions such as Fourier series. These properties include the concepts of **self-adjoint operators**, **eigenvalues and eigenfunctions**, and the **orthogonality of solutions**.

1.2.1 Self-adjoint operators

A **self-adjoint operator** is a differential operator that satisfies the condition:

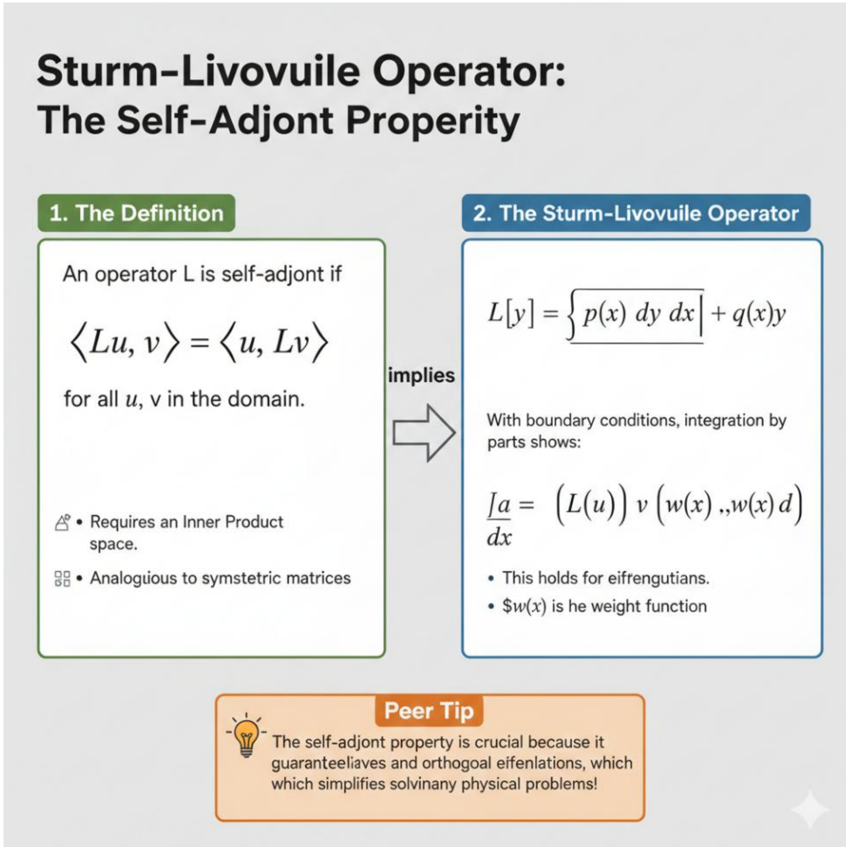
$$\langle Lf,g\rangle=\langle f,Lg\rangle$$

for all suitable functions fand g, where $\langle\cdot,\cdot\rangle$denotes the inner product. In Sturm–Liouville problems, the operator

$$L[y]=ddxp(x)dydx-q(x)y$$

is self-adjoint under the appropriate boundary conditions. This property ensures that eigenvalues are real and eigenfunctions form a complete set.

Fill in the blank: A self-adjoint operator guarantees that all eigenvalues are __________.



Self-adjoint operator property in Sturm–Liouville systems.

1.2.2 Eigenvalues and eigenfunctions

An **eigenvalue** is a constant that arises when solving the Sturm–Liouville equation, while an **eigenfunction** is the corresponding solution y(x). Each eigenvalue has one or more associated eigenfunctions.

Key properties:

- Eigenvalues are real due to the self-adjoint nature of the operator.
- Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function w(x).
- The set of eigenfunctions can be used to expand other functions defined on the same interval.

Fill in the blank: Eigenfunctions corresponding to distinct eigenvalues are __________ with respect to the weight function.

Properties of eigenvalues and eigenfunctions

Property	Explanation
Eigenvalues	Always real
Eigenfunctions	Orthogonal with respect to w(x)
Completeness	Can expand other functions

1.2.3 Orthogonality of solutions

The **orthogonality property** means that for two eigenfunctions ym(x)and yn(x)corresponding to different eigenvalues, we have:

abym(x)yn(x)w(x) dx=0

This property is fundamental because it allows us to construct series expansions of functions, similar to Fourier series, using eigenfunctions as basis functions.

Fill in the blank: Orthogonality of eigenfunctions means that their weighted inner product is equal to __________.

Orthogonality property of Sturm–Liouville eigenfunctions

- Orthogonality ensures independence of solutions
- Weighted inner product equals zero for distinct eigenfunctions
- Basis for function expansions

2. Key Components of the Property

Component	Description	Why it Matters
Weight Function ($w(x)$)	The specific function $w(x)$ from the S-L equation.	Without $w(x)$, the functions are not necessarily "perpendicular." It acts as a scaling factor for the inner product.
Distinct Eigenvalues	Orthogonality is guaranteed for eigenfunctions of different λ.	This ensures that each "mode" of the system is independent of the others.
Inner Product Space	The integral $\langle f, g \rangle_w = \int fgw\, dx$.	It creates a mathematical "geometry" where we can use concepts like length (norm) and angle.

Pilot questions 1.2

1. A self-adjoint operator guarantees that eigenvalues are: A. Real B. Complex C. Imaginary D. None of the above (SLO 1.2)

2. In Sturm–Liouville problems, eigenvalues are associated with: A. Eigenfunctions B. Weight functions C. Boundary conditions only D. None of the above (SLO 1.2)
3. Eigenfunctions corresponding to distinct eigenvalues are: A. Orthogonal with respect to the weight function B. Identical C. Always complex D. None of the above (SLO 1.2)
4. The operator L[y]=ddx(p(x)dydx)-q(x)yis: A. Self-adjoint B. Non-linear C. Non-self-adjoint D. None of the above (SLO 1.2)
5. Orthogonality of eigenfunctions means their weighted inner product is equal to: A. Zero B. Infinity C. One D. None of the above (SLO 1.2)

1.3 Applications of Sturm–Liouville Problems

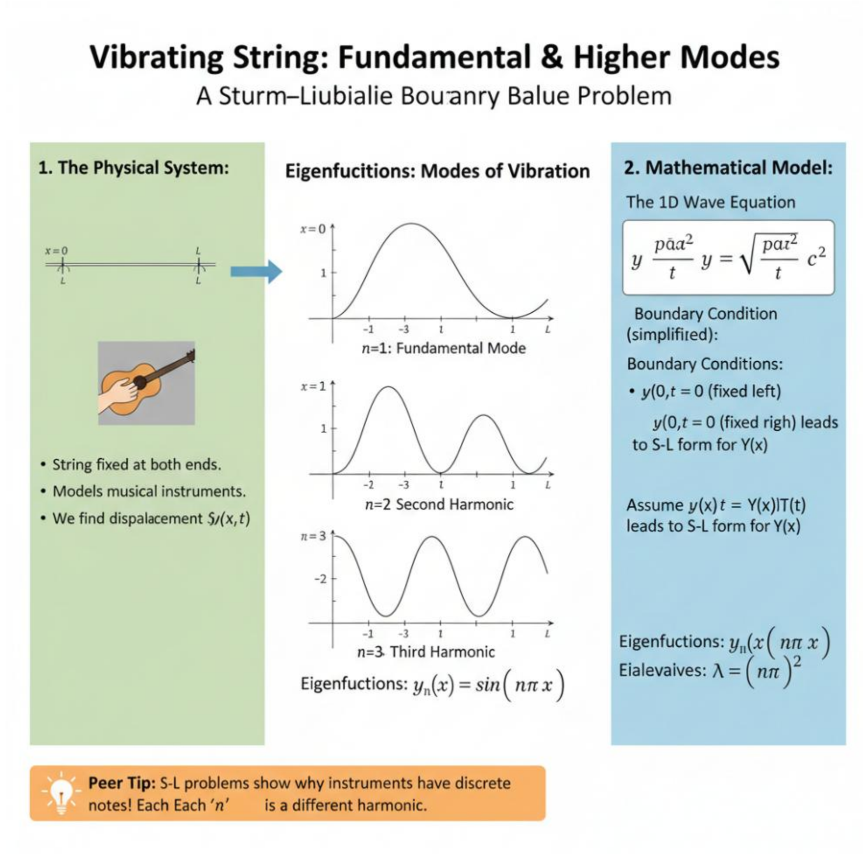
Sturm–Liouville problems are not only theoretical constructs, they have wide-ranging applications in physics and engineering. Their ability to generate orthogonal sets of eigenfunctions makes them particularly useful in solving **boundary value problems** and in expanding functions into series.

1.3.1 Boundary value problems

Boundary value problems arise when a differential equation must satisfy conditions at the boundaries of a domain. Sturm–Liouville theory provides a systematic way to solve these problems by reducing them to eigenvalue problems.

For example, consider the equation governing a vibrating string fixed at both ends. The boundary conditions require the displacement to be zero at the endpoints. Sturm–Liouville theory allows us to find eigenfunctions that satisfy these conditions, representing the modes of vibration.

Fill in the blank: In a vibrating string problem, the displacement at the fixed ends must be equal to __________.



Vibrating string as a boundary value problem.

1.3.2 Heat conduction examples

In heat conduction, the temperature distribution along a rod can be modelled using partial differential equations. By applying separation of variables, the spatial part of the solution reduces to a Sturm–Liouville problem. The eigenfunctions describe the temperature modes, while eigenvalues determine the rate of decay of each mode.

Fill in the blank: In heat conduction problems, eigenvalues determine the rate of __________ of each mode.

Applications of Sturm–Liouville problems in boundary value contexts

Problem	**Application**	**Notes**
Vibrating string	Modes of vibration	Displacement zero at ends

Heat conduction	Temperature distribution	Eigenvalues control decay rates

1.3.3 Connection to Fourier series

Sturm–Liouville problems are closely connected to **Fourier series** because the eigenfunctions form an orthogonal basis. This means that any function defined on the interval can be expanded in terms of these eigenfunctions, just as functions can be expanded in terms of sines and cosines in Fourier series.

This connection is powerful because it allows us to solve complex partial differential equations by expanding solutions in terms of eigenfunctions.

Fill in the blank: Sturm–Liouville eigenfunctions form an __________ basis, similar to sines and cosines in Fourier series.

[Insert Box here] Caption: Box 1.3 Connection between Sturm–Liouville problems and Fourier series

- Eigenfunctions form orthogonal basis
- Functions can be expanded in terms of eigenfunctions
- Provides method for solving PDEs AI prompt: "Generate a box explaining the connection between Sturm–Liouville problems and Fourier series." Search string: "Sturm Liouville Fourier series connection OER"

Pilot questions 1.3

1. In a vibrating string problem, the displacement at the fixed ends must be: A. Zero B. Infinite C. Constant D. None of the above (SLO 1.3)
2. In heat conduction problems, eigenvalues determine the rate of: A. Decay of each mode B. Growth of each mode C. Vibration of each mode D. None of the above (SLO 1.3)
3. Sturm–Liouville problems reduce boundary value problems to: A. Eigenvalue problems B. Algebraic equations C. Polynomial expansions D. None of the above (SLO 1.3)
4. Sturm–Liouville eigenfunctions form an: A. Orthogonal basis B. Parallel basis C. Non-orthogonal set D. None of the above (SLO 1.3)
5. The connection between Sturm–Liouville problems and Fourier series lies in: A. Orthogonality of eigenfunctions B. Use of boundary conditions C. Presence of eigenvalues D. None of the above (SLO 1.3)

1.4 Solution Techniques

Having explored the definition, properties, and applications of Sturm–Liouville problems, we now turn to the practical methods used to solve them. These techniques allow us to find eigenvalues and eigenfunctions systematically, and to apply them in real-world contexts.

1.4.1 Separation of variables

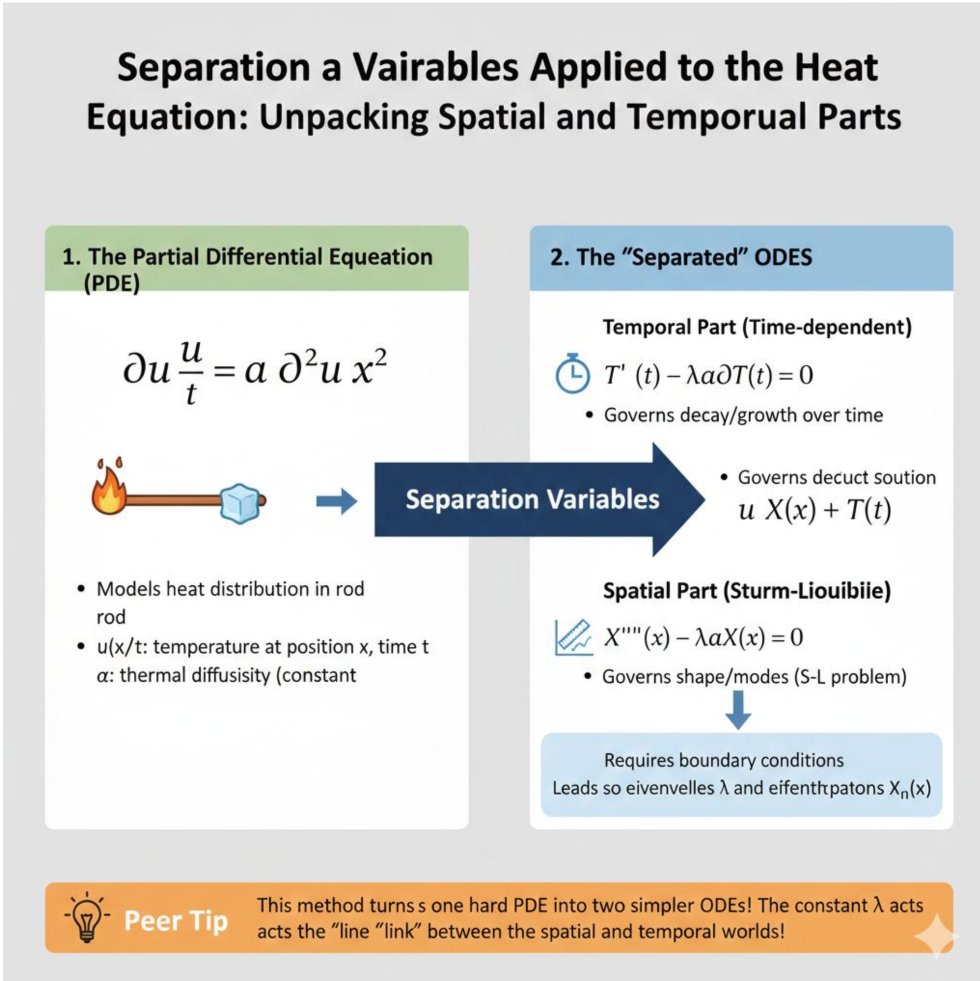
The **separation of variables** method involves assuming that the solution to a partial differential equation can be written as a product of functions, each depending on a single variable. When applied to problems such as the heat or wave equation, the spatial part of the solution reduces to a Sturm–Liouville problem.

For example, in the heat equation:

$$u(x,t)=X(x)T(t)$$

Substituting into the equation separates the variables, leading to an ordinary differential equation in $X(x)$ that is of Sturm–Liouville type.

Fill in the blank: In separation of variables, the solution is expressed as a __________ of functions, each depending on a single variable.



Separation of variables leading to a Sturm–Liouville problem.

1.4.2 Normalisation of eigenfunctions

Normalisation is the process of scaling eigenfunctions so that their inner product with themselves equals one. This ensures consistency when using eigenfunctions as basis functions in expansions.

For an eigenfunction y(x), normalisation requires:

$$aby(x)2w(x)\ dx=1$$

This step is important when constructing series solutions, as it simplifies coefficients and ensures orthogonality is preserved.

Fill in the blank: Normalisation of eigenfunctions ensures that their weighted inner product with themselves equals __________.

Steps in normalising eigenfunctions

- Compute weighted inner product
- Scale eigenfunction appropriately
- Ensure result equals one

2. Step-by-Step Normalization Process

Step	Action	Mathematical Goal
1	**Solve the S-L Problem**	Find the general eigenfunction $\phi_n(x)$ (which usually contains an arbitrary constant C).
2	**Identify the Weight**	Extract $w(x)$ from the original Sturm–Liouville equation format.
3	**Set up the Integral**	Calculate the "norm squared" ($
4	**Solve for the Constant**	Set $C^2 \times (\text{result of integral}) = 1$ and solve for C. Usually, $C = \frac{1}{\sqrt{I}}$.
5	**Define Normalized Form**	Write the final normalized eigenfunction: $y_n(x) = C\phi_n(x)$.

1.4.3 Worked examples

Let us consider a simple Sturm–Liouville problem:

d2ydx2+λy=0,y(0)=0,y(π)=0

The solutions are:

- Eigenvalues: n=n2, where n=1,2,3,...
- Eigenfunctions: yn(x)=sin(nx)

These eigenfunctions are orthogonal on the interval with respect to the weight function w(x)=1.

Fill in the blank: For the Sturm–Liouville problem with boundary conditions y(0)=0and y(π)=0, the eigenfunctions are __________.

Example Sturm–Liouville problem solutions

Eigenvalue	Eigenfunction	Notes
1=1	sin(x)	Fundamental mode

2=4	sin(2x)	Second mode
3=9	sin(3x)	Third mode

Pilot questions 1.4

1. In separation of variables, the solution is expressed as a: A. Product of functions B. Sum of functions C. Difference of functions D. None of the above (SLO 1.4)
2. Normalisation of eigenfunctions ensures that their weighted inner product with themselves equals: A. One B. Zero C. Infinity D. None of the above (SLO 1.4)
3. The eigenfunctions for the problem d2ydx2+λy=0, with y(0)=0and y(π)=0, are: A. sin(nx)B. cos(nx)C. enxD. None of the above (SLO 1.4)
4. In the heat equation, separation of variables reduces the spatial part to a: A. Sturm–Liouville problem B. Polynomial equation C. Algebraic identity D. None of the above (SLO 1.4)
5. Normalisation is important because it: A. Simplifies coefficients and preserves orthogonality B. Eliminates eigenvalues C. Removes boundary conditions D. None of the above (SLO 1.4)

Series Summary

In this Series, we explored the concept of **Sturm–Liouville problems**, which are fundamental in mathematical methods and widely applied in physics and engineering. We began by introducing the definition, historical background, and general form of these problems, highlighting their role as boundary value problems. We then examined the properties of Sturm–Liouville systems, focusing on self-adjoint operators, eigenvalues, eigenfunctions, and the orthogonality of solutions. Following that, we considered practical applications such as vibrating strings, heat conduction, and the connection to Fourier series. Finally, we studied solution techniques, including separation of variables, normalisation of eigenfunctions, and worked examples.

By completing this Series, you should now be able to **define** Sturm–Liouville problems and explain their general form (SLO 1.1), **analyse** their properties (SLO 1.2), **apply** Sturm–Liouville theory to boundary value problems (SLO 1.3), and **demonstrate** solution techniques such as separation of variables and normalisation (SLO 1.4). These skills provide a strong foundation for later topics in Fourier analysis and eigenfunction expansions.

Glossary of Terms

- **Boundary value problem**: A differential equation that requires solutions to satisfy conditions at the boundaries of the domain.
- **Eigenfunction**: A function that satisfies a Sturm–Liouville equation for a particular eigenvalue.
- **Eigenvalue**: A constant associated with an eigenfunction in a Sturm–Liouville problem.
- **Orthogonality**: A property where the weighted inner product of two distinct eigenfunctions equals zero.
- **Self-adjoint operator**: A differential operator that ensures eigenvalues are real and eigenfunctions form a complete set.
- **Separation of variables**: A method of solving partial differential equations by expressing the solution as a product of functions, each depending on a single variable.
- **Normalisation**: The process of scaling eigenfunctions so that their weighted inner product with themselves equals one.
- **Weight function**: A positive function used in Sturm–Liouville problems to define the inner product space.

Concept Check Questions 1

Objective questions

1. Sturm–Liouville problems are a type of __________ problem. (Linked to SLO 1.1)
2. A self-adjoint operator guarantees that eigenvalues are __________. (Linked to SLO 1.2)
3. Eigenfunctions corresponding to distinct eigenvalues are __________ with respect to the weight function. (Linked to SLO 1.2)
4. In heat conduction problems, eigenvalues determine the rate of __________ of each mode. (Linked to SLO 1.3)
5. Normalisation of eigenfunctions ensures that their weighted inner product with themselves equals __________. (Linked to SLO 1.4)

Short-answer questions

6. Define a Sturm–Liouville problem and state its general form. (Linked to SLO 1.1)

7. Explain why orthogonality of eigenfunctions is important in mathematical physics. (Linked to SLO 1.2)

8. Give one example of a physical system where Sturm–Liouville problems are applied. (Linked to SLO 1.3)

9. Describe the process of normalising an eigenfunction. (Linked to SLO 1.4)

Long-answer questions

10. Analyse the properties of Sturm–Liouville systems, explaining how self-adjointness leads to real eigenvalues and orthogonal eigenfunctions. (Linked to SLO 1.2)

11. Evaluate the usefulness of Sturm–Liouville problems in solving boundary value problems, with reference to vibrating strings and heat conduction. (Linked to SLO 1.3)

Answers to Concept Check Questions 1

Objective questions

1. Boundary value
2. Real
3. Orthogonal
4. Decay
5. One

Short-answer questions

6. A Sturm–Liouville problem is a second-order linear differential equation of the form ddx(p(x)dydx)+[λw(x)-q(x)]y=0, subject to boundary conditions.

7. Orthogonality ensures independence of solutions and allows functions to be expanded in terms of eigenfunctions, similar to Fourier series.

8. Vibrating string problems, where eigenfunctions represent modes of vibration.

9. Normalisation involves scaling the eigenfunction so that its weighted inner product with itself equals one.

Long-answer questions

10. *Basic response*: Self-adjoint operators ensure eigenvalues are real and eigenfunctions are orthogonal. *Value-added response*: This property allows eigenfunctions to form a complete basis, enabling expansions of arbitrary functions and providing a framework for spectral theory.

11. *Basic response*: Sturm–Liouville problems reduce boundary value problems to eigenvalue problems, making them solvable. *Value-added response*: In vibrating strings, eigenfunctions represent vibration modes, while in heat conduction, they describe temperature distributions. This demonstrates their broad applicability in physics and engineering.

Answers to Pilot Questions 1

Part 1.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.2**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.3**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.4**: 1-A, 2-A, 3-A, 4-A, 5-A

Series 2: Orthogonal Polynomials and Functions

Series outline

- **Part 2.1: Introduction to Orthogonal Polynomials and Functions**
  - Definition and general concept of orthogonality
  - Importance in mathematical methods and physics
- **Part 2.2: Classical Orthogonal Polynomials**
  - Legendre polynomials
  - Chebyshev polynomials
  - Hermite and Laguerre polynomials
- **Part 2.3: Properties and Applications of Orthogonal Functions**
  - Orthogonality relations
  - Recurrence relations

    - Applications in approximation and expansions
- **Part 2.4: Worked Examples and Problem Solving**
    - Solving problems using orthogonal polynomials
    - Demonstrating expansions and approximations
    - Practice exercises

Introduction

Orthogonal polynomials and functions form a central part of mathematical methods because they provide powerful tools for representing and approximating functions. The concept of **orthogonality** ensures that these functions can serve as bases for expansions, much like Fourier series, and they appear naturally in solutions to many physical and engineering problems. By studying this Series, you will see how orthogonal polynomials such as Legendre, Chebyshev, Hermite, and Laguerre play a vital role in applied mathematics.

The aim of this Series is to introduce you to the theory of orthogonal polynomials and functions, explain their properties, and demonstrate their applications in approximation and problem solving.

We shall commence this Series by defining orthogonality and explaining its importance. Next, we will study classical orthogonal polynomials, including Legendre, Chebyshev, Hermite, and Laguerre. Following that, we will examine the properties and applications of orthogonal functions, focusing on orthogonality relations and recurrence relations. Finally, we will conclude with worked examples and problem solving, where you will apply these concepts to practical exercises.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 2.1** define orthogonality and explain its importance in mathematical methods,
- **SLO 2.2** describe classical orthogonal polynomials such as Legendre, Chebyshev, Hermite, and Laguerre,
- **SLO 2.3** analyse the properties and applications of orthogonal functions, including orthogonality and recurrence relations, and

- **SLO 2.4** demonstrate problem solving using orthogonal polynomials and apply them in expansions and approximations.

2.1 Introduction to Orthogonal Polynomials and Functions

Orthogonal polynomials and functions are central to mathematical methods because they provide a systematic way of representing and approximating other functions. The concept of **orthogonality** means that two functions are independent in a certain sense, specifically that their weighted inner product equals zero. This property allows orthogonal functions to serve as building blocks for expansions, much like sine and cosine functions in Fourier series.

Fill in the blank: Two functions are said to be orthogonal if their weighted __________ equals zero.

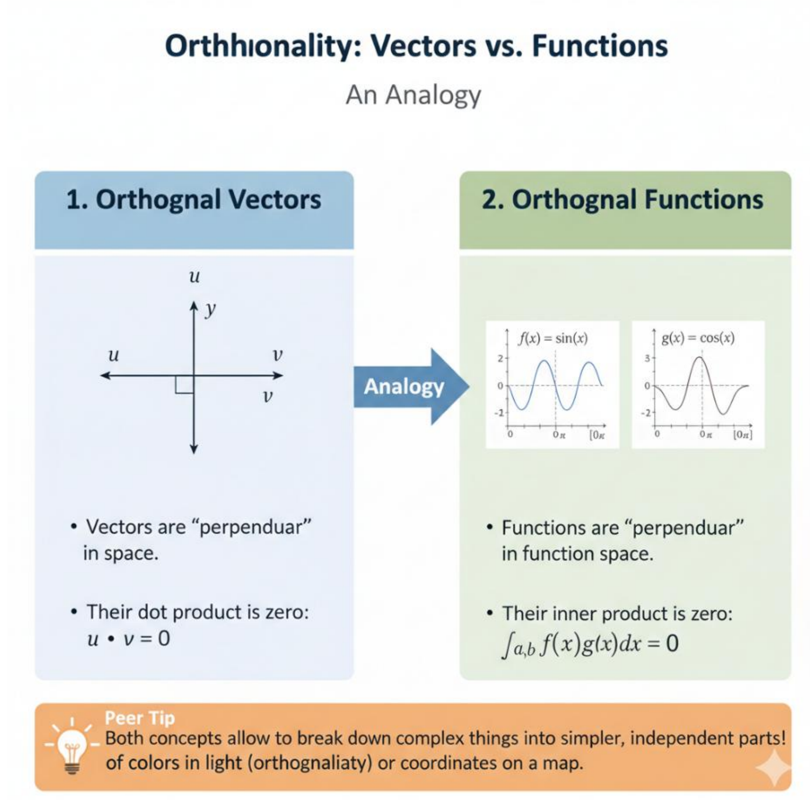
2.1.1 Definition and general concept of orthogonality

Orthogonality in mathematics refers to the property that the integral of the product of two functions, multiplied by a weight function, is zero over a given interval. Formally, for functions f(x)and g(x),

$$abf(x)g(x)w(x)\ dx=0$$

where w(x)is the **weight function**. This property is analogous to perpendicular vectors in geometry, which have a dot product equal to zero.

Fill in the blank: Orthogonality of functions is analogous to __________ vectors in geometry.



Orthogonality in vectors and functions.

2.1.2 Importance in mathematical methods and physics

Orthogonal polynomials and functions are important because they:

- Provide bases for function expansions.
- Simplify the solution of differential equations.
- Appear naturally in physical problems such as quantum mechanics, wave propagation, and heat conduction.
- Allow efficient approximation of functions in numerical methods.

For example, **Legendre polynomials** arise in solving Laplace's equation in spherical coordinates, while **Hermite polynomials** appear in quantum mechanics when solving the Schrödinger equation for the harmonic oscillator.

Fill in the blank: Legendre polynomials arise in solving __________ equation in spherical coordinates.

Importance of orthogonal polynomials and functions

- Basis for expansions
- Simplify differential equations
- Appear in physical problems
- Useful in numerical approximations

Pilot questions 2.1

1. Two functions are said to be orthogonal if their weighted: A. Inner product equals zero B. Sum equals zero C. Difference equals zero D. None of the above (SLO 2.1)
2. Orthogonality of functions is analogous to: A. Perpendicular vectors in geometry B. Parallel vectors in geometry C. Complex numbers D. None of the above (SLO 2.1)
3. The weight function in orthogonality is used to: A. Define the inner product space B. Eliminate eigenvalues C. Remove boundary conditions D. None of the above (SLO 2.1)
4. Legendre polynomials arise in solving: A. Laplace's equation in spherical coordinates B. Heat equation in Cartesian coordinates C. Schrödinger equation in quantum mechanics D. None of the above (SLO 2.1)
5. Orthogonal polynomials are important because they: A. Provide bases for expansions B. Complicate differential equations C. Ignore physical problems D. None of the above (SLO 2.1)

2.2 Classical Orthogonal Polynomials

Orthogonal polynomials are families of polynomials that satisfy specific orthogonality relations with respect to a weight function over a given interval. They arise naturally in solving differential equations and in approximation theory. In this Part, we will focus on some of the most important classical orthogonal polynomials: **Legendre**, **Chebyshev**, **Hermite**, and **Laguerre**.

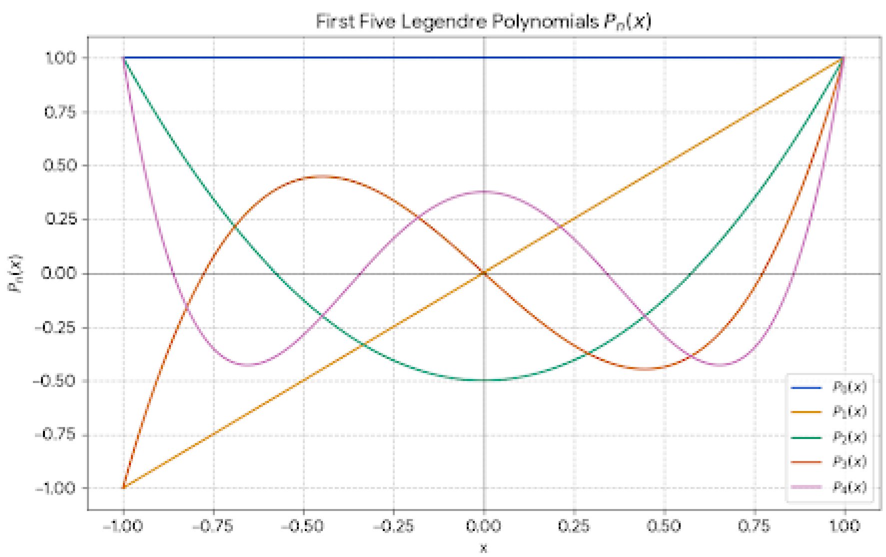
2.2.1 Legendre polynomials

Legendre polynomials are solutions to Legendre's differential equation:

$$(1\text{-}x2)d2ydx2\text{-}2xdydx+n(n+1)y=0$$

They are orthogonal on the interval -1,1with weight function w(x)=1. These polynomials are widely used in physics, particularly in solving Laplace's equation in spherical coordinates.

Fill in the blank: Legendre polynomials are orthogonal on the interval __________ with weight function w(x)=1.



Legendre polynomials of low degree.

2.2.2 Chebyshev polynomials

Chebyshev polynomials are defined by the relation:

$$T_n(x)=\cos(n\cos^{-1}(x))$$

They are orthogonal on -1,1with weight function w(x)=11-x2. Chebyshev polynomials are particularly important in numerical analysis because they minimise the maximum error in polynomial approximation.

Fill in the blank: Chebyshev polynomials are orthogonal on -1,1with weight function w(x)=11-x2.

Key features of Chebyshev polynomials

- Defined by trigonometric relation
- Orthogonal with weight function 1/1-x2

- Useful in minimising approximation error \

1. Fundamental Mathematical Properties

Feature	Definition / Property	Practical Benefit
Trigonometric Form	$T_n(\cos\theta) = \cos(n\theta)$	Allows for extremely fast calculation using the properties of cosines.
Recurrence Relation	$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$	Enables stable, recursive generation of higher-order polynomials ($T_0 = 1, T_1 = x$).
Minimax Property	Of all polynomials of degree n with a leading coefficient of 1, T_n has the smallest maximum value.	**Equal Error:** Spreads approximation errors evenly across the interval, avoiding "spikes" at the edges.
Weight Function	$w(x) = \frac{1}{\sqrt{1-x^2}}$	Defines the specific "S-L" inner product that makes these polynomials orthogonal.

2.2.3 Hermite polynomials

Hermite polynomials are solutions to Hermite's differential equation:

d2ydx2-2xdydx+2ny=0

They are orthogonal on -,with weight function w(x)=e-x2. Hermite polynomials appear in quantum mechanics, particularly in the solution of the Schrödinger equation for the harmonic oscillator.

Fill in the blank: Hermite polynomials are orthogonal on -,with weight function __________.

Classical orthogonal polynomials and their properties

Polynomial	**Interval**	**Weight function**	**Application**
Legendre	-1,1	1	Laplace's equation
Chebyshev	-1,1	1/1-x2	Approximation theory
Hermite	-,	e-x2	Quantum mechanics
Laguerre	[0,)	e-x	Radial Schrödinger equation

2.2.4 Laguerre polynomials

Laguerre polynomials are solutions to Laguerre's differential equation:

xd2ydx2+(1-x)dydx+ny=0

They are orthogonal on [0,)with weight function w(x)=e-x. Laguerre polynomials are used in physics, especially in solving the radial part of the Schrödinger equation for the hydrogen atom.

Fill in the blank: Laguerre polynomials are orthogonal on [0,)with weight function __________.

[Insert Figure here] Short description: Graph of the first few Laguerre polynomials. Caption: Figure 2.3 Laguerre polynomials of low degree. AI prompt: "Generate a plot showing the first few Laguerre polynomials on the interval [0,∞]." Search string: "Laguerre polynomials graph OER"

Pilot questions 2.2

1. Legendre polynomials are orthogonal on: A. -1,1with weight function 1B. [0,)with weight function e-xC. -,with weight function e-x2D. None of the above (SLO 2.2)
2. Chebyshev polynomials are defined by: A. Tn(x)=cos(ncos-1(x))B. Tn(x)=sin(ncos-1(x))C. Tn(x)=exD. None of the above (SLO 2.2)
3. Hermite polynomials are orthogonal on: A. -,with weight function e-x2B. -1,1with weight function 1C. [0,)with weight function e-xD. None of the above (SLO 2.2)
4. Laguerre polynomials are used in solving: A. Radial Schrödinger equation for hydrogen atom B. Laplace's equation in spherical coordinates C. Heat equation in Cartesian coordinates D. None of the above (SLO 2.2)
5. Which classical orthogonal polynomial is most useful in approximation theory? A. Chebyshev B. Legendre C. Hermite D. Laguerre (SLO 2.2)

2.3 Properties and Applications of Orthogonal Functions

Orthogonal functions possess distinctive properties that make them highly effective in mathematical analysis and practical applications. These properties include **orthogonality relations**, **recurrence relations**, and their use in approximation and expansions. Together, they provide a framework for solving differential equations and representing complex functions in simpler forms.

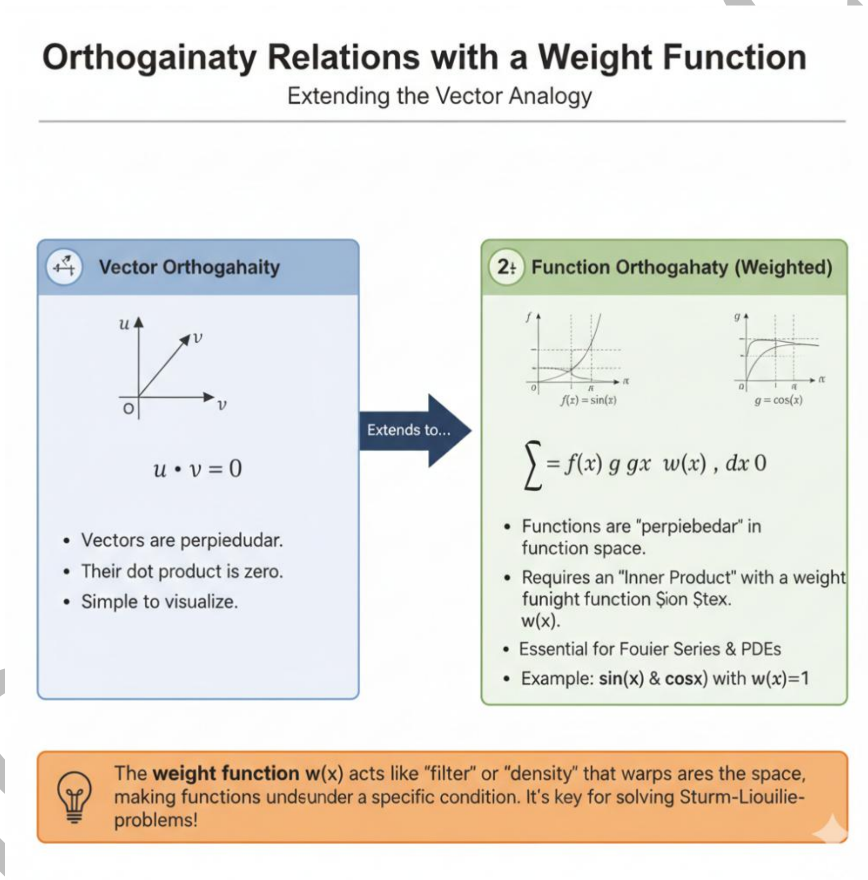
2.3.1 Orthogonality relations

Orthogonality relations ensure that different members of a family of orthogonal functions are independent of one another. For functions fm(x)and fn(x), the relation is:

abfm(x)fn(x)w(x) dx=0for mn

This property is fundamental because it allows functions to be expanded uniquely in terms of orthogonal bases, simplifying analysis and computation.

Fill in the blank: Orthogonality relations ensure that the integral of the product of two distinct functions, multiplied by the __________ function, equals zero.



Orthogonality relations of functions.

2.3.2 Recurrence relations

Recurrence relations are formulas that relate polynomials of different orders within the same family. They provide a convenient way to generate higher-order polynomials from lower-order ones.

For example, Legendre polynomials satisfy the recurrence relation:

n+1)Pn+1(x)=(2n+1)xPn(x)-nPn-1(x

Recurrence relations are useful because they reduce computational effort and provide insight into the structure of polynomial families.

Fill in the blank: Recurrence relations allow higher-order polynomials to be generated from __________-order ones.

Example recurrence relation for Legendre polynomials
n+1)Pn+1(x)=(2n+1)xPn(x)-nPn-1(x

1. The Standard (Bonnet's) Recurrence Relation

The most fundamental relation, used to generate the sequence of polynomials starting from $P_0(x) = 1$ and $P_1(x) = x$, is:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Rearranged for $P_{n+1}(x)$:

$$P_{n+1}(x) = \frac{(2n+1)xP_n(x) - nP_{n-1}(x)}{n+1}$$

2. Derivative Recurrence Relations

These relations are essential when solving differential equations or performing numerical integration where the slope of the polynomial is required:

Type	Formula
Relation A	$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$
Relation B	$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$
Relation C	$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$

2.3.3 Applications in approximation and expansions

Orthogonal functions are widely applied in approximation theory. By expanding a function in terms of orthogonal polynomials, one can approximate complex functions with high accuracy. This is particularly useful in numerical analysis, physics, and engineering.

For example:

- Chebyshev polynomials minimise maximum error in polynomial approximation.
- Legendre polynomials are used in spherical harmonics for solving Laplace's equation.
- Hermite polynomials appear in probability theory and quantum mechanics.

Fill in the blank: Chebyshev polynomials are especially useful in approximation because they minimise the maximum __________.

Applications of orthogonal functions in approximation and expansions

Polynomial	Application	Notes
Chebyshev	Polynomial approximation	Minimises maximum error
Legendre	Spherical harmonics	Solves Laplace's equation
Hermite	Quantum mechanics	Harmonic oscillator solutions
Laguerre	Hydrogen atom	Radial Schrödinger equation

Pilot questions 2.3

1. Orthogonality relations ensure that the integral of the product of two distinct functions, multiplied by the weight function, equals: A. Zero B. One C. Infinity D. None of the above (SLO 2.3)
2. Recurrence relations allow higher-order polynomials to be generated from: A. Lower-order ones B. Random functions C. Complex numbers D. None of the above (SLO 2.3)
3. Legendre polynomials satisfy the recurrence relation: A. n+1)Pn+1(x)=(2n+1)xPn(x)-nPn-1(xB. Pn+1(x)=Pn(x)+Pn-1(x)C. Pn+1(x)=xPn(x)D. None of the above (SLO 2.3)

4. Chebyshev polynomials are especially useful in approximation because they minimise: A. Maximum error B. Minimum error C. Orthogonality D. None of the above (SLO 2.3)
5. Hermite polynomials appear in: A. Quantum mechanics B. Heat conduction C. Laplace's equation D. None of the above (SLO 2.3)

2.4 Worked Examples and Problem Solving

In this Part, we will consolidate the ideas from earlier sections by working through examples that demonstrate how orthogonal polynomials can be applied to solve problems. These examples will show how orthogonal functions are used in expansions, approximations, and solutions to differential equations.

2.4.1 Solving problems using orthogonal polynomials

Let us consider the approximation of a function $f(x)$on the interval -1,1using **Legendre polynomials**. The expansion takes the form:

$$f(x)\approx n=0NanPn(x)$$

where $Pn(x)$are Legendre polynomials and the coefficients anare given by:

$$an=2n+12-11f(x)Pn(x)\ dx$$

This method allows us to approximate functions with high accuracy using a finite number of terms.

Fill in the blank: In Legendre polynomial expansions, the coefficients anare obtained by integrating the product of $f(x)$and __________ over the interval -1,1.

[Insert Figure here] Short description: Diagram showing approximation of a function using Legendre polynomials. Caption: Figure 2.5 Function approximation using Legendre polynomial expansion. AI prompt: "Generate a diagram showing approximation of a function using Legendre polynomial expansion." Search string: "Legendre polynomial expansion example OER"

2.4.2 Demonstrating expansions and approximations

Chebyshev polynomials are often used in numerical analysis to approximate functions because they minimise maximum error. For example, approximating $f(x)=\cos(x)$on -1,1using Chebyshev polynomials gives:

$$f(x)\approx n=0NbnTn(x)$$

where Tn(x)are Chebyshev polynomials and bnare coefficients determined by integration with the weight function w(x)=11-x2.

Fill in the blank: Chebyshev polynomials are especially useful in approximation because they minimise the maximum __________.

Steps in approximating functions using Chebyshev polynomials

- Select interval -1,1
- Compute coefficients using weight function
- Expand function in terms of Chebyshev polynomials
- Truncate series for approximation

2. Step-by-Step Approximation Process

Step	Action	Description
1	**Map the Interval**	If your function is on $[a, b]$, map it to $[-1, 1]$ using $x = \frac{2t-(a+b)}{b-a}$.
2	**Select the Degree (N)**	Choose how many polynomials to use. Higher N means higher accuracy.
3	**Identify Chebyshev Nodes**	Calculate the $N+1$ roots: $x_k = \cos\left(\frac{\pi(k+0.5)}{N+1}\right)$ for $k = 0, \ldots, N$.
4	**Evaluate the Function**	Calculate the value of your function $f(x)$ at each of these specific nodes x_k.
5	**Compute Coefficients (c_n)**	Use the Discrete Cosine Transform formula: $c_n = \frac{2}{N+1} \sum_{k=0}^{N} f(x_k) \cos\left(\frac{n\pi(k+0.5)}{N+1}\right)$.
6	**Sum the Series**	Construct the final polynomial $P_N(x) = \frac{c_0}{2} + \sum_{n=1}^{N} c_n T_n(x)$.

2.4.3 Practice exercises

To reinforce your learning, here are some practice exercises:

1. Expand f(x)=x2on -1,1using Legendre polynomials up to P2(x).
2. Approximate f(x)=sin(x)on -1,1using Chebyshev polynomials up to T3(x).

3. Show that Hermite polynomials are orthogonal with respect to the weight function e-x2on -,.

Fill in the blank: Hermite polynomials are orthogonal with respect to the weight function __________ on -,.

Practice exercises with orthogonal polynomials

Exercise	Polynomial	Interval	Weight function
Expand x2	Legendre	-1,1	1
Approximate sin(x)	Chebyshev	-1,1	1/1-x2
Orthogonality proof	Hermite	-,	e-x2

Pilot questions 2.4

1. In Legendre polynomial expansions, the coefficients anare obtained by integrating the product of f(x)and: A. Legendre polynomial Pn(x)B. Chebyshev polynomial Tn(x)C. Hermite polynomial D. None of the above (SLO 2.4)
2. Chebyshev polynomials are especially useful in approximation because they minimise: A. Maximum error B. Minimum error C. Orthogonality D. None of the above (SLO 2.4)
3. Hermite polynomials are orthogonal with respect to the weight function: A. e-x2B. e-xC. 1D. None of the above (SLO 2.4)
4. Expanding f(x)=x2on -1,1using Legendre polynomials involves terms up to: A. P2(x)B. P3(x)C. P4(x)D. None of the above (SLO 2.4)
5. In Chebyshev polynomial approximation, coefficients are determined using the weight function: A. 1/1-x2B. e-x2C. e-xD. None of the above (SLO 2.4)

Series Summary

In this Series, we examined the concept of **orthogonal polynomials and functions**, which are essential tools in mathematical methods and widely applied in physics, engineering, and numerical analysis. We began by introducing the definition of orthogonality and its importance in providing bases for expansions. We then studied classical orthogonal polynomials, including Legendre, Chebyshev, Hermite, and Laguerre, highlighting their intervals, weight functions, and applications. Following that, we explored the properties of orthogonal functions, focusing on orthogonality relations, recurrence relations, and their role in approximation. Finally, we worked through examples that demonstrated how orthogonal polynomials are applied in expansions and problem solving.

By completing this Series, you should now be able to **define** orthogonality and explain its importance (SLO 2.1), **describe** classical orthogonal polynomials (SLO 2.2), **analyse** the properties and applications of orthogonal functions (SLO 2.3), and **demonstrate** problem solving using orthogonal polynomials in expansions and approximations (SLO 2.4).

Glossary of Terms

- **Chebyshev polynomials**: Orthogonal polynomials defined by a trigonometric relation, useful in minimising maximum error in approximation.
- **Hermite polynomials**: Orthogonal polynomials on -,with weight function e-x2, used in quantum mechanics and probability theory.
- **Laguerre polynomials**: Orthogonal polynomials on [0,)with weight function e-x, applied in the radial Schrödinger equation for hydrogen atoms.
- **Legendre polynomials**: Orthogonal polynomials on -1,1with weight function 1, used in solving Laplace's equation in spherical coordinates.
- **Orthogonality**: A property where the weighted inner product of two distinct functions equals zero.
- **Recurrence relation**: A formula that relates polynomials of different orders within the same family, allowing higher-order polynomials to be generated from lower-order ones.
- **Weight function**: A positive function used to define the inner product space in orthogonality relations.

Concept Check Questions 2

Objective questions

1. Orthogonality of functions means their weighted inner product equals __________. (Linked to SLO 2.1)
2. Legendre polynomials are orthogonal on the interval -1,1with weight function __________. (Linked to SLO 2.2)

3. Chebyshev polynomials are especially useful in approximation because they minimise __________. (Linked to SLO 2.3)
4. Hermite polynomials are orthogonal with respect to the weight function __________. (Linked to SLO 2.2)
5. Recurrence relations allow higher-order polynomials to be generated from __________-order ones. (Linked to SLO 2.3)

Short-answer questions

6. Define orthogonality in the context of functions. (Linked to SLO 2.1)

7. State the recurrence relation for Legendre polynomials. (Linked to SLO 2.3)

8. Give one physical application of Laguerre polynomials. (Linked to SLO 2.2)

9. Explain why Chebyshev polynomials are important in numerical analysis. (Linked to SLO 2.3)

Long-answer questions

10. Analyse the properties of classical orthogonal polynomials, highlighting their intervals, weight functions, and applications. (Linked to SLO 2.2)

11. Evaluate the usefulness of orthogonal polynomials in approximation and expansions, with examples from Legendre and Chebyshev polynomials. (Linked to SLO 2.4)

Answers to Concept Check Questions 2

Objective questions

1. Zero
2. 1
3. Maximum error
4. e-x2
5. Lower

Short-answer questions

6. Orthogonality means the integral of the product of two functions, multiplied by a weight function, equals zero over a given interval.

7. n+1)Pn+1(x)=(2n+1)xPn(x)-nPn-1(x.

8. Laguerre polynomials are used in solving the radial Schrödinger equation for the hydrogen atom.

9. Chebyshev polynomials minimise maximum error, making them efficient for polynomial approximation in numerical analysis.

Long-answer questions

10. *Basic response*: Legendre polynomials are orthogonal on -1,1with weight 1, Chebyshev on -1,1with weight 1/1-x2, Hermite on -,with weight e-x2, and Laguerre on [0,)with weight e-x. *Value-added response*: These polynomials are applied in Laplace's equation, approximation theory, quantum mechanics, and the hydrogen atom problem, showing their broad utility.

11. *Basic response*: Orthogonal polynomials allow functions to be expanded and approximated efficiently. *Value-added response*: Legendre polynomials provide spherical harmonics for Laplace's equation, while Chebyshev polynomials minimise maximum error in approximation, demonstrating their importance in both theoretical and numerical contexts.

Answers to Pilot Questions 2

Part 2.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.2**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.3**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.4: 1-A, 2-A, 3-A, 4-A, 5-A**

Series 3: Fourier Series and Integrals

Series outline

- **Part 3.1: Introduction to Fourier Series**
  - Definition and concept of periodic functions
  - General form of Fourier series
  - Importance in mathematical methods
- **Part 3.2: Properties of Fourier Series**
  - Convergence and conditions
  - Even and odd functions
  - Parseval's theorem

- **Part 3.3: Fourier Integrals and Transforms**
  - Transition from Fourier series to Fourier integrals
  - Fourier transform definition and properties
  - Applications in solving differential equations
- **Part 3.4: Worked Examples and Applications**
  - Expansion of functions using Fourier series
  - Use of Fourier integrals in physical problems
  - Practice exercises

Introduction

Fourier series and integrals are powerful tools for representing functions and solving differential equations. A **Fourier series** expresses a periodic function as a sum of sines and cosines, while **Fourier integrals** and **Fourier transforms** extend this idea to non-periodic functions. These methods are widely used in physics, engineering, and applied mathematics, particularly in problems involving heat conduction, wave propagation, and signal processing.

The aim of this Series is to introduce you to Fourier series and integrals, explain their properties, and demonstrate their applications in solving mathematical and physical problems.

We shall commence this Series by defining Fourier series and explaining their importance in representing periodic functions. Next, we will study the properties of Fourier series, including convergence, even and odd functions, and Parseval's theorem. Following that, we will explore Fourier integrals and transforms, focusing on their definitions, properties, and applications. Finally, we will conclude with worked examples and applications, where you will practise expanding functions and applying Fourier methods to physical problems.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 3.1** define Fourier series and explain their importance in representing periodic functions,
- **SLO 3.2** analyse the properties of Fourier series, including convergence, even and odd functions, and Parseval's theorem,

- **SLO 3.3** explain Fourier integrals and transforms, and apply them to solving differential equations, and
- **SLO 3.4** demonstrate problem solving using Fourier series and integrals, supported by worked examples and applications.

3.1 Introduction to Fourier Series

Fourier series provide a way to represent periodic functions as sums of sine and cosine terms. A **periodic function** is one that repeats its values at regular intervals, such as the sine and cosine functions themselves. By expressing a function in terms of trigonometric components, we can analyse and solve problems in physics, engineering, and applied mathematics more effectively.

Fill in the blank: A periodic function is one that __________ its values at regular intervals.

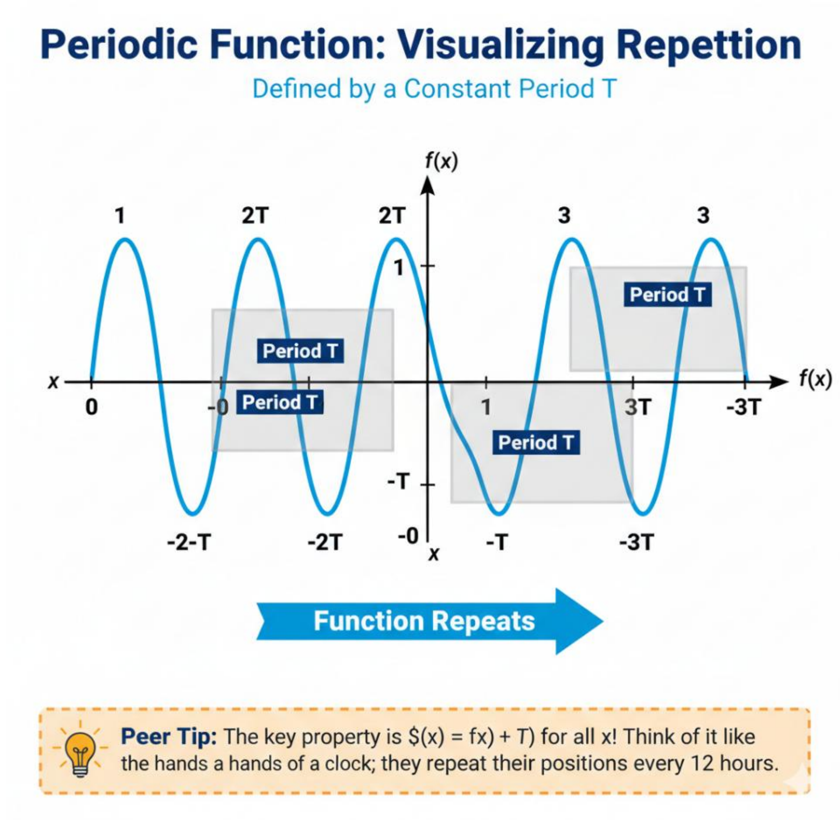
3.1.1 Definition and concept of periodic functions

A **periodic function** is defined by the property:

$$f(x+T)=f(x)$$

for all x, where Tis the **period**. Examples include trigonometric functions such as sin(x)and cos(x). Fourier series allow us to represent any reasonably well-behaved periodic function as a sum of sines and cosines.

Fill in the blank: The constant Tin the definition of a periodic function is called the __________.



Example of a periodic function.

3.1.2 General form of Fourier series

The general form of a Fourier series for a function f(x)with period 2πis:

$$f(x)=a02+n=1ancos(nx)+bnsin(nx)$$

where the coefficients are given by:

$$an=1-\pi f(x)cos(nx)\ dx,bn=1-\pi f(x)sin(nx)\ dx$$

These coefficients determine the contribution of each sine and cosine term to the overall function.

Fill in the blank: In a Fourier series, the coefficients anand bnare obtained by evaluating specific __________.

General form of Fourier series

- Function expressed as sum of sines and cosines

- Coefficients determined by integration
- Applicable to periodic functions

2. The Fourier Coefficients (Euler's Formulas)

The coefficients a_0, a_n, and b_n represent the "weight" or amplitude of each frequency component. They are calculated using the orthogonality of sine and cosine:

Coefficient	Integral Formula	Role in the Series
a_0	$\frac{1}{L}\int_{-L}^{L} f(x)\,dx$	The **DC Offset** or average value of the function over one period.
a_n	$\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$	The amplitudes of the **even** (cosine) components.
b_n	$\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$	The amplitudes of the **odd** (sine) components.

3.1.3 Importance in mathematical methods

Fourier series are important because they:

- Provide a systematic way to represent periodic functions.
- Simplify the solution of partial differential equations such as the heat and wave equations.
- Form the basis for Fourier integrals and transforms, which extend the method to non-periodic functions.
- Are widely applied in physics, engineering, and signal processing.

Fill in the blank: Fourier series are widely applied in physics, engineering, and ___________ processing.

Applications of Fourier series

Field	Application	Notes
Physics	Heat equation	Temperature distribution
Engineering	Wave equation	Vibrations and acoustics
Signal processing	Function representation	Frequency analysis

Pilot questions 3.1

1. A periodic function is one that: A. Repeats its values at regular intervals B. Is constant everywhere C. Has no defined period D. None of the above (SLO 3.1)
2. The constant Tin the definition of a periodic function is called the: A. Period B. Frequency C. Amplitude D. None of the above (SLO 3.1)
3. In a Fourier series, the coefficients anand bnare obtained by evaluating: A. Integrals B. Derivatives C. Limits D. None of the above (SLO 3.1)
4. Fourier series express a function as a sum of: A. Sines and cosines B. Polynomials C. Exponentials only D. None of the above (SLO 3.1)
5. Fourier series are widely applied in: A. Signal processing B. Geometry C. Number theory D. None of the above (SLO 3.1)

3.2 Properties of Fourier Series

Fourier series have several important properties that determine how effectively they represent periodic functions. These properties include **convergence conditions**, the treatment of **even and odd functions**, and **Parseval's theorem**, which connects Fourier coefficients to energy or power in a signal.

3.2.1 Convergence and conditions

A Fourier series converges to the original function under certain conditions, known as **Dirichlet's conditions**. These state that a function can be represented by a Fourier series if:

- It is periodic.
- It is piecewise continuous.
- It has a finite number of maxima and minima in any given interval.

When these conditions are satisfied, the Fourier series converges to the function at all points of continuity and to the average of the left and right limits at points of discontinuity.

Fill in the blank: A Fourier series converges to the average of the left and right limits at points of __________.

[Insert Box here] Caption: Box 3.2 Dirichlet's conditions for Fourier series convergence

- Function must be periodic
- Function must be piecewise continuous

- Function must have finite maxima and minima AI prompt: “Generate a box listing Dirichlet’s conditions for Fourier series convergence.” Search string: “Dirichlet conditions Fourier series OER”

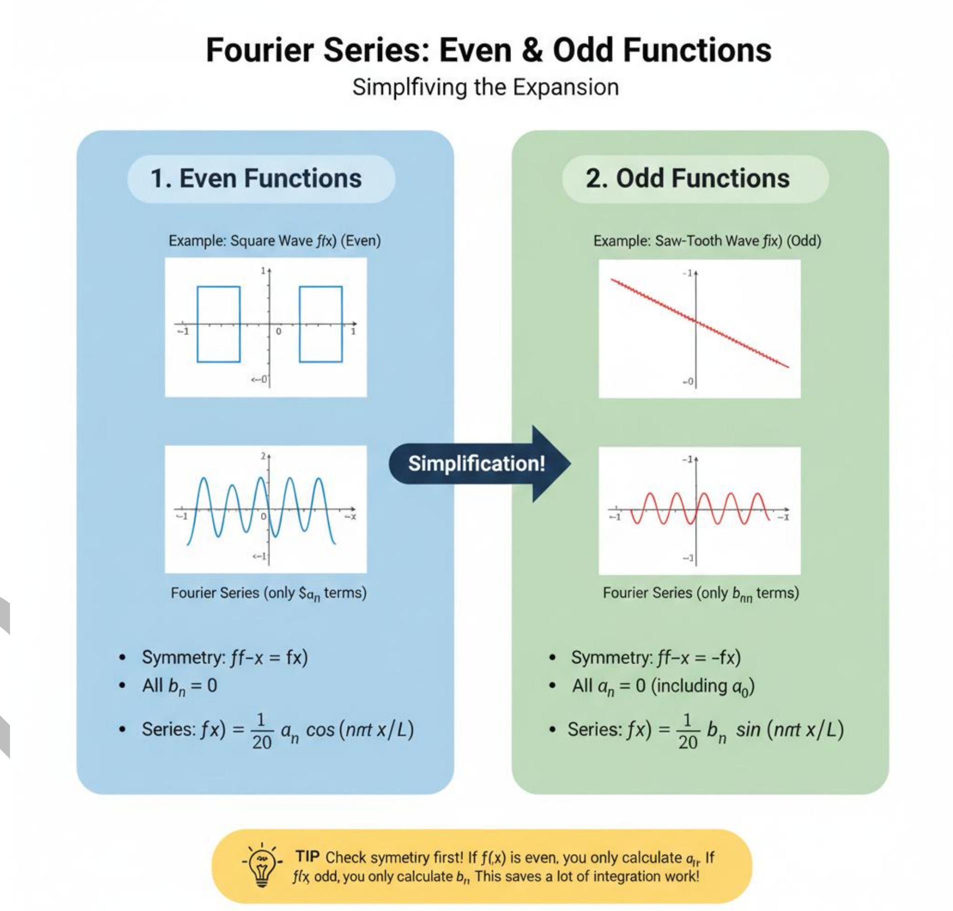
3.2.2 Even and odd functions

Fourier series simplify when applied to **even** and **odd** functions:

- For an **even function** (f(-x)=f(x)), only cosine terms appear in the Fourier series.
- For an **odd function** (f(-x)=-f(x)), only sine terms appear.

This property reduces computation and highlights the symmetry of the function being represented.

Fill in the blank: In a Fourier series, only __________ terms appear when the function is odd.



Fourier series representation of even and odd functions.

3.2.3 Parseval's theorem

Parseval's theorem states that the total energy (or power) of a function over one period is equal to the sum of the squares of its Fourier coefficients. Mathematically:

$$1\text{-}\pi|f(x)|2dx=a022+n=1an2+bn2$$

This theorem is important in signal processing and physics because it connects the time domain representation of a function to its frequency domain representation.

Fill in the blank: Parseval's theorem relates the total energy of a function to the sum of the squares of its __________.

Properties of Fourier series

Property	**Explanation**	**Application**
Convergence	Dirichlet's conditions	Ensures series represents function correctly
Even functions	Only cosine terms	Simplifies computation
Odd functions	Only sine terms	Highlights symmetry
Parseval's theorem	Energy equals sum of coefficient squares	Signal processing

Pilot questions 3.2

1. A Fourier series converges to the average of the left and right limits at points of: A. Discontinuity B. Continuity C. Infinity D. None of the above (SLO 3.2)
2. Dirichlet's conditions require a function to be: A. Periodic and piecewise continuous B. Constant everywhere C. Non-periodic D. None of the above (SLO 3.2)
3. In a Fourier series, only cosine terms appear when the function is: A. Even B. Odd C. Discontinuous D. None of the above (SLO 3.2)
4. In a Fourier series, only sine terms appear when the function is: A. Odd B. Even C. Continuous D. None of the above (SLO 3.2)

5. Parseval’s theorem relates the total energy of a function to the sum of the squares of its: A. Fourier coefficients B. Derivatives C. Integrals D. None of the above (SLO 3.2)

3.3 Fourier Integrals and Transforms

Fourier integrals and transforms extend the concept of Fourier series to non-periodic functions. While Fourier series represent periodic functions as sums of sines and cosines, Fourier integrals and transforms allow us to analyse functions defined on infinite intervals. This makes them powerful tools in physics, engineering, and applied mathematics.

3.3.1 Transition from Fourier series to Fourier integrals

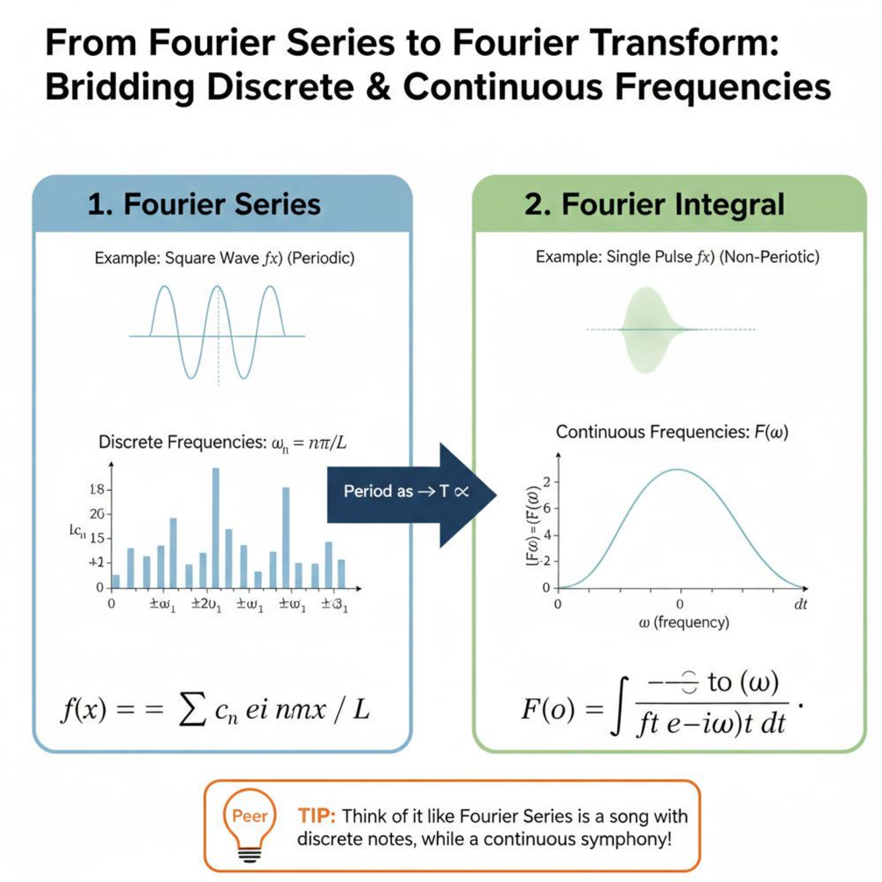
When the period of a function becomes infinitely large, the Fourier series representation transitions into a **Fourier integral**. Instead of discrete sums of sines and cosines, the representation involves continuous integrals over frequency.

The Fourier integral representation of a function f(x)is:

$$f(x)=12\pi\text{-}A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x)d\omega$$

This allows non-periodic functions to be expressed in terms of continuous frequency components.

Fill in the blank: When the period of a function becomes infinitely large, the Fourier series transitions into a Fourier __________.



Transition from Fourier series to Fourier integrals.

3.3.2 Fourier transform definition and properties

The **Fourier transform** is a mathematical operation that converts a function from the time domain into the frequency domain. It is defined as:

$$F(\omega)=-f(x)e-i\omega x\,dx$$

The inverse Fourier transform recovers the original function:

$$f(x)=12\pi-F(\omega)ei\omega x\,d\omega$$

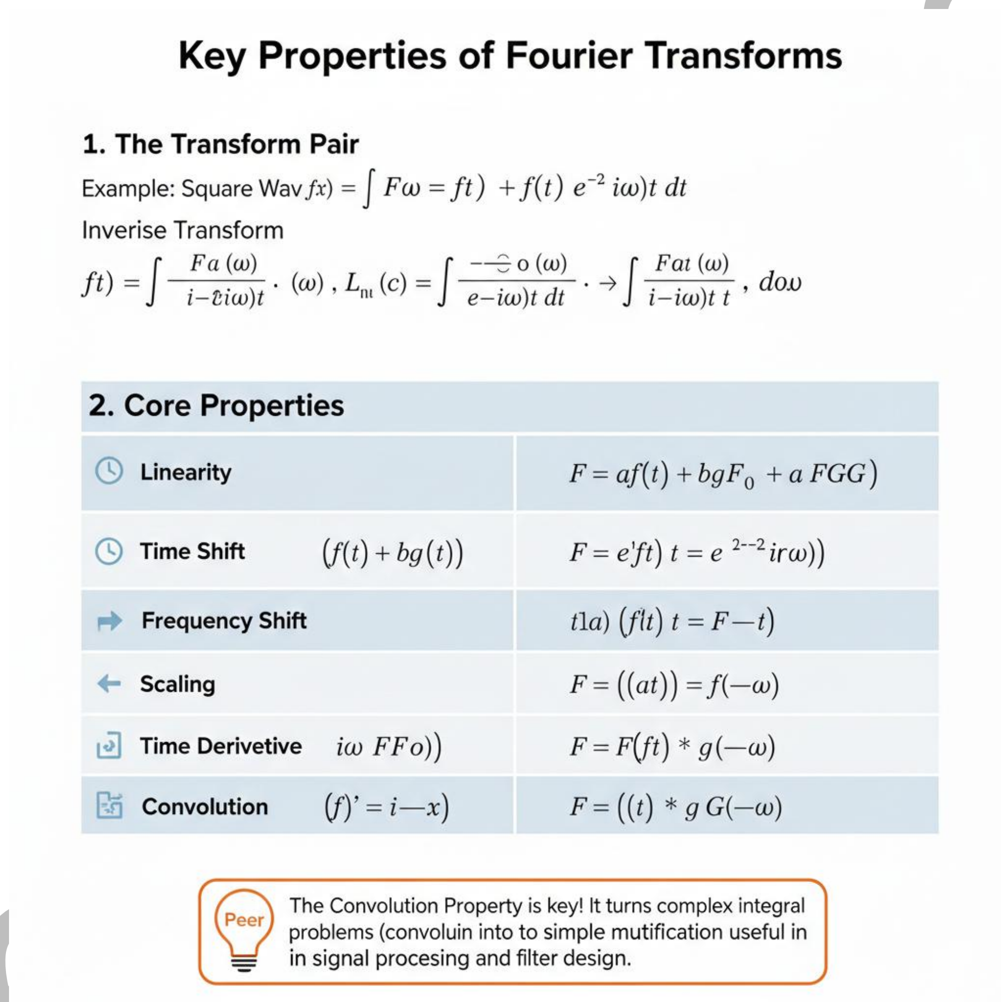
Key properties of Fourier transforms include:

- **Linearity**: The transform of a sum is the sum of the transforms.
- **Scaling**: Stretching a function in time compresses its transform in frequency.
- **Shift**: Shifting a function in time introduces a phase factor in frequency.

Fill in the blank: The Fourier transform converts a function from the time domain into the __________ domain.

Properties of Fourier transforms

- Linearity
- Scaling
- Shift property



3.3.3 Applications in solving differential equations

Fourier integrals and transforms are widely used to solve differential equations, especially in physics and engineering. For example:

- In **heat conduction**, Fourier transforms simplify the heat equation by converting derivatives into algebraic terms.

- In **wave propagation**, they allow analysis of signals in terms of frequency components.
- In **signal processing**, Fourier transforms are used to analyse and filter signals.

Fill in the blank: Fourier transforms simplify the heat equation by converting derivatives into __________ terms.

Applications of Fourier integrals and transforms

Field	**Application**	**Notes**
Heat conduction	Heat equation	Converts derivatives into algebraic terms
Wave propagation	Signal analysis	Frequency components
Signal processing	Filtering	Frequency domain representation

Pilot questions 3.3

1. When the period of a function becomes infinitely large, the Fourier series transitions into a: A. Fourier integral B. Fourier polynomial C. Fourier coefficient D. None of the above (SLO 3.3)
2. The Fourier transform converts a function from the time domain into the: A. Frequency domain B. Space domain C. Energy domain D. None of the above (SLO 3.3)
3. The inverse Fourier transform recovers the: A. Original function B. Fourier coefficients only C. Derivative of the function D. None of the above (SLO 3.3)
4. Fourier transforms simplify the heat equation by converting derivatives into: A. Algebraic terms B. Polynomial terms C. Trigonometric terms D. None of the above (SLO 3.3)
5. Which property of Fourier transforms states that stretching a function in time compresses its transform in frequency? A. Scaling B. Linearity C. Shift D. None of the above (SLO 3.3)

3.4 Worked Examples and Applications

In this Part, we will consolidate the concepts of Fourier series and transforms by working through examples that demonstrate their practical use. These examples will show how Fourier methods can be applied to expand functions, solve differential equations, and analyse signals.

3.4.1 Expansion of functions using Fourier series

Consider the function f(x)=xdefined on the interval -π,π. Since this function is odd, its Fourier series expansion will contain only sine terms:

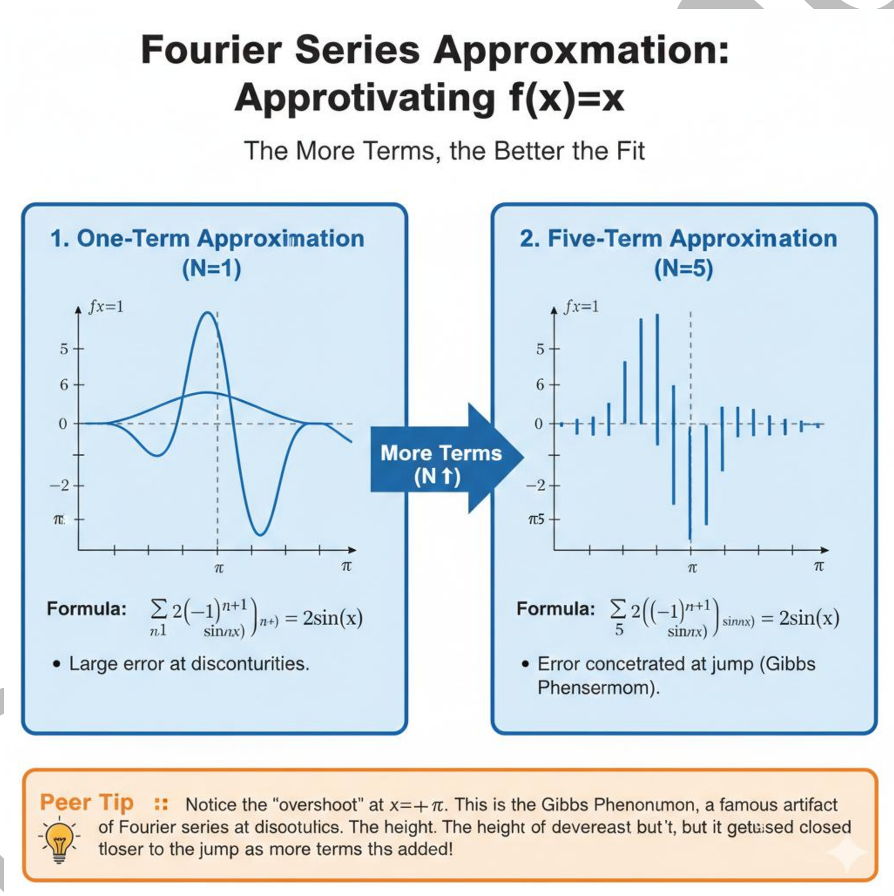
$$f(x)=n=1bnsin(nx)$$

where

$$bn=1-\pi xsin(nx)\ dx$$

This expansion allows us to represent the linear function xusing trigonometric components.

Fill in the blank: Since f(x)=xis an odd function, its Fourier series expansion contains only __________ terms.



Fourier series expansion of f(x)=x.

3.4.2 Use of Fourier integrals in physical problems

Fourier integrals are particularly useful in solving problems involving non-periodic functions. For example, in **heat conduction**, the temperature distribution in an infinite rod can be expressed using Fourier integrals. The transform converts derivatives into algebraic terms, making the equation easier to solve.

Fill in the blank: Fourier integrals are especially useful in solving problems involving __________ functions.

Steps in solving heat conduction problems using Fourier integrals

- Express temperature distribution as Fourier integral
- Convert derivatives into algebraic terms
- Solve algebraic equation in frequency domain
- Apply inverse transform to obtain solution in time domain

Steps in Solving Heat Conduction Problems Using Fourier Integrals

For Infinite Domains & Initial Conditions

Step		Action	Mathematical Formula / Description
1	Formulate the Problem $t\,a^2$ Initial Condition: $u(x,0) = f(x)$.	Transform the PDE with respeact x. Use Use properties x, Use properties: $\{F\ U\} = (i\omega^2) = iU$ **Result:** $-a\,U = \omega > U$	$for\ (_n = (-a\omega u,\ xx),\ t > 0.$ $i\omega^2) = -d^2\ u\ U$ $\mid\Rightarrow C(\omega) = F(\omega)$
2	**Solve the ODE**	**Inverse Fourier Transform**	
3	Solve the ODE	This is a first-order linear ODE in with parameter. Solution: $ux,0){=}f(x), = F\{\{fx)\} = F(\omega)$	
4	Applye Fourier Transform to get u, t).	Recall $F^{-1}(\omega), t = F(\omega m)G(\omega v), t) =$ $(f + g(x)$. $u(2m) = F(\omega m)e - -\omega a^2\ \omega m^2\ te^{\alpha}\ i\omega m x$ The inverse transfo̶rm of G is G(x)) $= a\sqrt{4p}\ t\ e{-}x^2/4a^2\,t$. This is the This is Heat Kernel (or Fundamental Solution)	
5	**Use Convolution Property**	Recall $F^{-1\perp}\}\ F(\omega m)G(\omega v)\} = (f + g) + g(x)$.	
6	Final Solution $u(x,t)$ convolution of the inith the Heat Kernel.	$u(x_n\ t\ ux, t, t = 1\,1\,a\ \ \sqrt{4p}\ t$ of the initial temperature profile with $e{-}x^2/4a^2\,t\ e{-}si^2/4a^2\,t\ ,\ ,\ d_{\beta\epsilon})$	

Peer Tip:
This final solnal solution is incredibly powerful! It shows the temperature at any point (x) is average of aεverage of inal temperatures $f\epsilon$), weighted by Gaussion-shaped) function thpreads out over time. This the fundamental physics of heat diffusion!

Would you like to see a specific example through, like an initial condition with sharp heat pulse (a Delta function)?

3.4.3 Practice exercises

To reinforce your learning, here are some practice exercises:

1. Expand f(x)=|x|on -π,πusing Fourier series.
2. Use Fourier integrals to solve the heat equation for an infinite rod with initial temperature distribution f(x)=e-x2.
3. Apply the Fourier transform to f(x)=e-ax2and show that the result is another Gaussian function.

Fill in the blank: The Fourier transform of a Gaussian function is another __________ function.

Practice exercises with Fourier series and integrals

Exercise	**Method**	**Notes**		
Expand \(	x	\)	Fourier series	Even function, cosine terms only
Heat equation	Fourier integrals	Infinite rod problem		
Transform of Gaussian	Fourier transform	Result is Gaussian		

Pilot questions 3.4

1. Since f(x)=xis an odd function, its Fourier series expansion contains only: A. Sine terms B. Cosine terms C. Exponential terms D. None of the above (SLO 3.4)
2. Fourier integrals are especially useful in solving problems involving: A. Non-periodic functions B. Periodic functions only C. Polynomial functions D. None of the above (SLO 3.4)
3. In heat conduction problems, Fourier integrals convert derivatives into: A. Algebraic terms B. Polynomial terms C. Trigonometric terms D. None of the above (SLO 3.4)
4. The Fourier transform of a Gaussian function is another: A. Gaussian function B. Polynomial function C. Trigonometric function D. None of the above (SLO 3.4)

5. Expanding $f(x)=|x|$on $-\pi,\pi$using Fourier series involves: A. Cosine terms only B. Sine terms only C. Both sine and cosine terms D. None of the above (SLO 3.4)

Series Summary

In this Series, we explored the powerful methods of **Fourier series and integrals**, which allow us to represent functions in terms of trigonometric or frequency components. We began by introducing Fourier series and their role in representing periodic functions. We then examined their properties, including convergence under Dirichlet's conditions, simplifications for even and odd functions, and Parseval's theorem. Next, we extended the concept to Fourier integrals and transforms, which apply to non-periodic functions and enable analysis in the frequency domain. Finally, we worked through examples and applications, showing how Fourier methods are used in expansions, solving differential equations, and signal analysis.

By completing this Series, you should now be able to **define** Fourier series and explain their importance (SLO 3.1), **analyse** their properties (SLO 3.2), **explain and apply** Fourier integrals and transforms (SLO 3.3), and **demonstrate** problem solving using Fourier methods in practical contexts (SLO 3.4).

Glossary of Terms

- **Dirichlet's conditions**: Criteria that ensure convergence of a Fourier series, requiring periodicity, piecewise continuity, and finite maxima and minima.
- **Even function**: A function satisfying $f(-x)=f(x)$, leading to cosine terms only in its Fourier series.
- **Fourier coefficients**: Constants $a_n$and $b_n$that determine the contribution of sine and cosine terms in a Fourier series.
- **Fourier integral**: Representation of a non-periodic function as a continuous integral of sine and cosine terms.
- **Fourier series**: Expansion of a periodic function as a sum of sine and cosine terms.
- **Fourier transform**: Mathematical operation converting a function from the time domain into the frequency domain.
- **Odd function**: A function satisfying $f(-x)=-f(x)$, leading to sine terms only in its Fourier series.

- **Parseval's theorem**: A property stating that the total energy of a function equals the sum of the squares of its Fourier coefficients.

Concept Check Questions 3

Objective questions

1. A periodic function is one that __________ its values at regular intervals. (Linked to SLO 3.1)
2. The constant Tin the definition of a periodic function is called the __________. (Linked to SLO 3.1)
3. In a Fourier series, only cosine terms appear when the function is __________. (Linked to SLO 3.2)
4. Parseval's theorem relates the total energy of a function to the sum of the squares of its __________. (Linked to SLO 3.2)
5. The Fourier transform converts a function from the time domain into the __________ domain. (Linked to SLO 3.3)

Short-answer questions

6. State Dirichlet's conditions for Fourier series convergence. (Linked to SLO 3.2)

7. Define the Fourier transform and its inverse. (Linked to SLO 3.3)

8. Explain why Fourier integrals are useful for non-periodic functions. (Linked to SLO 3.3)

9. Give one example of a physical application of Fourier series. (Linked to SLO 3.1)

Long-answer questions

10. Analyse the properties of Fourier series, explaining convergence, even and odd functions, and Parseval's theorem. (Linked to SLO 3.2)

11. Evaluate the usefulness of Fourier transforms in solving differential equations, with examples from heat conduction and signal processing. (Linked to SLO 3.3)

Answers to Concept Check Questions 3

Objective questions

1. Repeats

2. Period
3. Even
4. Fourier coefficients
5. Frequency

Short-answer questions

6. Dirichlet’s conditions: function must be periodic, piecewise continuous, and have finite maxima and minima.

7. Fourier transform: F(ω)=-f(x)e-iωx dx. Inverse: f(x)=12π-F(ω)eiωx dω.

8. Fourier integrals allow non-periodic functions to be expressed in terms of continuous frequency components.

9. Fourier series are used in solving the heat equation to represent temperature distributions.

Long-answer questions

10. *Basic response*: Fourier series converge under Dirichlet’s conditions, simplify for even and odd functions, and Parseval’s theorem relates energy to coefficient squares. *Value-added response*: These properties ensure accurate representation of functions, highlight symmetry, and connect time and frequency domains, making Fourier series essential in analysis.

11. *Basic response*: Fourier transforms convert differential equations into algebraic equations, simplifying solutions. *Value-added response*: In heat conduction, transforms reduce derivatives to algebraic terms, while in signal processing they enable filtering and frequency analysis, showing their broad utility.

Answers to Pilot Questions 3

Part 3.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 3.2**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 3.3**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 3.4**: 1-A, 2-A, 3-A, 4-A, 5-A

Series 4 Equation Types in Mathematical Models

Series outline

- **Part 4.1: Algebraic Equations in Models**
  - 4.1.1 Definition and characteristics of algebraic equations
  - 4.1.2 Examples in physical and economic models
  - 4.1.3 Strengths and limitations
- **Part 4.2: Differential Equations in Models**
  - 4.2.1 Definition and types (ordinary, partial)
  - 4.2.2 Applications in population growth, physics, and epidemiology
  - 4.2.3 Strengths and limitations
- **Part 4.3: Difference Equations in Models**
  - 4.3.1 Definition and characteristics
  - 4.3.2 Applications in discrete systems (finance, ecology)
  - 4.3.3 Strengths and limitations
- **Part 4.4: Integral Equations in Models**
  - 4.4.1 Definition and characteristics
  - 4.4.2 Applications in physics and engineering
  - 4.4.3 Strengths and limitations
- **Part 4.5: Functional Equations in Models**
  - 4.5.1 Definition and characteristics
  - 4.5.2 Applications in mathematics and computer science
  - 4.5.3 Strengths and limitations

Introduction

Mathematical models often rely on different types of equations to represent relationships between variables. Each equation type has unique characteristics and applications, making it suitable for particular systems. For example, algebraic equations may describe static relationships, while differential equations capture dynamic changes over time.

The aim of this Series is to introduce you to the **five major types of equations used in mathematical modelling**: algebraic, differential, difference, integral, and functional. By understanding these equation types, you will be able to select the most appropriate mathematical structure for a given problem.

We shall commence this Series by examining **algebraic equations**, followed by **differential equations**, **difference equations**, **integral equations**, and finally **functional equations**. Each Part will include definitions, examples, strengths, and limitations, ensuring a comprehensive understanding of their role in modelling.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 4.1** define algebraic equations and explain their role in modelling,
- **SLO 4.2** describe differential equations and their applications,
- **SLO 4.3** explain difference equations and their use in discrete systems,
- **SLO 4.4** analyse integral equations and their applications, and
- **SLO 4.5** evaluate functional equations and their role in mathematical modelling.

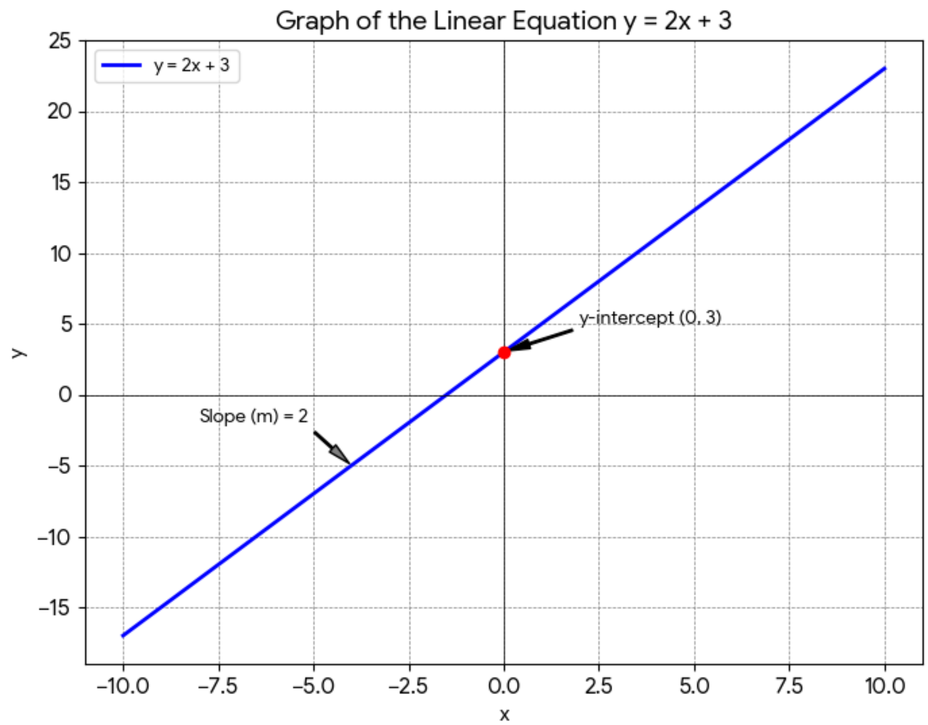
4.1 Algebraic Equations in Models

Algebraic equations are among the simplest and most widely used mathematical structures in modelling. They describe **static relationships** between variables, meaning they do not involve rates of change or time dependence. In this Part, we will define algebraic equations, explore examples in physical and economic models, and consider their strengths and limitations.

4.1.1 Definition and characteristics of algebraic equations

An **algebraic equation** is an equation that relates variables using algebraic operations such as addition, subtraction, multiplication, division, and powers. Unlike differential or difference equations, algebraic equations do not involve derivatives or discrete steps.

Fill in the blank: An algebraic equation relates variables using __________ operations such as addition and multiplication.



Example of an algebraic equation

4.1.2 Examples in physical and economic models

Algebraic equations appear in many disciplines:

- **Physics**: Ohm's law V=IR, relating voltage, current, and resistance.
- **Economics**: Linear supply–demand equations, e.g., Qd=a-bP.
- **Chemistry**: Stoichiometric equations balancing reactants and products.

Fill in the blank: Ohm's law in physics is expressed as __________, relating voltage, current, and resistance.

Examples of algebraic equations in models

Discipline	**Equation**	**Notes**
Physics	V=IR	Ohm's law
Economics	Qd=a-bP	Demand function
Chemistry	Balanced chemical equation	Stoichiometry

4.1.3 Strengths and limitations

Strengths:

- Simple and easy to solve.
- Useful for static relationships.
- Widely applicable across disciplines.

Limitations:

- Cannot represent dynamic processes over time.
- Oversimplifies systems with feedback or change.
- Limited predictive power for complex phenomena.

Fill in the blank: A limitation of algebraic equations is that they cannot represent __________ processes over time.

Strengths and limitations of algebraic equations

- Strength: Simple and easy to solve
- Strength: Useful for static relationships
- Limitation: Cannot represent dynamic processes
- Limitation: Oversimplifies complex systems

1. Strengths: Precision and Scalability

Strength	Impact on Modelling
Exactness	Unlike verbal descriptions, equations provide a precise, unambiguous relationship (e.g., $E = mc^2$).
Predictive Power	They allow for "What-if" analysis by changing input variables to see exactly how the output reacts.
Generality	A single equation can describe a vast array of scenarios (e.g., the Pythagorean theorem applies to every right triangle in existence).
Computational Efficiency	Equations can be easily programmed into software to handle massive datasets or simulate complex systems.

2. Limitations: The Rigidity of Symbols

- **Oversimplification:** Real-world systems are often "noisy." An equation might model a smooth trend but fail to account for outliers or random fluctuations.
- **Assumption Sensitivity:** Most models rely on assumptions (like "linear growth"). If the underlying reality is non-linear or chaotic, the algebraic model becomes increasingly inaccurate over time.

Pilot questions 4.1

1. An algebraic equation relates variables using: A. Algebraic operations such as addition and multiplication B. Derivatives C. Discrete steps D. None of the above (SLO 4.1)
2. Which of the following is an example of an algebraic equation in physics? A. V=IRB. dydt=kyC. Qt+1=rQtD. None of the above (SLO 4.1)
3. In economics, a demand function can be expressed as: A. Qd=a-bPB. dydt=kyC. ∫f(x)dxD. None of the above (SLO 4.1)
4. One strength of algebraic equations is that they are: A. Simple and easy to solve B. Always dynamic C. Based on derivatives D. None of the above (SLO 4.1)
5. A limitation of algebraic equations is that they cannot represent: A. Dynamic processes over time B. Static relationships C. Simple systems D. None of the above (SLO 4.1)

4.2 Differential Equations in Models

Differential equations are fundamental in mathematical modelling because they describe **dynamic processes**—systems that change continuously over time. They capture how variables evolve and interact, making them essential for modelling growth, motion, and many natural phenomena.

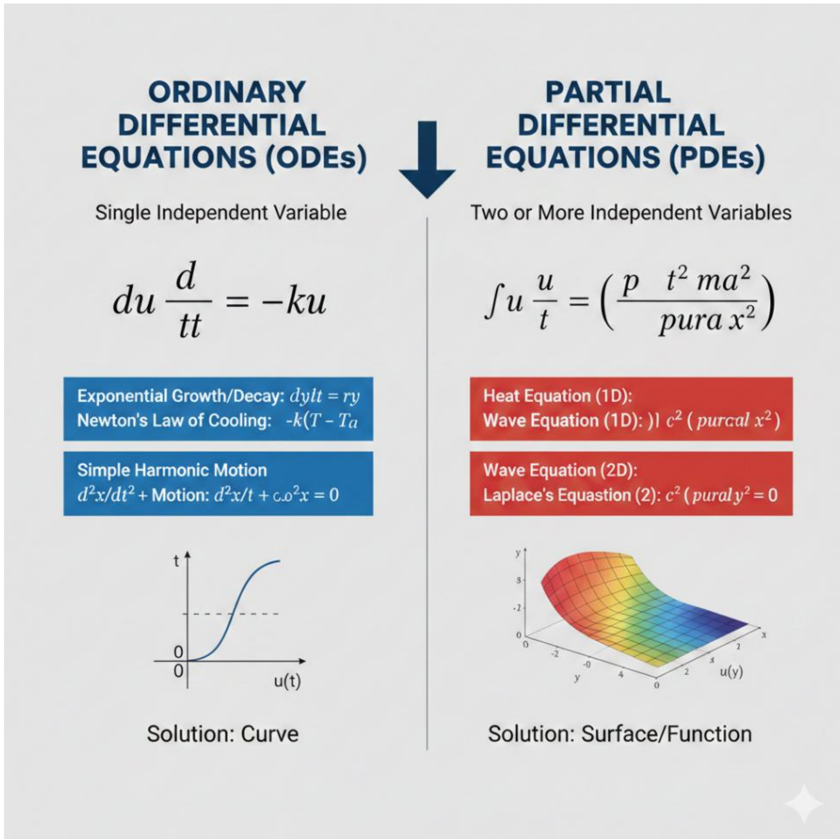
4.2.1 Definition and types (ordinary, partial)

A **differential equation** is an equation that involves derivatives of a function. It expresses how a quantity changes with respect to another, usually time or space.

- **Ordinary Differential Equations (ODEs)**: Involve derivatives with respect to a single variable (e.g., time).

- **Partial Differential Equations (PDEs)**: Involve derivatives with respect to multiple variables (e.g., time and space).

Fill in the blank: A differential equation is an equation that involves __________ of a function.



Ordinary vs. Partial Differential Equations.

4.2.2 Applications in population growth, physics, and epidemiology

Differential equations are widely applied:

- **Population growth**: Logistic growth model dPdt=rPPK.
- **Physics**: Newton's second law d2xdt2=Fm.
- **Epidemiology**: SIR model dSdt=-βSI,dIdt=βSI-γI,dRdt=γI.

Fill in the blank: The SIR model in epidemiology uses differential equations to describe the spread of __________.

[Insert Table here] Caption: Table 4.2 Applications of differential equations in modelling

Discipline	Equation	Notes
Population growth	dPdt=rP(1-P/K)	Logistic growth
Physics	d2xdt2=F/m	Newton's second law
Epidemiology	SIR model equations	Disease spread

4.2.3 Strengths and limitations

Strengths:

- Capture dynamic processes over time.
- Provide predictive power for continuous systems.
- Widely applicable in science and engineering.

Limitations:

- Solutions may be complex or unsolvable analytically.
- Require numerical methods for many real-world problems.
- Sensitive to initial conditions and parameter values.

Fill in the blank: A limitation of differential equations is that many real-world problems require __________ methods for solutions.

Strengths and limitations of differential equations

- Strength: Capture dynamic processes
- Strength: Provide predictive power
- Limitation: Often require numerical methods
- Limitation: Sensitive to initial conditions

1. Strengths: Capturing Dynamic Systems

Strength	Impact on Research & Engineering
Modelling Change	They are uniquely suited to describe rates of change (e.g., velocity, acceleration, or population growth rates).
Causality	DEs naturally represent cause-and-effect relationships over time (e.g., Newton's Second Law, $F = ma$).
Continuous Time	Unlike algebraic models, they allow for a continuous representation of reality rather than discrete snapshots.
Universality	The same equation structure can model diverse phenomena; for example, the **Heat Equation** models both thermal diffusion and the spread of stock prices.

2. Limitations: Complexity and Sensitivity

- **Analytical Difficulty:** Many non-linear differential equations are impossible to solve "exactly" with pen and paper, requiring complex numerical methods and computer simulations.
- **Sensitivity to Initial Conditions:** In "Chaotic Systems" (like weather), a tiny change in the starting value can lead to wildly different outcomes over time—known as the **Butterfly Effect**.
- **Requirement of Continuous Data:** DEs assume that variables change smoothly. They struggle to model systems with sudden, discrete "jumps" or "shocks" (like a sudden market crash or a singular biological mutation).
- **High Parameter Demands:** To work accurately, DEs require very precise physical constants (like friction coefficients or growth rates), which are often difficult to measure in the real world.

Pilot questions 4.2

1. A differential equation involves: A. Derivatives of a function B. Only algebraic operations C. Discrete steps D. None of the above (SLO 4.2)
2. An ordinary differential equation involves derivatives with respect to: A. A single variable B. Multiple variables C. No variables D. None of the above (SLO 4.2)
3. The logistic growth model is an example of: A. A differential equation in population growth B. An algebraic equation C. A difference equation D. None of the above (SLO 4.2)

4. The SIR model in epidemiology describes: A. Disease spread using differential equations B. Supply and demand C. Projectile motion D. None of the above (SLO 4.2)
5. A limitation of differential equations is that many require: A. Numerical methods for solutions B. Simple algebraic manipulation C. Ignoring initial conditions D. None of the above (SLO 4.2)

4.3 Difference Equations in Models

Difference equations are used to describe **discrete systems**, where changes occur in steps rather than continuously. They are especially useful in modelling processes that evolve in distinct intervals, such as financial transactions, population counts, or seasonal cycles.

4.3.1 Definition and characteristics

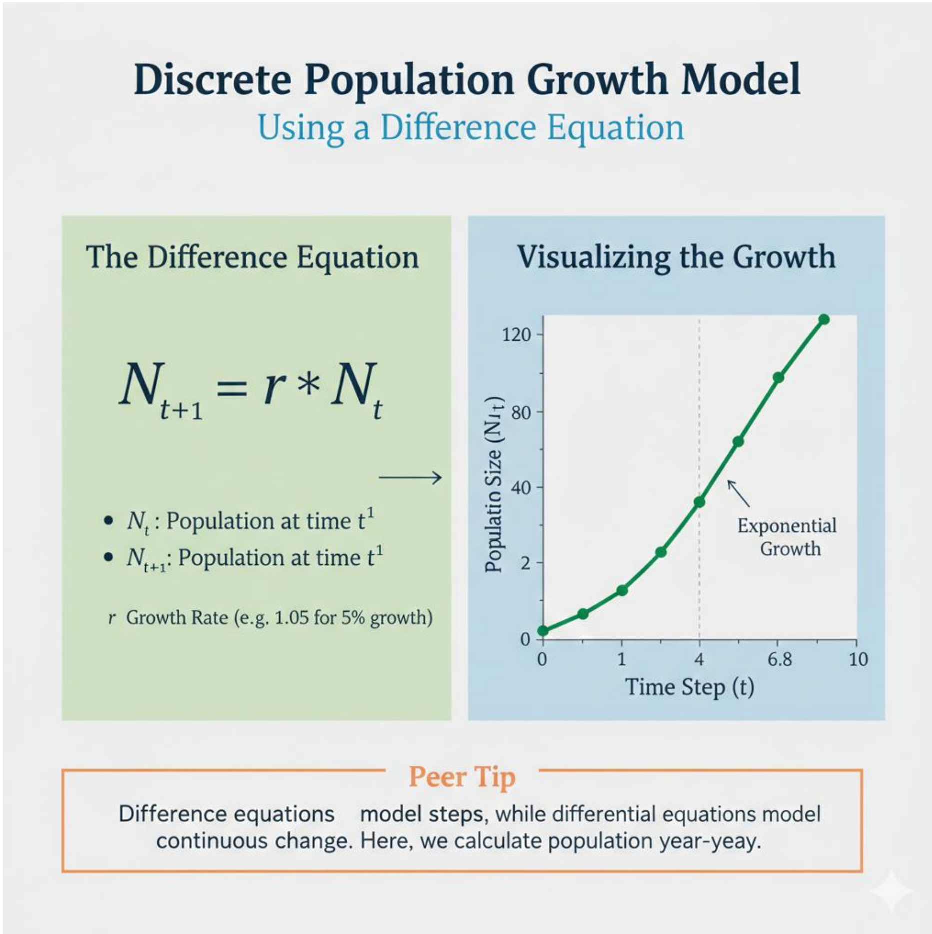
A **difference equation** relates the value of a variable at one point in time to its value at previous points. Unlike differential equations, which deal with continuous change, difference equations capture **step-by-step evolution**.

Example:

$$xt+1=rxt(1\text{-}xt)$$

This is the logistic difference equation, often used in ecology to model population growth in discrete generations.

Fill in the blank: A difference equation relates the value of a variable at one point in time to its value at __________ points.



Example of a difference equation showing step-by-step growth.

4.3.2 Applications in discrete systems (finance, ecology)

Difference equations are widely applied in systems where data is naturally discrete:

- **Finance**: Compound interest, An+1=An(1+r).
- **Ecology**: Population growth in species with non-overlapping generations.
- **Economics**: Cobweb model for supply and demand in discrete time.

Fill in the blank: In finance, compound interest can be modelled using a __________ equation.

[Insert Table here] Caption: Table 4.3 Applications of difference equations in modelling

Discipline	Equation	Notes

Finance	An+1=An(1+r)	Compound interest
Ecology	xt+1=rxt(1-xt)	Logistic growth
Economics	Cobweb model equations	Supply–demand cycles

4.3.3 Strengths and limitations

Strengths:

- Suitable for systems with discrete time steps.
- Easier to compute than differential equations.
- Useful for modelling iterative processes.

Limitations:

- Cannot capture continuous change.
- May oversimplify systems with smooth dynamics.
- Sensitive to step size and initial conditions.

Fill in the blank: A limitation of difference equations is that they cannot capture __________ change.

Strengths and limitations of difference equations

- Strength: Suitable for discrete systems
- Strength: Easier to compute
- Limitation: Cannot capture continuous change
- Limitation: Sensitive to step size

1. Strengths: The Power of Discrete Steps

Strength	Impact on Modelling
Natural Fit for Iteration	Perfect for phenomena that occur in cycles, such as compound interest (monthly), biological generations (yearly), or digital signal processing.
Computational Simplicity	They are inherently "computer-ready." You can calculate the next state simply by plugging the current state into a formula without needing complex integration.
Capturing Lag Effects	Easy to model systems where the current state depends on a specific moment in the past (e.g., $x_n = x_{n-1} + x_{n-2}$ in Fibonacci sequences).
Handling "Chunky" Data	Most real-world data (census, stock prices, test scores) is collected at discrete intervals, making these equations a direct match for raw data.

2. Limitations: Resolution and Stability

- **Resolution Loss:** Because they jump from one point to the next, they can miss "micro-behaviors" that happen between steps. If the time step is too large, the model becomes inaccurate.
- **Sensitivity to Parameters:** Small changes in the growth rate constant (r) in a difference equation can cause the model to shift from stable growth to chaotic, unpredictable oscillation (the "Butterfly Effect" in discrete time).
- **Accumulated Error:** In long-term simulations, small rounding errors in each step can compound, leading to significant "drift" from the actual theoretical value.
- **Analytical Complexity:** While easy to simulate on a computer, finding a "closed-form solution" (a single formula that predicts x_n for any n without iterating) can be mathematically difficult for non-linear equations.

Pilot questions 4.3

1. A difference equation relates a variable's value at one point to: A. Its value at previous points B. Continuous derivatives C. Integrals D. None of the above (SLO 4.3)

2. The logistic difference equation is often used in: A. Ecology to model population growth B. Physics to model motion C. Chemistry to balance reactions D. None of the above (SLO 4.3)
3. In finance, compound interest can be modelled using: A. A difference equation B. A differential equation C. An integral equation D. None of the above (SLO 4.3)
4. One strength of difference equations is that they are: A. Suitable for discrete systems B. Always continuous C. Based on derivatives D. None of the above (SLO 4.3)
5. A limitation of difference equations is that they cannot capture: A. Continuous change B. Discrete steps C. Iterative processes D. None of the above (SLO 4.3)

4.4 Integral Equations in Models

Integral equations are powerful tools in mathematical modelling because they describe systems where the value of a function depends on an **integral of the function itself**. They are particularly useful in physics and engineering, where cumulative effects or distributed interactions must be considered.

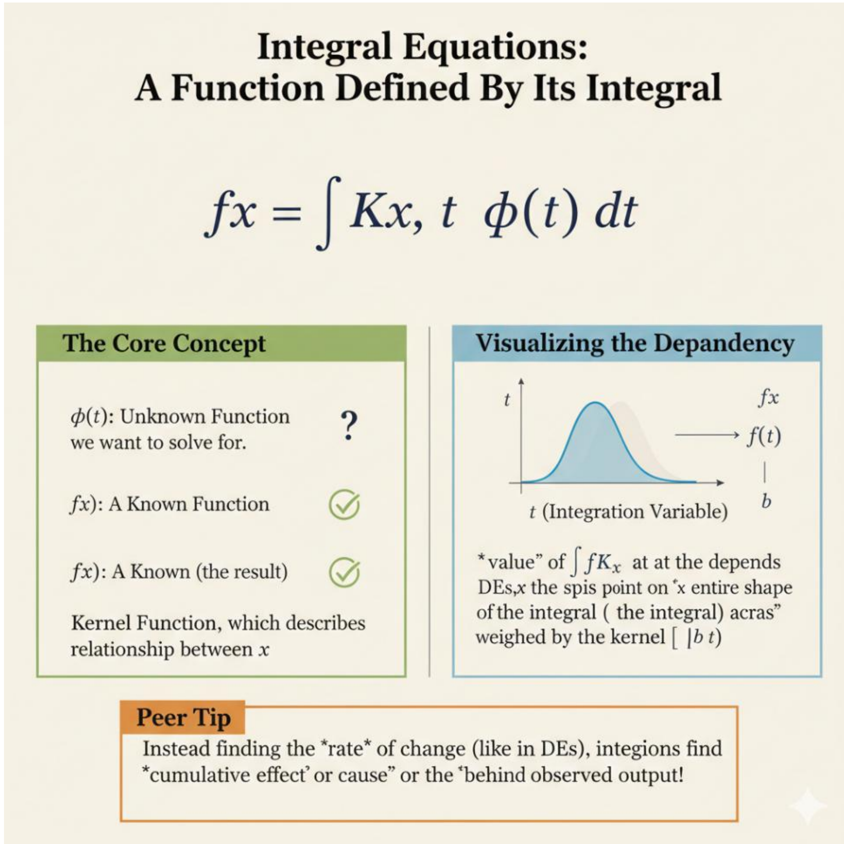
4.4.1 Definition and characteristics

An **integral equation** is an equation in which the unknown function appears under an integral sign. Unlike differential equations, which focus on rates of change, integral equations capture cumulative or averaged effects.

Example:

$$f(x)=\lambda abK(x,t)f(t)\ dt+g(x)$$

Fill in the blank: An integral equation is an equation in which the unknown function appears under an __________ sign.



Structure of an integral equation.

4.4.2 Applications in physics and engineering

Integral equations are widely applied in systems involving cumulative effects:

- **Physics**: Potential theory, where potentials depend on integrals of charge distributions.
- **Engineering**: Heat conduction problems, where temperature depends on integrals of heat sources.
- **Electromagnetism**: Maxwell's equations can be expressed in integral form.

Fill in the blank: In physics, potential theory uses integral equations to describe __________ distributions.

Applications of integral equations in modelling

Discipline	Example	Notes
Physics	Potential theory	Potentials from charge distributions

Engineering	Heat conduction	Temperature from heat sources
Electromagnetism	Maxwell's equations	Field integrals

4.4.3 Strengths and limitations

Strengths:

- Capture cumulative effects naturally.
- Provide alternative formulations to differential equations.
- Useful in boundary value problems.

Limitations:

- Often more complex to solve than differential equations.
- Require advanced numerical methods.
- Solutions may not be unique or easy to interpret.

Fill in the blank: A limitation of integral equations is that they often require advanced __________ methods for solutions.

Strengths and limitations of integral equations

- Strength: Capture cumulative effects
- Strength: Useful in boundary value problems
- Limitation: Complex to solve
- Limitation: Require advanced numerical methods

1. Strengths: Capturing History and Connectivity

Strength	Impact on Modelling
Smoothing Effect	Integration is a "smoothing" process. This makes integral equations more stable and less sensitive to small amounts of noise in data compared to differential equations.
Global Dependencies	They are perfect for modelling phenomena with "memory" or spatial interaction, where the current state depends on the sum of all past states or surrounding areas.
Inverse Problems	They are the standard tool for "undoing" an effect to find the cause (e.g., reconstructing a 3D image from 2D X-ray data in a CT scan).
Boundary Inclusion	In many physics problems, the boundary conditions are automatically built into the integral equation, rather than being added as separate constraints.

2. Limitations: The Complexity of "Unlocking" the Integral

- **The "Ill-Posed" Problem:** Specifically in *first-kind* integral equations, small changes in the output can lead to massive, erratic changes in the calculated input. This makes them very difficult to solve accurately without advanced "regularization" techniques.
- **Analytical Difficulty:** Most integral equations do not have simple solutions. Finding the "kernel" (the function that describes the relationship) and then solving for the unknown function often requires heavy computational power.
- **Kernel Dependency:** The model is only as good as the **Kernel Function** $K(x,t)$. If the relationship between points in space or time is misunderstood, the entire integral output will be skewed.
- **Computational Cost:** Solving these numerically often involves creating large matrices that can be memory-intensive for high-resolution models.

Pilot questions 4.4

1. An integral equation is an equation in which the unknown function appears under: A. An integral sign B. A derivative sign C. A summation sign D. None of the above (SLO 4.4)
2. In physics, integral equations are used in: A. Potential theory B. Algebraic equations C. Cobweb models D. None of the above (SLO 4.4)
3. In engineering, integral equations can describe: A. Heat conduction problems B. Chemical stoichiometry C. Linear demand functions D. None of the above (SLO 4.4)

4. One strength of integral equations is that they: A. Capture cumulative effects naturally B. Always ignore boundary conditions C. Are simpler than algebraic equations D. None of the above (SLO 4.4)
5. A limitation of integral equations is that they often require: A. Advanced numerical methods for solutions B. Simple algebraic manipulation C. Ignoring initial conditions D. None of the above (SLO 4.4)

4.5 Functional Equations in Models

Functional equations are a more abstract type of equation used in mathematical modelling. They describe relationships between functions rather than just variables, making them useful in advanced mathematics, computer science, and systems where the structure of the function itself is important.

4.5.1 Definition and characteristics

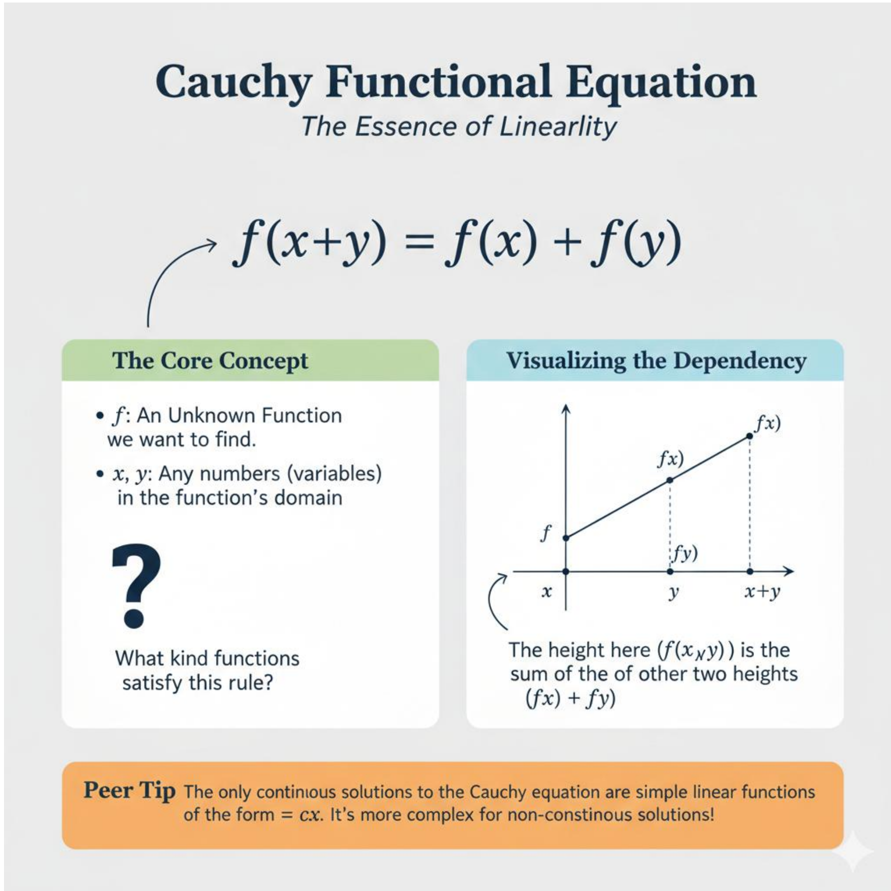
A **functional equation** is an equation in which the unknowns are functions, and the equation specifies relationships between the values of these functions at different points. Unlike algebraic or differential equations, functional equations focus on the properties of functions themselves.

Example:

$$f(x+y)=f(x)+f(y)$$

This is the **Cauchy functional equation**, which characterises additive functions.

Fill in the blank: A functional equation is an equation in which the unknowns are __________.



Example of a functional equation (Cauchy equation).

4.5.2 Applications in mathematics and computer science

Functional equations are applied in areas where functions themselves are the objects of study:

- **Mathematics**: Characterising trigonometric functions, e.g., $\sin(x+y)=\sin x\cos y+\cos x\sin y$.
- **Computer science**: Recursive algorithms, where functions are defined in terms of themselves.
- **Information theory**: Equations defining entropy functions.

Fill in the blank: In computer science, functional equations often appear in __________ algorithms.

Applications of functional equations in modelling

Discipline	Example	Notes
Mathematics	$\sin(x+y)=\sin x\cos y+\cos x\sin y$	Trigonometric identity
Computer science	Recursive definitions	Algorithms defined in terms of themselves
Information theory	Entropy equations	Measure of information content

4.5.3 Strengths and limitations

Strengths:

- Capture deep structural properties of functions.
- Useful in abstract mathematics and theoretical computer science.
- Provide general frameworks for defining functions.

Limitations:

- Often abstract and difficult to apply directly to real-world problems.
- Solutions may be non-unique or require advanced techniques.
- Less intuitive than algebraic or differential equations.

Fill in the blank: A limitation of functional equations is that they are often too ___________ for direct real-world application.

Strengths and limitations of functional equations

- Strength: Capture structural properties of functions
- Strength: Useful in abstract mathematics
- Limitation: Abstract and difficult to apply directly
- Limitation: Solutions may be non-unique

1. Strengths: Defining Fundamental Laws

Strength	Impact on Modelling
Characterizing Symmetry	They are unparalleled at capturing fundamental symmetries and scaling laws (e.g., if doubling the input always doubles the output).
Minimal Assumptions	You don't need a specific formula to start; you only need to know how the function *behaves* (e.g., "The interest on the sum is the sum of the interests").
Axiomatic Foundation	They allow scientists to derive complex laws from simple, intuitive axioms (principles), which provides a very "clean" theoretical base.
Cross-Disciplinary	The same functional equation can define the behavior of probability distributions, financial growth, and physical laws like Shannon Entropy.

2. Limitations: Abstractness and Difficulty

- **Non-Uniqueness of Solutions:** Without extra constraints (like "the function must be continuous" or "the function must be monotonic"), a functional equation can have infinitely many "pathological" or strange solutions that don't make sense in the real world.
- **Mathematical Complexity:** Solving them often requires specialized "mathematical gymnastics" (like substitution, fixed-point theory, or looking for invariants) that are not as standardized as solving differential equations.
- **Abstract Nature:** They describe *how* a system relates to itself rather than providing an immediate step-by-step path of change. This can make them harder to visualize for non-mathematicians.
- **Sensitivity to Domain:** A functional equation might be easy to solve for integers but incredibly complex or even unsolved for all real or complex numbers.

Pilot questions 4.5

1. A functional equation is an equation in which the unknowns are: A. Functions B. Variables only C. Derivatives D. None of the above (SLO 4.5)
2. The Cauchy functional equation is expressed as: A. f(x+y)=f(x)+f(y)B. V=IRC. dydt=kyD. None of the above (SLO 4.5)
3. In mathematics, functional equations can characterise: A. Trigonometric functions B. Compound interest C. Logistic growth D. None of the above (SLO 4.5)

4. In computer science, functional equations often appear in: A. Recursive algorithms B. Stoichiometric equations C. Cobweb models D. None of the above (SLO 4.5)
5. A limitation of functional equations is that they are often: A. Too abstract for direct real-world application B. Always simple to solve C. Based only on discrete steps D. None of the above (SLO 4.5)

Series 4 Summary

In this Series, we explored the **five major types of equations used in mathematical modelling**: algebraic, differential, difference, integral, and functional. Each type provides a unique way of representing relationships and processes in systems.

- **Algebraic equations (Part 4.1)** describe static relationships between variables using algebraic operations.
- **Differential equations (Part 4.2)** capture continuous dynamic processes, modelling rates of change over time or space.
- **Difference equations (Part 4.3)** represent discrete systems evolving step by step, useful in finance and ecology.
- **Integral equations (Part 4.4)** express cumulative effects, where functions depend on integrals of themselves, common in physics and engineering.
- **Functional equations (Part 4.5)** define relationships between functions themselves, often used in abstract mathematics and computer science.

By completing this Series, you should now be able to **define and apply algebraic equations (SLO 4.1)**, **describe and use differential equations (SLO 4.2)**, **explain difference equations in discrete systems (SLO 4.3)**, **analyse integral equations (SLO 4.4)**, and **evaluate functional equations (SLO 4.5)**.

Glossary of Terms

- **Algebraic equation**: Relates variables using algebraic operations (e.g., $V=IR$).
- **Differential equation**: Involves derivatives, describing continuous change.
- **Ordinary differential equation (ODE)**: Derivative with respect to one variable.

- **Partial differential equation (PDE)**: Derivatives with respect to multiple variables.
- **Difference equation**: Relates values of a variable at discrete time steps.
- **Integral equation**: Equation where the unknown function appears under an integral sign.
- **Functional equation**: Equation defining relationships between functions.
- **Numerical methods**: Techniques used to approximate solutions when exact solutions are difficult.

Concept Check Questions 4

Objective questions

1. An algebraic equation relates variables using __________ operations. (SLO 4.1)
2. A differential equation involves __________ of a function. (SLO 4.2)
3. A difference equation relates a variable's value at one point to its value at __________ points. (SLO 4.3)
4. An integral equation is an equation in which the unknown function appears under an __________ sign. (SLO 4.4)
5. A functional equation is an equation in which the unknowns are __________. (SLO 4.5)

Short-answer questions

6. Give one example of an algebraic equation in physics. (SLO 4.1)
7. State one application of differential equations in epidemiology. (SLO 4.2)
8. Explain how difference equations are used in finance. (SLO 4.3)
9. Describe one application of integral equations in engineering. (SLO 4.4)
10. Give one example of a functional equation in mathematics. (SLO 4.5)

Long-answer questions

11. Compare algebraic, differential, and difference equations in terms of their ability to model static vs. dynamic systems. (SLO 4.1–4.3)

12. Evaluate the importance of integral and functional equations in advanced modelling, highlighting their strengths and limitations. (SLO 4.4–4.5)

Answers to Pilot Questions 4

Part 4.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.2**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.3**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.4**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.5**: 1-A, 2-A, 3-A, 4-A, 5-A

Series 5: Eigenfunction Expansions and Variation Methods

Series Outline

- **Part 5.1: Introduction to Eigenfunction Expansions**
  - Concept of eigenfunctions and eigenvalues
  - Role in solving PDEs
  - General expansion in terms of eigenfunctions
- **Part 5.2: Properties of Eigenfunction Expansions**
  - Orthogonality and completeness
  - Expansion coefficients
  - Applications in boundary value problems
- **Part 5.3: Variational Methods**
  - Principle of stationary action
  - Rayleigh–Ritz method
  - Applications in approximating eigenvalues and eigenfunctions
- **Part 5.4: Worked Examples and Applications**
  - Eigenfunction expansions in PDEs
  - Variational methods applied to physical problems
  - Practice exercises

Introduction

Eigenfunction expansions and variational methods are powerful techniques in mathematical physics. **Eigenfunctions** arise naturally when solving boundary value problems, and they form complete sets that allow functions to be expanded in terms of them. This is similar to Fourier series, but tailored to the specific PDE and boundary conditions.

Variational methods provide approximate solutions to problems where exact solutions are difficult to obtain. By minimising or maximising certain functionals, one can approximate eigenvalues and eigenfunctions effectively. These methods are widely used in quantum mechanics, elasticity, and other areas of applied mathematics.

In this Series, we will begin by introducing eigenfunction expansions and their role in solving PDEs. Next, we will study their properties, including orthogonality and completeness. We will then explore variational methods, focusing on the Rayleigh–Ritz method and its applications. Finally, we will work through examples and practice problems to consolidate understanding.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 5.1** define eigenfunction expansions and explain their role in solving PDEs,
- **SLO 5.2** analyse the properties of eigenfunction expansions, including orthogonality and completeness,
- **SLO 5.3** explain variational methods and apply them to approximate eigenvalues and eigenfunctions, and
- **SLO 5.4** demonstrate problem solving using eigenfunction expansions and variational methods through worked examples and applications.

5.1 Introduction to Eigenfunction Expansions

Eigenfunction expansions are a powerful method for solving boundary value problems in partial differential equations (PDEs). The idea is to express a function as a series of eigenfunctions, which are solutions to certain differential equations subject to boundary conditions. These eigenfunctions form a complete set, allowing us to represent complex functions in terms of simpler building blocks.

Fill in the blank: Eigenfunction expansions express a function as a series of __________ that satisfy boundary conditions.

5.1.1 Concept of eigenfunctions and eigenvalues

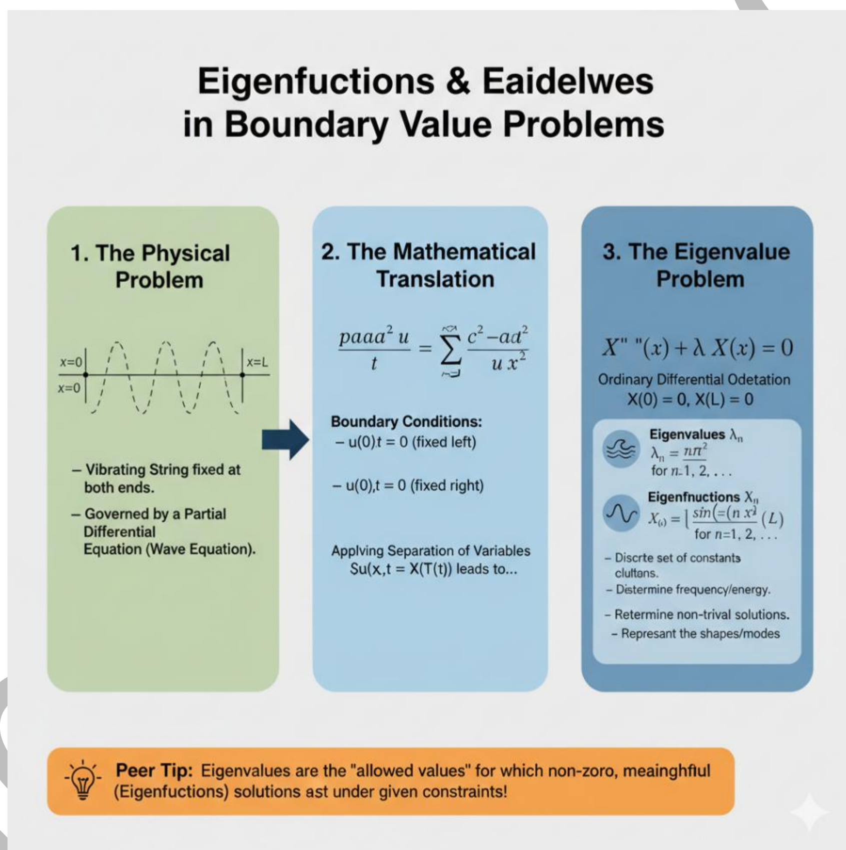
An **eigenfunction** is a non-trivial solution to a differential equation that satisfies given boundary conditions, associated with a constant called an **eigenvalue**. For example, in Sturm–Liouville problems, eigenfunctions arise naturally and are orthogonal with respect to a weight function.

Formally, for a differential operator L, an eigenfunction y(x)satisfies:

$$L[y(x)]=\lambda y(x)$$

where is the eigenvalue.

Fill in the blank: In eigenfunction problems, the constant associated with an eigenfunction is called the __________.



Eigenfunctions and eigenvalues in boundary value problems.

5.1.2 Role in solving PDEs

Eigenfunction expansions are crucial in solving PDEs because they:

- Provide a systematic way to represent solutions.
- Simplify PDEs into algebraic problems involving eigenvalues.

- Allow expansions similar to Fourier series but tailored to boundary conditions.
- Appear naturally in physical problems such as vibrations, heat conduction, and quantum mechanics.

Fill in the blank: Eigenfunction expansions are similar to __________ series but adapted to boundary conditions.

Role of eigenfunction expansions in PDEs

- Represent solutions systematically
- Simplify PDEs into algebraic problems
- Adapt to boundary conditions
- Appear in physical problems

1. Core Functions in PDE Solving

Role	Description	Practical Outcome
Basis Construction	Eigenfunctions (y_n) act as a "coordinate system" for functions.	Any valid physical state can be represented as a sum: $u(x,t) = \sum c_n(t)y_n(x)$.
Variable Separation	It handles the spatial part of a PDE after separation of variables.	Turns a PDE into a set of simpler Ordinary Differential Equations (ODEs).
Initial Condition Fitting	Coefficients are calculated to match the starting state of the system.	Allows the model to reflect specific starting temperatures or displacements.
Handling Source Terms	Used in "Method of Eigenfunction Expansion" for non-homogeneous PDEs.	Solves equations with external forces or heat sources (e.g., $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x,t)$).

5.1.3 General expansion in terms of eigenfunctions

If yn(x)is a set of eigenfunctions, any function f(x)satisfying the same boundary conditions can be expanded as:

f(x)=n=1anyn(x)

where the coefficients anare determined using orthogonality relations:

an=abf(x)yn(x)w(x) dxabyn2(x)w(x) dx

This expansion is analogous to Fourier series but uses eigenfunctions specific to the problem.

Fill in the blank: The coefficients in eigenfunction expansions are determined using __________ relations.

General form of eigenfunction expansions

Component	**Description**
Eigenfunctions	Solutions to boundary value problems
Eigenvalues	Constants associated with eigenfunctions
Expansion	Function expressed as series of eigenfunctions
Coefficients	Determined using orthogonality relations

Pilot questions 5.1

1. Eigenfunction expansions express a function as a series of: A. Eigenfunctions B. Polynomials C. Exponentials D. None of the above (SLO 5.1)
2. In eigenfunction problems, the constant associated with an eigenfunction is called the: A. Eigenvalue B. Coefficient C. Weight function D. None of the above (SLO 5.1)
3. Eigenfunction expansions are similar to: A. Fourier series B. Taylor series C. Polynomial expansions D. None of the above (SLO 5.1)
4. The coefficients in eigenfunction expansions are determined using: A. Orthogonality relations B. Differentiation C. Limits D. None of the above (SLO 5.1)
5. Eigenfunction expansions are important because they: A. Simplify PDEs into algebraic problems B. Complicate boundary conditions C. Ignore physical problems D. None of the above (SLO 5.1)

5.2 Properties of Eigenfunction Expansions

Eigenfunction expansions have several important properties that make them powerful tools in solving PDEs and boundary value problems. These properties include **orthogonality**, **completeness**, and the determination of **expansion coefficients**.

5.2.1 Orthogonality

Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to a weight function w(x). This means:

abym(x)yn(x)w(x) dx=0for mn

Orthogonality ensures that eigenfunctions form independent building blocks, simplifying the calculation of expansion coefficients.

Fill in the blank: Eigenfunctions corresponding to distinct eigenvalues are __________ with respect to a weight function.

5.2.2 Completeness

A set of eigenfunctions is said to be **complete** if any function satisfying the same boundary conditions can be represented as a series of these eigenfunctions. Completeness guarantees that eigenfunction expansions can approximate arbitrary functions within the problem's domain.

Fill in the blank: Completeness means that any function satisfying the same __________ can be represented as a series of eigenfunctions.

5.2.3 Expansion coefficients

The coefficients in an eigenfunction expansion are determined using orthogonality relations. For a function f(x), the coefficient anis:

an=abf(x)yn(x)w(x) dxabyn2(x)w(x) dx

These coefficients measure how much each eigenfunction contributes to the overall expansion.

Fill in the blank: Expansion coefficients are determined using __________ relations.

5.2.4 Applications in boundary value problems

Eigenfunction expansions are widely applied in boundary value problems, such as:

- Vibrating string problems (wave equation).
- Heat conduction in rods (heat equation).
- Steady-state distributions (Laplace's equation).

In each case, eigenfunctions naturally arise from the boundary conditions and provide a systematic way to construct solutions.

Fill in the blank: Eigenfunction expansions are widely applied in solving __________ value problems.

Properties of eigenfunction expansions

Property	Explanation	Application
Orthogonality	Distinct eigenfunctions are orthogonal	Simplifies coefficient calculation
Completeness	Any function can be represented	Ensures generality of expansion
Expansion coefficients	Determined using orthogonality	Measure contribution of eigenfunctions
Boundary value problems	Natural applications	Heat, wave, Laplace equations

Pilot questions 5.2

1. Eigenfunctions corresponding to distinct eigenvalues are: A. Orthogonal B. Parallel C. Identical D. None of the above (SLO 5.2)
2. Completeness means that any function satisfying the same: A. Boundary conditions B. Initial conditions C. Polynomial form D. None of the above (SLO 5.2)
3. Expansion coefficients are determined using: A. Orthogonality relations B. Differentiation C. Limits D. None of the above (SLO 5.2)
4. Eigenfunction expansions are widely applied in solving: A. Boundary value problems B. Algebraic equations C. Polynomial expansions D. None of the above (SLO 5.2)
5. The coefficient anin an eigenfunction expansion measures: A. Contribution of each eigenfunction B. The eigenvalue itself C. The weight function D. None of the above (SLO 5.2)

5.3 Variational Methods

Variational methods are powerful techniques used to approximate solutions to problems where exact solutions are difficult to obtain. They are especially useful in finding approximate eigenvalues and eigenfunctions in mathematical physics.

Fill in the blank: Variational methods provide __________ solutions when exact solutions are difficult to obtain.

5.3.1 Principle of stationary action

The foundation of variational methods lies in the **principle of stationary action**, which states that the path taken by a physical system is the one that makes a certain functional stationary (usually minimised or maximised). In mathematics, this principle is applied to functionals to derive approximate solutions.

Example: In mechanics, the action integral is stationary for the actual motion of a system.

Fill in the blank: The principle of stationary action states that the path taken by a system makes a certain __________ stationary.

5.3.2 Rayleigh–Ritz method

The **Rayleigh–Ritz method** is a variational technique used to approximate eigenvalues and eigenfunctions. The idea is to assume a trial function that satisfies boundary conditions and then minimise a functional to obtain approximate eigenvalues.

For a Sturm–Liouville problem:

$$\lambda \approx ab(y')2+q(x)y2dxaby2dx$$

where $y(x)$is the trial function.

This method is widely used in quantum mechanics and structural engineering.

Fill in the blank: The Rayleigh–Ritz method approximates eigenvalues by minimising a __________.

5.3.3 Applications in approximating eigenvalues and eigenfunctions

Variational methods are applied in:

- **Quantum mechanics**: Approximating ground state energies of atoms and molecules.
- **Structural engineering**: Estimating vibration frequencies of beams and plates.
- **Mathematical physics**: Approximating eigenvalues in boundary value problems.

Fill in the blank: In quantum mechanics, variational methods are used to approximate __________ state energies.

Applications of variational methods

Field	Application	Notes
Quantum mechanics	Ground state energy	Trial wavefunctions used
Structural engineering	Vibration frequencies	Beams and plates
Mathematical physics	Eigenvalue approximation	Sturm–Liouville problems

Pilot questions 5.3

1. Variational methods provide: A. Approximate solutions B. Exact solutions always C. No solutions D. None of the above (SLO 5.3)
2. The principle of stationary action states that the path taken by a system makes a certain: A. Functional stationary B. Derivative stationary C. Polynomial stationary D. None of the above (SLO 5.3)
3. The Rayleigh–Ritz method approximates eigenvalues by minimising a: A. Functional B. Polynomial C. Derivative D. None of the above (SLO 5.3)
4. In quantum mechanics, variational methods are used to approximate: A. Ground state energies B. Excited state probabilities only C. Polynomial coefficients D. None of the above (SLO 5.3)
5. Variational methods are widely applied in: A. Quantum mechanics, structural engineering, and mathematical physics B. Geometry only C. Number theory only D. None of the above (SLO 5.3)

5.4 Worked Examples and Applications

In this Part, we will apply **eigenfunction expansions** and **variational methods** to practical problems. These examples will show how eigenfunctions provide exact solutions to PDEs and how variational methods yield useful approximations when exact solutions are difficult.

5.4.1 Worked Example: Eigenfunction expansion in the heat equation

Consider the one-dimensional heat equation:

$$ut=\alpha 2ux2,0<x<L,\ t>0$$

with boundary conditions $u(0,t)=u(L,t)=0$.

Step 1: Assume eigenfunction expansion

$$u(x,t)=n=1An(t)\sin n\pi xL$$

Step 2: Substitute into PDE This reduces the PDE to ODEs for $An(t)$:

$$An'(t)=-\alpha n\pi L2An(t)$$

Step 3: Solve ODEs

$$An(t)=An(0)e\text{-}\alpha n\pi L2t$$

Final solution:

$$u(x,t)=n=1An(0)\sin n\pi xLe\text{-}\alpha n\pi L2t$$

Fill in the blank: In the heat equation eigenfunction expansion, the spatial part involves __________ functions.

5.4.2 Worked Example: Rayleigh–Ritz method for eigenvalue approximation

Consider the Sturm–Liouville problem:

$$y''+\lambda y=0,y(0)=y(\pi)=0$$

Exact eigenvalues are $n=n2$.

Step 1: Choose a trial function Let $y(x)=\sin(x)$, which satisfies the boundary conditions.

Step 2: Apply Rayleigh–Ritz formula

$$\lambda\approx 0(y')2dx0y2dx$$

Step 3: Compute integrals

- Numerator: $0(\cos(x))2dx=2$
- Denominator: $0(\sin(x))2dx=2$

Thus, $\lambda\approx 1$, which matches the exact first eigenvalue.

Fill in the blank: The Rayleigh–Ritz method approximates eigenvalues using a chosen __________ function.

5.4.3 Practice exercises

1. Use eigenfunction expansions to solve the vibrating string problem with fixed ends.
2. Apply the Rayleigh–Ritz method to approximate the lowest eigenvalue of $y''+\lambda y=0$with boundary conditions $y(0)=0, y(1)=0$.
3. Expand $f(x)=x$on 1using eigenfunctions $\sin(n\pi x)$.

Fill in the blank: Eigenfunction expansions are especially useful in solving __________ problems with boundary conditions.

Practice exercises with eigenfunction expansions and variational methods

Exercise	**Method**	**Notes**
Vibrating string	Eigenfunction expansion	Wave equation with fixed ends
Lowest eigenvalue	Rayleigh–Ritz	Approximation using trial function
Expansion of $f(x)=x$	Eigenfunction expansion	Orthogonality used for coefficients

Pilot questions 5.4

1. In the heat equation eigenfunction expansion, the spatial part involves: A. Sine functions B. Cosine functions C. Exponential functions D. None of the above (SLO 5.4)
2. The Rayleigh–Ritz method approximates eigenvalues using a chosen: A. Trial function B. Polynomial C. Derivative D. None of the above (SLO 5.4)
3. Eigenfunction expansions are especially useful in solving: A. Boundary value problems B. Algebraic equations C. Polynomial expansions D. None of the above (SLO 5.4)
4. In the Rayleigh–Ritz method, eigenvalues are approximated by minimising a: A. Functional B. Polynomial C. Limit D. None of the above (SLO 5.4)
5. In the heat equation solution, the time part involves: A. Exponential decay B. Oscillations C. Harmonic functions D. None of the above (SLO 5.4)

Series Summary

In this Series, we explored **eigenfunction expansions** and **variational methods**, two central techniques in mathematical physics. We began by introducing eigenfunctions and eigenvalues, explaining how they arise naturally in boundary value problems. We then studied the properties of eigenfunction expansions, including orthogonality, completeness, and the determination of coefficients. Next, we examined variational methods, focusing on the principle of stationary action and the Rayleigh–Ritz method. Finally, we worked through examples and practice problems, showing how eigenfunction expansions provide exact solutions to PDEs and how variational methods yield useful approximations when exact solutions are difficult.

By completing this Series, you should now be able to **define eigenfunction expansions** (SLO 5.1), **analyse their properties** (SLO 5.2), **explain and apply variational methods** (SLO 5.3), and **demonstrate problem solving using both techniques** (SLO 5.4).

Glossary of Terms

- **Eigenfunction**: A non-trivial solution to a differential equation subject to boundary conditions, associated with an eigenvalue.
- **Eigenvalue**: The constant associated with an eigenfunction in an eigenvalue problem.
- **Orthogonality**: Property where eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to a weight function.
- **Completeness**: Property ensuring that any function satisfying the same boundary conditions can be represented as a series of eigenfunctions.
- **Rayleigh–Ritz method**: A variational technique used to approximate eigenvalues and eigenfunctions using trial functions.
- **Stationary action principle**: Principle stating that the actual path of a system makes a certain functional stationary.
- **Variational methods**: Techniques that provide approximate solutions by minimising or maximising functionals.

Concept Check Questions 5

Objective questions

1. Eigenfunction expansions express a function as a series of __________. (SLO 5.1)

2. The constant associated with an eigenfunction is called the __________. (SLO 5.1)
3. Eigenfunctions corresponding to distinct eigenvalues are __________. (SLO 5.2)
4. The Rayleigh–Ritz method approximates eigenvalues by minimising a __________. (SLO 5.3)
5. In the heat equation eigenfunction expansion, the time part involves __________ decay. (SLO 5.4)

Short-answer questions

6. State the orthogonality property of eigenfunctions. (SLO 5.2)

7. Explain the principle of stationary action. (SLO 5.3)

8. Give one physical application of variational methods. (SLO 5.3)

9. Describe how eigenfunction expansions are used in solving PDEs. (SLO 5.1)

Long-answer questions

10. Analyse the properties of eigenfunction expansions, including orthogonality, completeness, and coefficients. (SLO 5.2)

11. Evaluate the usefulness of variational methods in approximating eigenvalues, with examples from quantum mechanics and structural engineering. (SLO 5.3)

Answers to Concept Check Questions 5

Objective questions

1. Eigenfunctions
2. Eigenvalue
3. Orthogonal
4. Functional
5. Exponential

Short-answer questions

6. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to a weight function.

7. The principle of stationary action states that the actual path of a system makes a certain functional stationary (minimised or maximised).

8. Variational methods are used in quantum mechanics to approximate ground state energies.

9. Eigenfunction expansions express solutions to PDEs as series of eigenfunctions tailored to boundary conditions.

Long-answer questions

10. *Basic response*: Orthogonality ensures independence, completeness guarantees representation, and coefficients are determined using orthogonality relations. *Value-added response*: These properties make eigenfunction expansions systematic and reliable, allowing accurate solutions to PDEs in physics and engineering.

11. *Basic response*: Variational methods approximate eigenvalues using trial functions. *Value-added response*: In quantum mechanics, they approximate ground state energies; in engineering, they estimate vibration frequencies. This shows their broad utility in applied mathematics.

Answers to Pilot Questions 5

Part 5.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.2**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.3**: 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.4**: 1-A, 2-A, 3-A, 4-A, 5-A