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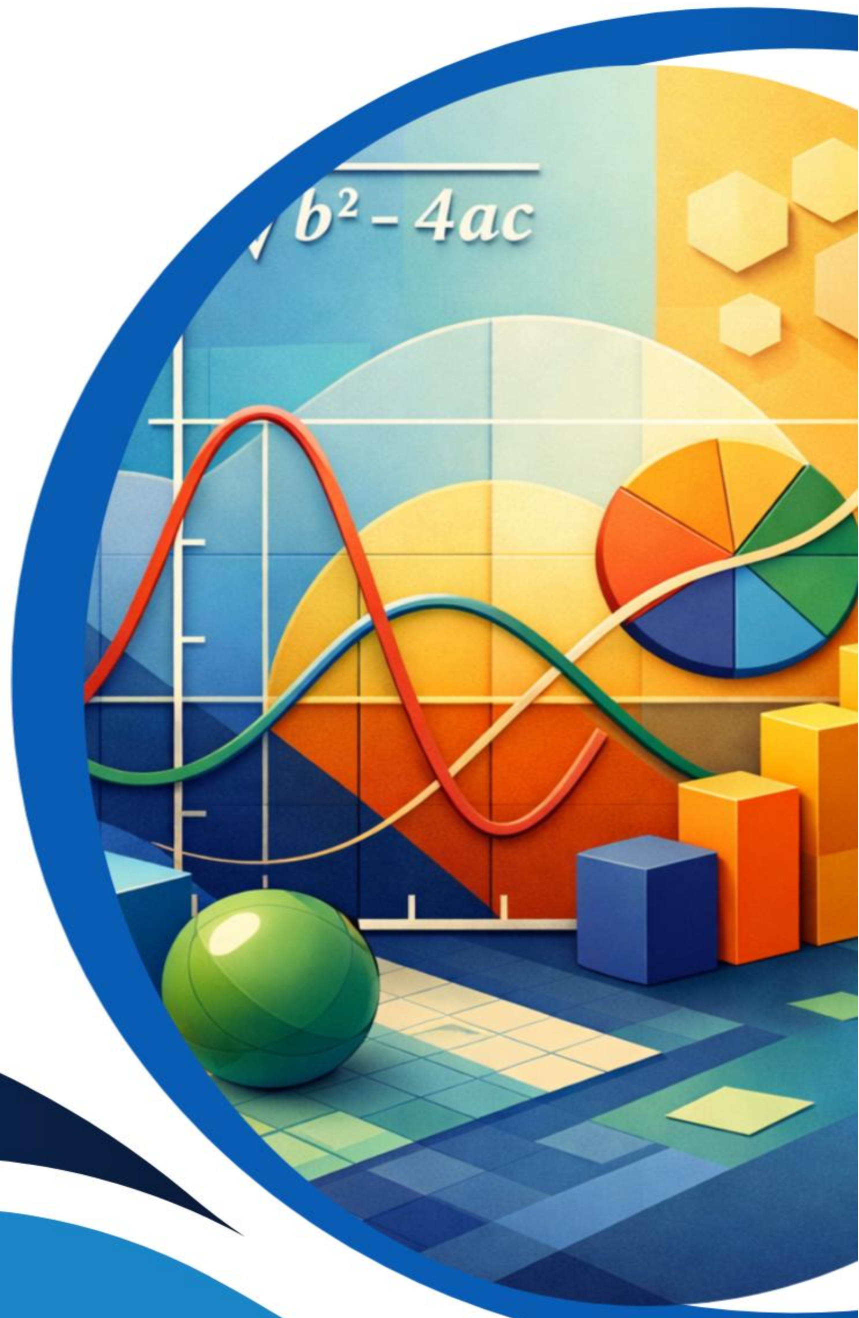
ECO104

INTRODUCTORY MATHEMATICS II



Course code: ECO 104

Course title: Introductory Mathematics II

Course credit load: 2 Units

Series 1: Derivatives of Trigonometric Functions and Expansions

Series 2: Sequences and Series in Economic Applications

Series 3: Partial and Total Derivatives with Optimisation

Series 4: Linear and Matrix Algebra in Economics

Series 5: Mathematical Relationships and Graphical Analysis of Economic Concepts

Suggested Outline

Part 1: Derivatives of trigonometric functions

- Definition of trigonometric functions in calculus
- Rules for differentiation of sine, cosine, tangent, and reciprocal functions
- Higher-order derivatives of trigonometric functions
- Applications of trigonometric derivatives in economics (e.g., modelling cycles and periodic behaviour)

Part 2: Expansions and approximations

- Concept of expansions in mathematics
- Binomial theorem and its applications
- Taylor and Maclaurin series expansions
- Use of expansions for approximating economic functions

Part 3: Sequences and series in economic contexts

- Definition of sequences and series
- Arithmetic and geometric progressions
- Convergence and divergence of series
- Application of series in modelling economic growth and investment returns

Part 4: Integration of derivatives, expansions, and series in economic analysis

- Linking derivatives and expansions to economic modelling
- Practical examples of using series and expansions in forecasting
- Case studies of mathematical tools applied to introductory economics

Introduction

Mathematics provides the language through which many economic ideas are expressed, and calculus in particular allows us to describe change and relationships with precision. Trigonometric functions, expansions, and series are not only central to mathematics but also have practical applications in economics, such as modelling cycles, approximating functions, and analysing growth patterns. By engaging with these concepts, you will strengthen your ability to translate economic statements into mathematical form and interpret mathematical results in economic terms.

The aim of this Series is to equip you with the skills to differentiate trigonometric functions, apply expansions, and work with sequences and series, while also appreciating how these tools can be applied to economic analysis.

We shall commence this Series by exploring the derivatives of trigonometric functions and their rules of differentiation. Next, we will move on to expansions, beginning with the binomial theorem and continuing with Taylor and Maclaurin series. Following that, we will examine sequences and series, focusing on arithmetic and geometric progressions as well as convergence and divergence. Finally, we will bring the Series to a close by linking these mathematical tools together, showing how derivatives, expansions, and series can be integrated into economic modelling and forecasting.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 1.1 **define** trigonometric functions and **demonstrate** how to differentiate sine, cosine, tangent, and reciprocal functions,

SLO 1.2 **explain** the rules for higher-order derivatives of trigonometric functions and **apply** them to simple problems,

SLO 1.3 **describe** the concept of expansions and **apply** the binomial theorem to mathematical and economic expressions,

SLO 1.4 **demonstrate** how to use Taylor and Maclaurin series expansions to approximate functions,

SLO 1.5 **define** sequences and series and **distinguish** between arithmetic and geometric progressions,

SLO 1.6 **analyse** the convergence and divergence of series with reference to economic contexts, and

SLO 1.7 **apply** derivatives, expansions, and series together to represent and solve introductory economic statements and forecasting problems.

1.1 Derivatives of Trigonometric Functions

1.1.1 Definition of trigonometric functions

Trigonometric functions are mathematical functions that relate the angles of a right-angled triangle to the ratios of its sides. The primary functions are sine (sin), cosine (cos), and tangent (tan). Their reciprocal functions are cosecant (csc), secant (sec), and cotangent (cot). These functions are periodic, meaning they repeat their values at regular intervals. This property makes them useful in economics for modelling cycles such as seasonal demand or fluctuations in investment.

For example, the sine function oscillates between –1 and +1, representing a repeating pattern. In economics, this can be used to model demand that rises and falls over time.

Fill in the blank: The reciprocal of sine is called __________.

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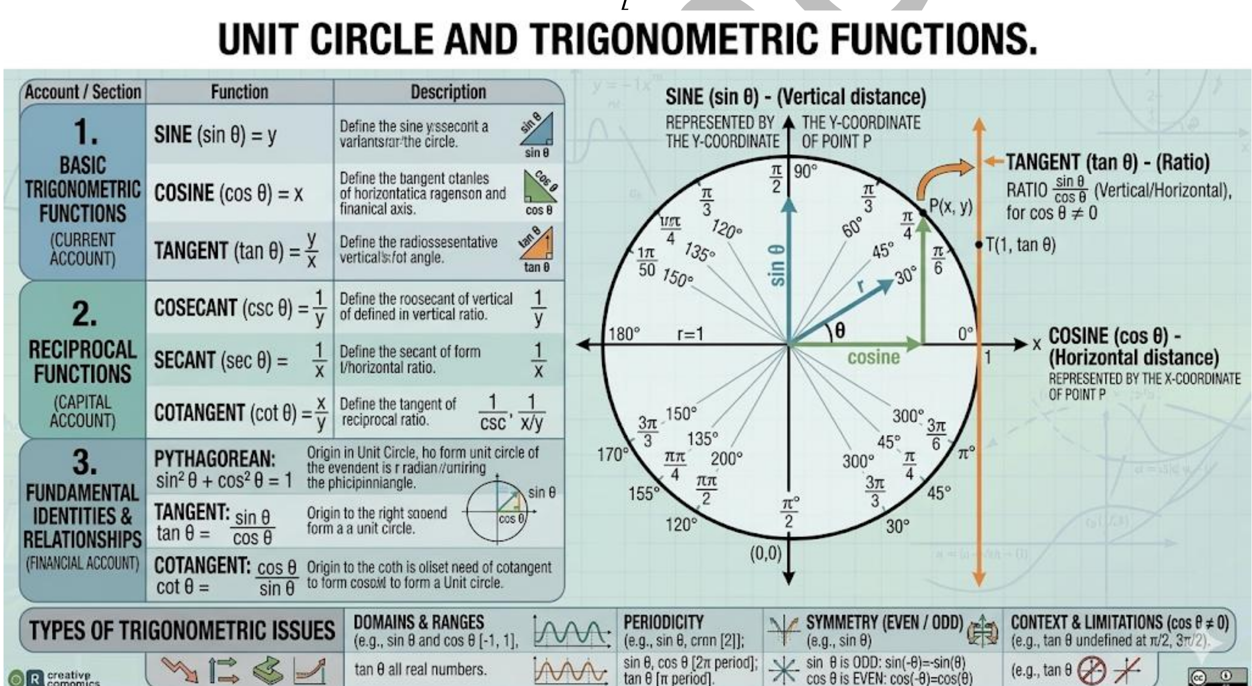

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Diagram showing the unit circle with sine, cosine, and tangent labelled.

Figure 1.1: Unit circle representation of trigonometric functions

1.1.2 Rules for differentiation of sine, cosine, tangent, and reciprocal functions

Differentiation is the process of finding the rate of change of a function. For trigonometric functions, the derivatives follow predictable rules:

- d/dx (sin x) = cos x
- d/dx (cos x) = –sin x
- d/dx (tan x) = sec^{2}x
- d/dx (cot x) = –csc^{2}x
- d/dx (sec x) = sec x tan x
- d/dx (csc x) = –csc x cot x

These rules are essential for solving problems involving periodic behaviour. For instance, if demand is modelled as a sine function, its rate of change can be found using the derivative.

Fill in the blank: The derivative of tan x is __________.

[Insert Table here]
Table 1.1: Derivatives of basic trigonometric functions
Function | Derivative
sin x | cos x
cos x | –sin x
tan x | sec^{2}x
cot x | –csc^{2}x
sec x | sec x tan x
csc x | –csc x cot x
AI prompt suggestion: “Create a table listing trigonometric functions and their derivatives.”
Search string: “derivatives of trigonometric functions table OER mathematics”

1.1.3 Higher-order derivatives of trigonometric functions

A **higher-order derivative** is the derivative of a derivative. For trigonometric functions, these derivatives form cycles:

- Differentiating sin x repeatedly produces:
  sin x → cos x → –sin x → –cos x → sin x.
- Differentiating cos x repeatedly produces:
  cos x → –sin x → –cos x → sin x → cos x.

This cyclical behaviour is particularly useful in modelling phenomena that repeat over time. For example, seasonal demand in agriculture or tourism can be represented using sine and cosine functions, and their derivatives show how quickly demand is rising or falling.

Fill in the blank: Differentiating cos x four times returns the function back to __________.

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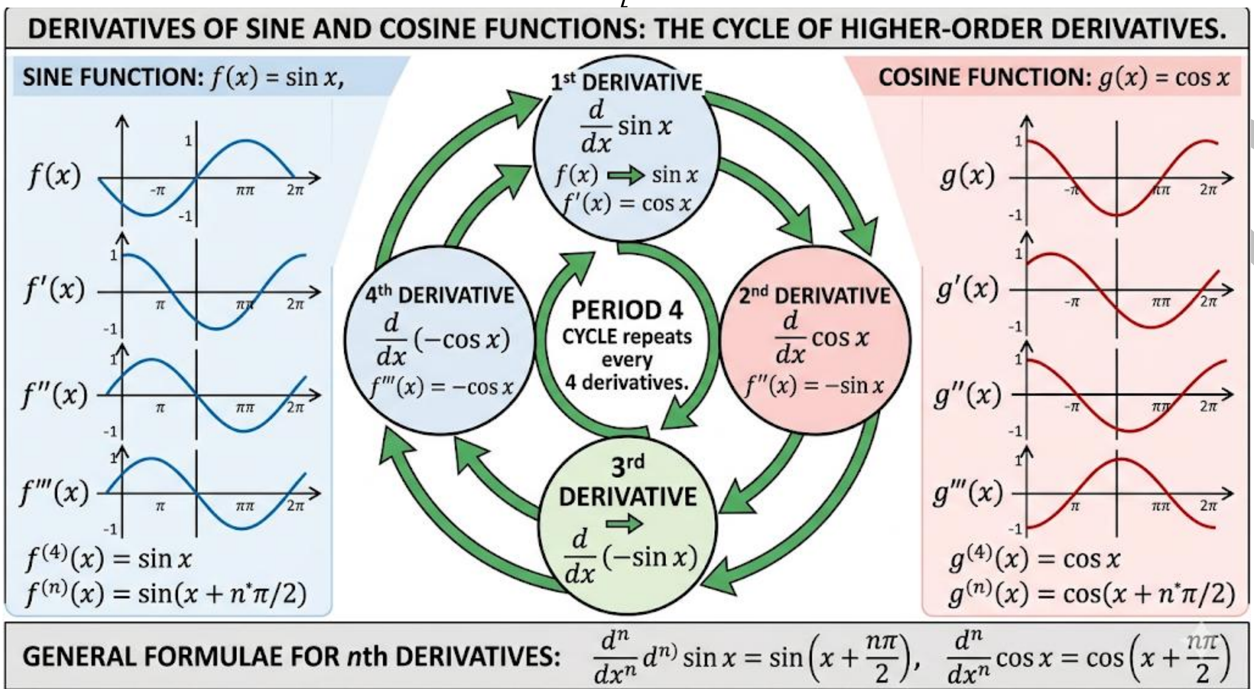

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Diagram showing the cycle of derivatives for sin x and cos x.

Figure 1.2: Cyclical nature of higher-order derivatives of sine and cosine

1.1.4 Applications of trigonometric derivatives in economics

Trigonometric derivatives are not just abstract mathematics, they have practical applications in economics:

- **Cyclical demand**: Sine and cosine functions can model demand that rises and falls periodically.
- **Seasonal production**: Agricultural output often follows seasonal cycles, which can be represented mathematically.
- **Oscillating growth models**: Investment returns or market cycles can be analysed using trigonometric derivatives.

For instance, if consumer demand follows a sine curve, the derivative shows whether demand is increasing or decreasing at a given point in time. This helps economists and businesses anticipate changes and plan accordingly.

Fill in the blank: Oscillating growth models can be analysed using __________ derivatives.

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Topic / Application Area	Description of Use	Derivative Use / Interpretation
Modeling Seasonal Demand	Many industries (tourism, agriculture, retail) face cyclical demand. If demand (Q) is modeled as $f(t) = \sin(t)$, the rate of change is seasonal.	The derivative, $\frac{dQ}{dt} = \cos(t)$, provides the instantaneous rate of change of demand, crucial for forecasting inventory and staffing.
Marginal Analysis of Periodic Revenue	Companies with cyclic income (e.g., ski resorts, summer festivals) model their total revenue (R) using sine waves.	Set Marginal Revenue ($MR = \frac{dR}{dt}$) to zero to find the exact timing of peaks and troughs in the seasonal cycle.
Business Cycle Dynamics	Macroeconomic models can use sine functions to represent the repeating phases of expansion and contraction (the Business Cycle).	Higher-order derivatives are used to define the boundaries (turning points), stability conditions, and period (duration) of the cycle.
Option Pricing & Cyclic Volatility	In finance, certain models (less standard than Black-Scholes) may use trigonometric functions to model predictable, repeating patterns of volatility in asset prices (e.g., time-of-day or seasonal effects).	Derivatives are used to calculate sensitivity (Greeks) related to these periodic components, which are essential for risk management.
Trade and Commodity Price Cycles	Commodity prices often follow seasonal or multi-year cycles. Models of these cycles assist with trade strategy.	Derivatives are used to find the rate of acceleration or deceleration of price changes at various stages of the cycle.

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Box 1.1: Applications of trigonometric derivatives in economics

Application | Description

Cyclical demand | Sine and cosine functions model periodic demand

Seasonal production | Trigonometric functions represent seasonal variations

Oscillating growth | Derivatives help analyse fluctuations in growth models

Pilot questions 1.1

1. Define trigonometric functions and give two examples.
2. State the derivative of cos x.
3. What is the derivative of tan x?
4. Explain the cyclical nature of higher-order derivatives of sine.
5. Give one example of how trigonometric derivatives are applied in economics.

1.2 Expansions and Approximations

1.2.1 Concept of expansions in mathematics

An **expansion** expresses a function as a sum of terms, often in powers of a variable. This process allows us to approximate complex functions with simpler polynomials. Expansions are particularly valuable because they make otherwise difficult functions easier to calculate, analyse, and interpret.

For instance, the exponential function (e^x) can be expanded into an infinite series:
$$[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots]$$
This expansion is widely used in economics when modelling continuous growth, such as compound interest or population growth.

Fill in the blank: The expansion of (e^x) begins with __________.

1.2.2 Binomial theorem and its applications

The **binomial theorem** provides a systematic way of expanding expressions of the form $((a + b)^n)$. It is expressed as:
$$[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k]$$
where $(\binom{n}{k})$ is the binomial coefficient, calculated as $(\frac{n!}{k!(n-k)!})$.

Worked Example: Expand $((1 + x)^3)$.
$$[(1 + x)^3 = 1 + 3x + 3x^2 + x^3]$$

Economic Application: Suppose a firm’s revenue grows by a fixed proportion each year. The binomial theorem can be used to expand expressions representing cumulative growth over several years.

Fill in the blank: The binomial coefficient is calculated using __________ factorials.

1.2.3 Taylor and Maclaurin series expansions

The **Taylor series** expands a function about a point (a):
$$[f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots]$$

The **Maclaurin series** is a special case where $(a = 0)$. For example:
$$[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots]$$

Worked Example: Approximate ($\sin(0.1)$) using the first three terms of its Maclaurin series.
$$\sin(0.1) \approx 0.1 - \frac{0.1^3}{6} + \frac{0.1^5}{120} = 0.09983$$
This approximation is very close to the actual value.

Economic Application: Taylor expansions are used to approximate production functions, making them easier to analyse when inputs change slightly.

Fill in the blank: The Maclaurin series expands a function about the point __________.

1.2.4 Use of expansions for approximating economic functions

Expansions simplify complex economic models:

- **Production functions**: Approximating Cobb-Douglas functions for small changes in inputs.
- **Utility functions**: Expanding utility functions to study marginal changes in consumption.
- **Cost functions**: Using expansions to approximate cost behaviour near equilibrium.
- **Forecasting models**: Applying series expansions to predict future values with manageable calculations.

Worked Example: A production function ($Q = L^{0.5}K^{0.5}$) can be approximated using a Taylor expansion around average values of labour (L) and capital (K) to estimate output changes.

Fill in the blank: Expansions are often used to approximate __________ functions in economics.

Pilot questions 1.2

1. Define expansions and give one example.
2. State the binomial theorem formula.
3. Expand ($(1 + x)^3$) using the binomial theorem.
4. Write the first three terms of the Maclaurin series for ($\sin x$).
5. Explain one application of expansions in production or utility functions.

1.3 Sequences and Series in Economic Contexts

1.3.1 Definition of sequences and series

A **sequence** is an ordered list of numbers following a rule, while a **series** is the sum of those terms. Sequences and series help us study growth, accumulation, and long-term behaviour.

Worked Example: The sequence 2, 4, 6, 8 follows the rule “add 2 each time.” The series is 2 + 4 + 6 + 8 = 20.

Economic Application: A sequence may represent annual profits, while the series represents cumulative profits over several years.

Fill in the blank: A sequence is an ordered __________ of numbers.

1.3.2 Arithmetic and geometric progressions

An **arithmetic progression (AP)** has a constant difference between terms.
Formula for nth term: ($a_n = a + (n-1)d$).
Formula for sum of n terms: ($S_n = \frac{n}{2}[2a + (n-1)d]$).

A **geometric progression (GP)** has a constant ratio between terms.
Formula for nth term: ($a_n = ar^{n-1}$).
Formula for sum of n terms: ($S_n = a \frac{1-r^n}{1-r}$), if ($|r| < 1$).

Worked Example: Find the sum of the first 5 terms of the GP 2, 4, 8, 16, 32.
$$S_5 = 2 \frac{1-2^5}{1-2} = 62$$

Economic Application: AP can represent steady wage increases, while GP models compound interest.

Fill in the blank: The nth term of an arithmetic progression is given by __________.

1.3.3 Convergence and divergence of series

A series **converges** if its sum approaches a finite value, and **diverges** if it grows without bound.

Worked Example: The series 1 + 1/2 + 1/4 + 1/8 + … converges to 2.
The series 1 + 2 + 3 + 4 + … diverges.

Economic Application: Convergence is important in investment models where returns stabilise, while divergence may represent unsustainable debt growth.

Fill in the blank: A series that grows without bound is said to __________.

1.3.4 Application of series in modelling economic growth and investment returns

Series are applied in economics to:

- **Investment returns**: GP models compound interest.
- **Economic growth**: Series represent cumulative growth.
- **Savings accumulation**: Series show how deposits build up.
- **Debt repayment**: Series calculate instalments and total repayment.

Worked Example: If you deposit ₦100 monthly at 5% interest compounded annually, the total savings after 12 months can be modelled as a geometric series.

Fill in the blank: Debt repayment schedules can be calculated using __________ series.

Pilot questions 1.3

1. Define a sequence and a series with examples.
2. State the formula for the nth term of an arithmetic progression.
3. Find the sum of the first 5 terms of the GP 2, 4, 8, 16, 32.
4. Explain the difference between convergence and divergence of series.

5. Give one application of series in savings or debt repayment.

1.4 Integration of Derivatives, Expansions, and Series in Economic Analysis

1.4.1 Linking derivatives and expansions to economic modelling

Derivatives are mathematical tools that measure the rate of change of a function. In economics, they are used to calculate marginal values, such as marginal cost or marginal utility. **Expansions**, on the other hand, approximate complex functions with simpler polynomial expressions, making them easier to analyse. When combined, derivatives and expansions allow economists to study how small changes in inputs affect outputs and to approximate functions that are otherwise difficult to handle.

Worked Example: Consider a production function (Q = L^{0.5}K^{0.5}). Expanding this function around average values of labour (L) and capital (K) using a Taylor series provides a polynomial approximation. Differentiating the expansion then gives estimates of marginal productivity.

Fill in the blank: Derivatives are used in economics to calculate __________ values such as marginal cost.

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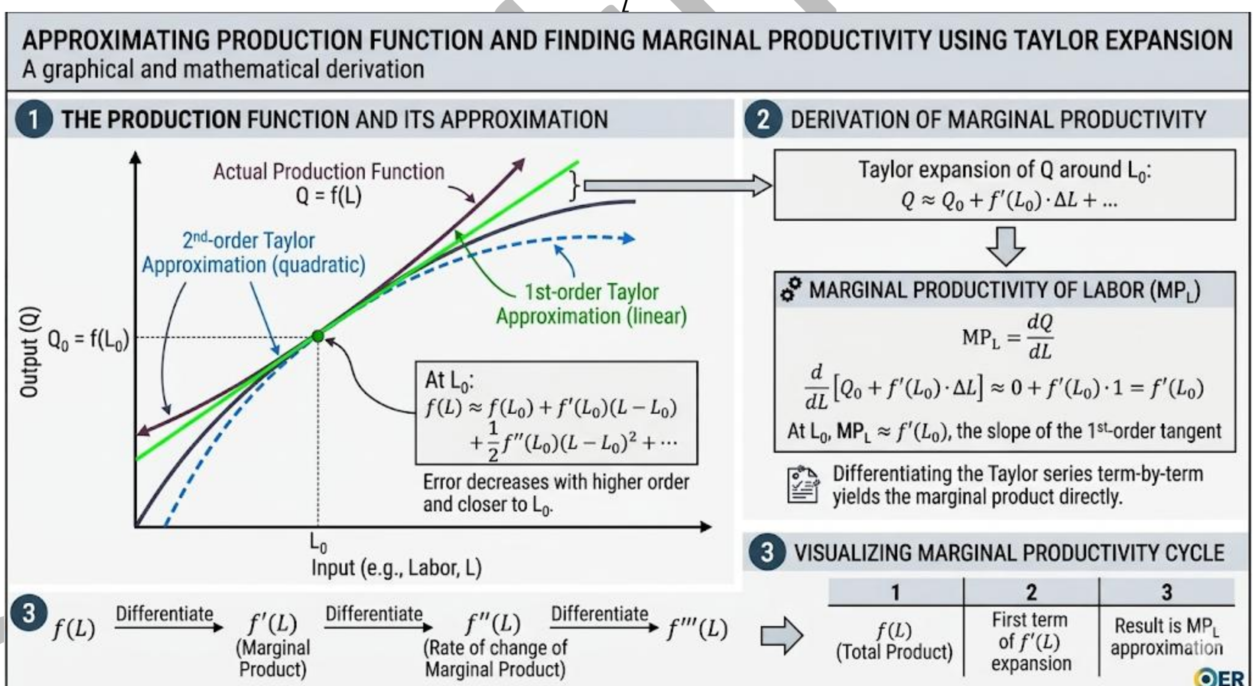

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Diagram showing a production function approximated by a Taylor expansion and differentiated to find marginal productivity.

Figure 1.7: Linking expansions and derivatives in economic modelling

1.4.2 Practical examples of using series and expansions in forecasting

Series represent cumulative behaviour over time, while expansions simplify functions for short-term analysis. Together, they provide powerful tools for forecasting.

Worked Example: Suppose a firm's revenue follows a cyclical pattern modelled by a sine function. Expanding the sine function using its Maclaurin series provides a polynomial approximation. Differentiating this approximation shows whether revenue is increasing or decreasing at a given point.

Economic Applications:

- Forecasting demand cycles using trigonometric series.
- Predicting savings accumulation using geometric series.
- Approximating cost functions with Taylor expansions for short-term analysis.
- Estimating revenue growth using polynomial approximations.

Fill in the blank: Savings accumulation over time can be modelled using __________ series.

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Forecasting Method / Area	Description of Use	Mathematical Approach (Series/Expansions)
Local Forecasting & Linearization	Simplifying complex, nonlinear models around a specific data point or steady state for near-term forecasting.	Taylor Series Expansions are used to linearize nonlinear equations (1st-order) or add curvature (2nd-order) for polynomial approximation.
DSGE Model Approximation	Making large structural macroeconomic models (Dynamic Stochastic General Equilibrium) tractable for simulating future scenarios.	Polynomial approximations, often up to the 2nd order, are applied to the Euler equations and first-order conditions to generate linear state-space models.
Long-Term Growth Steady State	Projecting the deterministic, long-term equilibrium path of an economy (e.g., Solow model).	Infinite Geometric Series are used to calculate the convergent sum of cumulative investment or technological progress over time.
Present Value & Asset Pricing	Forecasting the current fair value of investments, projects,	Infinite Series (Discounted Cash Flow Series), e.g., $V_0 = \sum_{t=1}^{\infty}

Forecasting Method / Area	Description of Use	Mathematical Approach (Series/Expansions)
	or stocks based on an infinite future income stream.	\frac{D_t}{(1+r)^t}$, are solved using formulas for infinite sums.
Recursive Time Series Models	Solving recursive structures to forecast multi-period ahead values (e.g., in vector autoregressions, VARs).	Solutions to difference equations in time series are often expressed as infinite sums of coefficients (e.g., MA(∞) or AR(∞) representations).
Volatility Convergence Analysis	In GARCH or volatility models, assessing if future variance forecasts will converge to a long-run level.	The convergence depends on the sum of coefficients in an infinite variance series, analogous to checking series convergence.

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Box 1.4: Forecasting applications of series and expansions

Application | Description

Demand cycles | Trigonometric series model periodic demand

Savings accumulation | Geometric series show build-up of deposits

Cost functions | Taylor expansions approximate short-term behaviour

Revenue forecasting | Series expansions predict future values

1.4.3 Case studies of mathematical tools applied to introductory economics

Case studies illustrate how derivatives, expansions, and series are applied in real-world economic contexts:

- **Case Study 1: Agricultural output**
  Seasonal crop yields can be modelled using sine functions. Differentiating the function shows the rate of increase or decrease in output at different times of the year, helping farmers plan harvests.
- **Case Study 2: Compound interest**
  Savings accounts with compound interest are modelled using geometric series. The sum of the series shows the total accumulation over time, which is essential for financial planning.
- **Case Study 3: Production function approximation**
  A Cobb-Douglas production function can be approximated using a Taylor expansion

around average input values. Differentiating the expansion provides marginal productivity estimates, guiding firms in resource allocation.

Worked Example: A firm deposits ₦100 monthly at 5% interest compounded annually. The total savings after 12 months can be modelled as a geometric series, showing both the accumulation and the effect of interest.

Fill in the blank: Seasonal crop yields can be modelled using __________ functions.

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Mathematical Tool	Primary Economic Application Area	Specific Case Study / Scenario	Description of Use	Derivative/Series Implementation
Derivatives (1st & 2nd Order)	**Microeconomics: Theory of the Firm**	Profit Maximization for a Monopolist	A firm with market power must find the output level that maximizes total profit ($\pi = TR - TC$).	Differentiate the profit function, set the **First-Order Condition (FOC)** to zero ($\frac{d\pi}{dQ} = MR - MC = 0$) to find critical points, and use the **Second-Order Condition (SOC)** ($\frac{d^2\pi}{dQ^2} < 0$) to confirm it is a maximum.
Derivatives (Partial)	**Microeconomics: Consumer Theory**	Constrained Utility Maximization	A consumer aims to maximize their satisfaction (utility, $U(x, y)$) given a limited budget.	Use partial derivatives within a Lagrange Multiplier setup ($\mathcal{L} = U(x, y) + \lambda(I - p_xx - p_yy)$) to find the optimal bundle (x^*, y^*) where the marginal rate of substitution equals the price ratio.
Expansions (Taylor Series)	**Macroeconomic Modeling & Forecasting**	Discretization of Dynamic Stochastic	Modern macroeconomic models rely on a	Apply a first- or second-order Taylor expansion to linearize or

Mathematical Tool	Primary Economic Application Area	Specific Case Study / Scenario	Description of Use	Derivative/Series Implementation
		General Equilibrium (DSGE) Models	system of complex nonlinear difference equations that are hard to solve.	quadratically approximate the nonlinear first-order conditions (like Euler equations) around a known steady state, transforming them into tractable equations for estimation and forecasting.
Series (Infinite Geometric)	**Macroeconomics: National Income & Finance**	The Keynesian Multiplier and Permanent Income Hypotheses	To determine how an initial change in autonomous spending (e.g., government investment, G) ripples through the economy.	If the Marginal Propensity to Consume (MPC) is 'c', the total change in GDP is an infinite geometric series: $\Delta Y = \Delta G + c\Delta G + c^2\Delta G + ...$ The sum is derived as $\Delta Y = \frac{\Delta G}{1-c}$, showing a convergent, predictable effect.
Series (Infinite Discounted)	**Finance & Investment Analysis**	Asset Pricing (e.g., Dividend Discount Model) and Capital Budgeting (NPV)	To determine the present value of a bond, project, or stock that provides a stream of future cash flows over an infinite horizon.	The current value is calculated as the sum of discounted future payments: $V_0 = \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^t}$ If dividends (D_t) are constant, this is solved using the convergent sum formula for an infinitely decreasing geometric series: $V_0 = \frac{D}{r}$.

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Table 1.4: Case studies of mathematical tools in economics

Case study	Mathematical tool	Economic application
Agricultural output	Trigonometric derivatives	Seasonal crop yields
Compound interest	Geometric series	Savings accumulation
Production function	Taylor expansion	Marginal productivity

Pilot questions 1.4

1. Explain how derivatives and expansions can be linked in economic modelling.
2. Give one example of using series in forecasting.
3. State the mathematical tool used to approximate production functions.
4. Which type of series models compound interest?
5. Describe one case study where trigonometric derivatives are applied in economics.

Series Summary

This Series has introduced you to the essential mathematical tools of derivatives of trigonometric functions, expansions, and sequences and series, all of which are highly relevant for economic analysis. We began by examining how trigonometric functions can be differentiated to reveal rates of change and cyclical behaviour, then moved on to expansions such as the binomial theorem and Taylor and Maclaurin series, which provide powerful ways of approximating complex functions. Following that, we explored sequences and series, including arithmetic and geometric progressions, convergence and divergence, and their applications in modelling savings, investment returns, and growth. Finally, we integrated these tools to show how they work together in forecasting, production analysis, and case studies drawn from economics.

By completing this Series, you should now be able to define and differentiate trigonometric functions, apply expansions to simplify mathematical and economic expressions, analyse sequences and series in both arithmetic and geometric forms, evaluate convergence and divergence, and apply these tools collectively to represent and solve introductory economic problems. In line with the Series Learning Outcomes, you are now equipped to demonstrate, explain, and apply these mathematical concepts with confidence in economic contexts.

Glossary of Terms

Arithmetic progression (AP): A sequence of numbers in which each term differs from the previous one by a constant difference.

Binomial theorem: A formula that provides the expansion of expressions of the form ((a + b)^n), using binomial coefficients.

Convergence: The property of a series where the sum of its terms approaches a finite value as more terms are added.

Derivative: A mathematical tool that measures the rate of change of a function with respect to a variable.

Expansion: A way of expressing a function as a sum of terms, often in powers of a variable, to simplify calculations or approximations.

Geometric progression (GP): A sequence of numbers in which each term is obtained by multiplying the previous term by a constant ratio.

Higher-order derivative: The derivative of a derivative, showing repeated rates of change of a function.

Maclaurin series: A special case of the Taylor series expansion of a function about zero.

Marginal productivity: The additional output produced when one more unit of input is added, often calculated using derivatives.

Sequence: An ordered list of numbers that follow a specific rule.

Series: The sum of the terms of a sequence.

Taylor series: An expansion of a function about a point, expressed as an infinite sum of derivatives evaluated at that point.

Trigonometric functions: Functions such as sine, cosine, and tangent that relate the angles of a triangle to the ratios of its sides, often used to model periodic behaviour.

Utility function: A mathematical representation of consumer preferences, showing the satisfaction derived from consuming goods or services.

Concept Check Questions Series 1: Derivatives of Trigonometric Functions and Expansions

Objective Questions

1. The derivative of (\cos x) is:
   a) (\sin x)
   b) (-\sin x)
   c) (\sec^2 x)
   d) (-\cos x)
   (Linked to SLO 1.1)
2. Which of the following is a geometric progression?
   a) 2, 4, 6, 8
   b) 3, 6, 12, 24
   c) 5, 10, 15, 20
   d) 1, 2, 3, 4
   (Linked to SLO 1.5)
3. The Maclaurin series is a special case of which expansion?
   a) Binomial theorem
   b) Taylor series
   c) Arithmetic progression

d) Geometric progression
(Linked to SLO 1.4)

4. A series that grows without bound is said to:
a) Converge
b) Diverge
c) Expand
d) Approximate
(Linked to SLO 1.6)

5. Which mathematical tool is most suitable for approximating production functions?
a) Derivatives
b) Expansions
c) Sequences
d) Trigonometric functions
(Linked to SLO 1.7)

Short-Answer Questions

1. Define a derivative and explain its importance in economics. (Linked to SLO 1.1)
2. Write the expansion of ((1 + x)^3) using the binomial theorem. (Linked to SLO 1.3)
3. State the formula for the nth term of an arithmetic progression. (Linked to SLO 1.5)
4. Give one example of a convergent series and explain why it converges. (Linked to SLO 1.6)
5. Describe one application of trigonometric derivatives in modelling economic cycles. (Linked to SLO 1.2)

Long-Answer Questions

1. Analyse how Taylor and Maclaurin series can be used to approximate economic functions, giving at least one worked example. (Linked to SLO 1.4)
    - **Basic response**: Explain the concept of Taylor and Maclaurin series, show how they approximate functions, and provide a simple example such as approximating (\sin x).
    - **Value-added response**: Extend the explanation by applying the approximation to an economic function, such as a production or utility function, and discuss the advantages and limitations of using series expansions in economic modelling.
2. Evaluate the role of sequences and series in modelling savings and investment returns, with reference to arithmetic and geometric progressions. (Linked to SLO 1.5, SLO 1.6)
    - **Basic response**: Define arithmetic and geometric progressions, explain their formulas, and show how they apply to savings or investment.
    - **Value-added response**: Provide a detailed worked example of compound interest using a geometric series, compare it with an arithmetic progression representing steady savings, and discuss the implications for long-term financial planning.
3. Design a simple economic model that integrates derivatives, expansions, and series to forecast demand or production. (Linked to SLO 1.7)

- **Basic response**: Outline how derivatives measure change, expansions approximate functions, and series represent cumulative behaviour, then combine them conceptually in a model.
- **Value-added response**: Present a worked example, such as modelling seasonal demand with a sine function, expanding it with a Maclaurin series, differentiating to find rate of change, and summing terms to forecast cumulative demand.

Answers to Concept Check Questions Series 1

Objective Questions

1. b) $(-\sin x)$
2. b) 3, 6, 12, 24
3. b) Taylor series
4. b) Diverge
5. b) Expansions

Short-Answer Questions

1. A derivative measures the rate of change of a function. In economics, it is used to calculate marginal values such as marginal cost or marginal utility.
2. $((1 + x)^3 = 1 + 3x + 3x^2 + x^3)$.
3. $(a_n = a + (n-1)d)$.
4. The series 1 + 1/2 + 1/4 + 1/8 + … converges to 2 because the sum approaches a finite value as more terms are added.
5. Trigonometric derivatives can model cyclical demand, showing whether demand is rising or falling at a given point in time.

Long-Answer Questions

1. **Basic response**: Taylor and Maclaurin series approximate functions by expressing them as polynomials. For example, $(\sin x = x - x^3/3! + x^5/5! - \dots)$.
   Value-added response: In economics, a production function can be approximated using a Taylor series around average input values. This makes it easier to analyse marginal productivity and forecast changes.

2. **Basic response**: Arithmetic progressions represent steady savings, while geometric progressions model compound interest.
   Value-added response: For example, depositing ₦100 monthly with compound interest forms a geometric series. The sum shows total savings accumulation, which grows faster than an arithmetic progression representing fixed deposits without interest.

3. **Basic response**: Derivatives measure change, expansions approximate functions, and series represent cumulative behaviour. Together, they can forecast demand or production.
   Value-added response: Seasonal demand can be modelled with a sine function, expanded using a Maclaurin series, differentiated to find the rate of change, and summed as a series to forecast cumulative demand over time.

Answers to Pilot Questions 1

Part 1: Derivatives of Trigonometric Functions

1. Trigonometric functions are mathematical functions that relate angles to side ratios of triangles. Examples include sine and cosine.
2. The derivative of tan x is $\sec^2 x$.
3. The derivative of cos x is –sin x.
4. Higher-order derivatives of sine follow a cycle: sin x → cos x → –sin x → –cos x → sin x.
5. Trigonometric derivatives are applied in economics to model cyclical demand patterns.

Part 2: Expansions and Approximations

1. Expansions express a function as a sum of terms, often in powers of a variable.
2. The binomial theorem formula is ($(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$).
3. The Maclaurin series for sin x up to the fifth term is ($x - \frac{x^3}{3!} + \frac{x^5}{5!}$).
4. The binomial theorem is applied in economics to calculate probabilities in repeated trials.
5. Expansions approximate production and utility functions to simplify analysis.

Part 3: Sequences and Series in Economic Contexts

1. A sequence is an ordered list of numbers, and a series is the sum of those numbers.
2. An example of an arithmetic progression is 2, 4, 6, 8, with a common difference of 2.
3. The common ratio in the geometric progression 3, 6, 12, 24 is 2.
4. Convergence means the sum approaches a finite value, while divergence means it grows without bound.
5. Series are applied in economics to model savings accumulation.

Part 4: Integration of Derivatives, Expansions, and Series in Economic Analysis

1. Derivatives and expansions are linked in economic modelling by approximating complex functions and then differentiating to find marginal values.
2. Series are used in forecasting, for example, geometric series model savings accumulation.
3. Production functions are approximated using Taylor expansions.
4. Compound interest is modelled using geometric series.
5. Trigonometric derivatives are applied in economics to model seasonal crop yields.

References and Attributions [Series 1]

Figures

- *Figure 1.1: Unit circle representation of trigonometric functions*
  AI-generated suggestion: "Generate a diagram of the unit circle showing sine, cosine, and tangent functions."
  Suggested OER search string: "unit circle trigonometric functions diagram OER mathematics."

- *Figure 1.2: Cyclical nature of higher-order derivatives of sine and cosine*
  AI-generated suggestion: "Generate a diagram showing the cycle of derivatives for sine and cosine functions."
  Suggested OER search string: "higher-order derivatives sine cosine cycle diagram OER mathematics."
- *Figure 1.3: Function approximation using expansion*
  AI-generated suggestion: "Generate a diagram showing a curve approximated by a polynomial expansion."
  Suggested OER search string: "function approximation polynomial expansion diagram OER mathematics."
- *Figure 1.4: Approximation of sine function using Maclaurin series*
  AI-generated suggestion: "Generate a diagram showing approximation of sine function using Maclaurin series expansion."
  Suggested OER search string: "Maclaurin series sine function approximation diagram OER mathematics."
- *Figure 1.5: Relationship between sequences and series*
  AI-generated suggestion: "Generate a diagram showing a sequence of numbers and the sum forming a series."
  Suggested OER search string: "sequence and series diagram OER mathematics."
- *Figure 1.6: Convergence and divergence of series*
  AI-generated suggestion: "Generate a diagram showing convergence of a series to a finite value and divergence to infinity."
  Suggested OER search string: "convergence divergence series diagram OER mathematics."
- *Figure 1.7: Linking expansions and derivatives in economic modelling*
  AI-generated suggestion: "Generate a diagram showing a production function approximated by Taylor expansion and differentiated to find marginal productivity."
  Suggested OER search string: "Taylor expansion derivative production function diagram OER economics."

Tables

- *Table 1.1: Derivatives of basic trigonometric functions*
  AI-generated suggestion: "Create a table listing trigonometric functions and their derivatives."
  Suggested OER search string: "derivatives of trigonometric functions table OER mathematics."
- *Table 1.2: Examples of binomial expansions*
  AI-generated suggestion: "Create a table showing examples of binomial expansions for powers 2, 3, and 4."
  Suggested OER search string: "binomial theorem expansion examples OER mathematics."
- *Table 1.3: Examples of arithmetic and geometric progressions*
  AI-generated suggestion: "Create a table showing examples of arithmetic and geometric progressions with rules."

Suggested OER search string: “arithmetic geometric progression examples OER mathematics.”

- *Table 1.4: Case studies of mathematical tools in economics*
  AI-generated suggestion: “Create a table summarising case studies of derivatives, expansions, and series in economics.”
  Suggested OER search string: “case studies derivatives expansions series economics OER.”

Boxes

- *Box 1.1: Applications of trigonometric derivatives in economics*
  AI-generated suggestion: “Create a text box summarising applications of trigonometric derivatives in economics.”
  Suggested OER search string: “applications of trigonometric derivatives economics OER.”
- *Box 1.2: Applications of expansions in economics*
  AI-generated suggestion: “Create a text box summarising applications of expansions in economics.”
  Suggested OER search string: “applications of expansions in economics OER.”
- *Box 1.3: Applications of series in economics*
  AI-generated suggestion: “Create a text box summarising applications of series in economics.”
  Suggested OER search string: “applications of series in economics OER.”
- *Box 1.4: Forecasting applications of series and expansions*
  AI-generated suggestion: “Create a text box summarising forecasting applications of series and expansions in economics.”
  Suggested OER search string: “applications of series expansions forecasting economics OER.”

General References

- Stewart, J. (2016). *Calculus: Early Transcendentals* (8th ed.). Boston: Cengage Learning.
- Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). New York: McGraw-Hill.
- OpenStax. (2016). *Calculus Volume 1–3*. Rice University. Retrieved from https://openstax.org/subjects/math
- Khan Academy. (n.d.). *Sequences, Series, and Taylor Expansions*. Retrieved from https://www.khanacademy.org

Course code: ECO 104

Course title: Introductory Mathematics II

Course credit load: 2 Units

Series 1: Derivatives of Trigonometric Functions and Expansions

Series 2: Sequences and Series in Economic Applications

Series 3: Partial and Total Derivatives with Optimisation

Series 4: Linear and Matrix Algebra in Economics

Series 5: Mathematical Relationships and Graphical Analysis of Economic Concepts

Suggested Outline

Part 2.1: Fundamentals of sequences and series

- Definition of sequences and series
- Distinction between finite and infinite series
- Common notation and terminology

Part 2.2: Arithmetic and geometric progressions

- General term and sum of arithmetic progression (AP)
- General term and sum of geometric progression (GP)
- Worked examples with economic interpretations (e.g., wages, savings)

Part 2.3: Convergence and divergence of series

- Concept of convergence and divergence
- Tests for convergence (basic level suitable for introductory learners)
- Economic implications of convergent and divergent series

Part 2.4: Applications of sequences and series in economics

- Modelling savings accumulation and investment returns
- Analysing growth rates and compounding
- Debt repayment schedules and annuities
- Case studies linking mathematical series to real-world economic scenarios

Introduction

In the last Series, we explored derivatives of trigonometric functions and expansions, seeing how these mathematical tools can be applied to model cyclical behaviour and approximate complex functions in economics. Building on that foundation, this Series turns our attention to **sequences and series**, which are equally important for understanding growth, accumulation, and long-term economic patterns such as savings, investment returns, and debt repayment. By studying these concepts, you will gain insight into how mathematics provides structure and predictability to economic analysis.

The aim of this Series is to equip you with the knowledge and skills to define, analyse, and apply sequences and series in economic contexts, enabling you to model financial and growth-related scenarios with confidence.

We shall commence this Series by examining the fundamentals of sequences and series, clarifying definitions and notation. Next, we will move on to arithmetic and geometric progressions, exploring their formulas and economic interpretations. Following that, we will consider convergence and divergence, highlighting their mathematical meaning and economic implications. Finally, we will conclude by applying sequences and series to practical economic problems such as savings accumulation, investment returns, and debt repayment schedules, supported by illustrative case studies.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 **define** sequences and series clearly, distinguishing between finite and infinite forms,

SLO 2.2 **explain** the notation and terminology commonly used in sequences and series,

SLO 2.3 **demonstrate** how to calculate the general term and sum of an arithmetic progression,

SLO 2.4 **apply** the formulas for the general term and sum of a geometric progression to practical problems,

SLO 2.5 **analyse** the convergence and divergence of series using basic tests and interpret their economic implications,

SLO 2.6 **evaluate** the role of sequences and series in modelling savings, investment returns, and growth rates, and

SLO 2.7 **design** simple economic models that use sequences and series to represent debt repayment schedules and annuities.

2.1 Fundamentals of Sequences and Series

2.1.1 Definition of sequences and series

A **sequence** is an ordered list of numbers that follow a specific rule. Each number in the list is called a **term**, and the position of the term is important because it determines how the sequence progresses. For example, 2, 4, 6, 8 is a sequence where each term increases by 2. A **series** is the sum of the terms of a sequence. For instance, adding the terms of the sequence 2, 4, 6, 8 gives the series 2 + 4 + 6 + 8 = 20.

Sequences and series are central to economics because they help us describe and analyse growth, savings, and investment patterns over time. They provide a mathematical framework for understanding how values accumulate or change across periods.

Fill in the blank: A sequence is an ordered __________ of numbers.

[

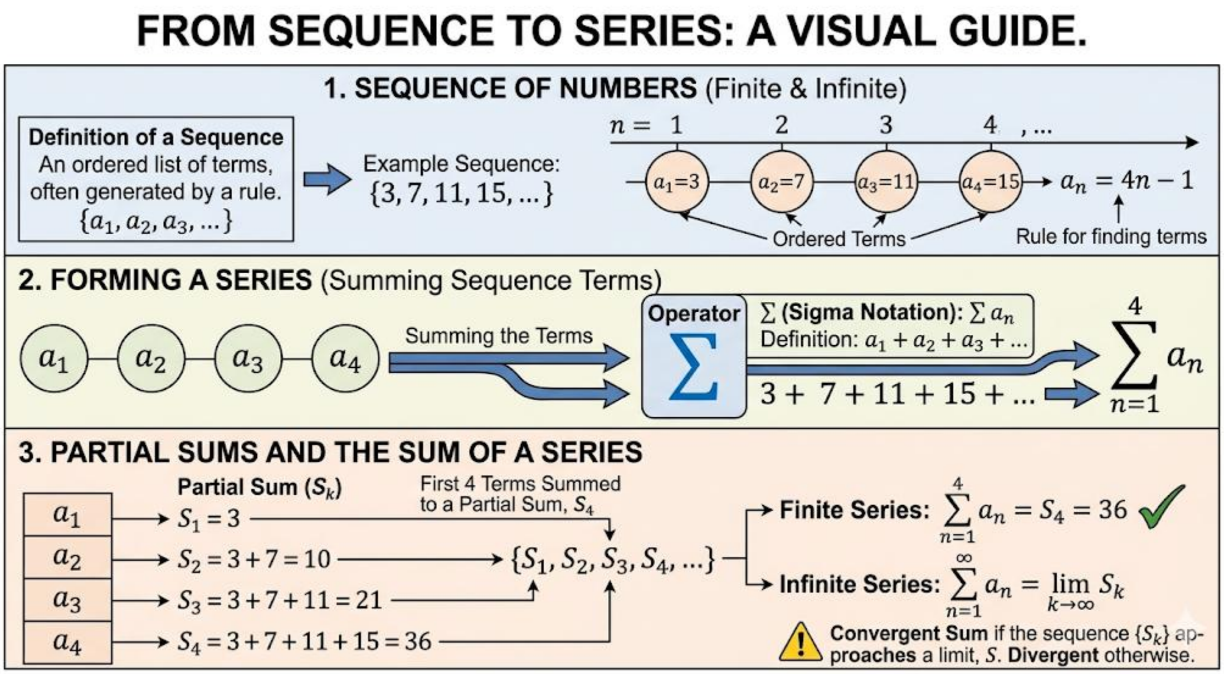

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Diagram showing a sequence of numbers and its corresponding series.

Figure 2.1: Relationship between sequences and series

2.1.2 Distinction between finite and infinite series

A **finite series** has a limited number of terms. For example, 2 + 4 + 6 + 8 is finite because it stops after four terms. An **infinite series** continues indefinitely. For example, 1 + 1/2 + 1/4 + 1/8 + … is infinite because the terms go on forever.

Infinite series are particularly useful in economics when modelling processes that continue over time, such as compound interest, depreciation, or long-term investment returns. Finite series, on the other hand, are often used to represent short-term scenarios such as fixed instalments or limited savings plans.

Fill in the blank: A series that continues indefinitely is called an __________ series.

[

Type of Series	Series Representation (Sigma Notation)	Expanded Series (Terms Summed)	Description and Behavior
Finite Series	$\sum_{i=1}^{5} (2i - 1)$	$1 + 3 + 5 + 7 + 9$	A finite arithmetic series. The sum is exactly 25. Used for summing a defined list.
Finite Series	$\sum_{k=0}^{4} 3^k$	$3^0 + 3^1 + 3^2 + 3^3 + 3^4 = 1 + 3 + 9 + 27 + 81$	A finite geometric series with a specific number of terms ($n=5$). The sum is 121.
Finite Series	$\sum_{j=1}^{10} 1$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	A constant finite series. The sum is simply the number of terms: 10.
Infinite Series (Convergent)	$\sum_{n=1}^{\infty} \frac{1}{2^n}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$	**Convergent Geometric Series:** The terms get infinitely small, approaching a sum of 1. A classic example where an infinite sum can be finite.
Infinite Series (Divergent)	$\sum_{n=1}^{\infty} n$	$1 + 2 + 3 + 4 + ...$	

]

Table 2.1: Examples of finite and infinite series

Type | Example | Description

Finite series | 2 + 4 + 6 + 8 | Stops after a fixed number of terms

Infinite series | 1 + 1/2 + 1/4 + 1/8 + ... | Continues without end

2.1.3 Common notation and terminology

When working with sequences and series, certain notation is widely used:

- a_1: the first term of a sequence.
- a_n: the nth term of a sequence.

- **n**: the number of terms.
- **S_n**: the sum of the first n terms of a series.
- **Σ (sigma notation)**: used to represent the sum of terms in a series. For example, (\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10).

This notation makes it easier to write and manipulate sequences and series in both mathematics and economics. For example, economists use sigma notation to represent cumulative savings or investment returns over time.

Fill in the blank: The symbol Σ is used to represent the __________ of terms in a series.

[

Notation Component	Symbol / Representation	Description / Example
Sequence	$(a_n)_{n=1}^\infty$, $\{a_n\}$, or $\{a_1, a_2, \dots\}$	An ordered list of numbers. Example: $a_n = 2n-1$ yields $\{1, 3, 5, \dots\}$.
Series	$\sum_{n=1}^\infty a_n$	The sum of the terms of a sequence. $\sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \dots$
Index of Summation	$n=1$ (or i, j, k)	The variable below Σ that specifies the starting term for the sum.
Lower Limit	The integer value next to the index (e.g., '1')	The starting point of the summation. Usually $n=1$ or $n=0$.
Upper Limit	The value above Σ (e.g., ∞, k)	The ending point of the summation (for finite series, e.g., k; for infinite series, ∞).
Partial Sum (S_k)	$S_k = \sum_{n=1}^k a_n$	The sum of the first k terms of a series. $S_3 = a_1 + a_2 + a_3$.
Convergence	$\lim_{k\to\infty} S_k = S$ (where S is finite)	The series sum approaches a specific finite number as $k \to \infty$.

Notation Component	Symbol / Representation	Description / Example
Divergence	$\lim_{k\to\infty} S_k$ does not exist or $\pm\infty$	The series sum does not approach a single finite limit.
Index Shift	$\sum_{n=1}^\infty a_n \equiv \sum_{m=0}^\infty a_{m+1}$	Changing the starting index, which changes the formula but keeps the same sum.

]

Box 2.1: Key notation in sequences and series

Notation | Meaning

a_1 | First term of a sequence

a_n | nth term of a sequence

n | Number of terms

S_n | Sum of the first n terms

Σ | Sum of terms in a series

2.1.4 Economic relevance of sequences and series

Sequences and series are not just abstract mathematical concepts; they have direct applications in economics. For example:

- **Savings accumulation**: A sequence can represent monthly deposits, while the series represents total savings.
- **Investment returns**: Geometric series model compound interest, showing how investments grow over time.
- **Debt repayment**: A finite series can represent fixed instalments paid over a set period.
- **Growth modelling**: Sequences and series can capture steady or exponential growth patterns in production or consumption.

Worked Example: If you deposit ₦100 monthly for 12 months, the sequence is 100, 100, 100, … (12 terms). The series is the sum of these terms, giving ₦1,200.

Fill in the blank: Compound interest is best modelled using __________ series.

[

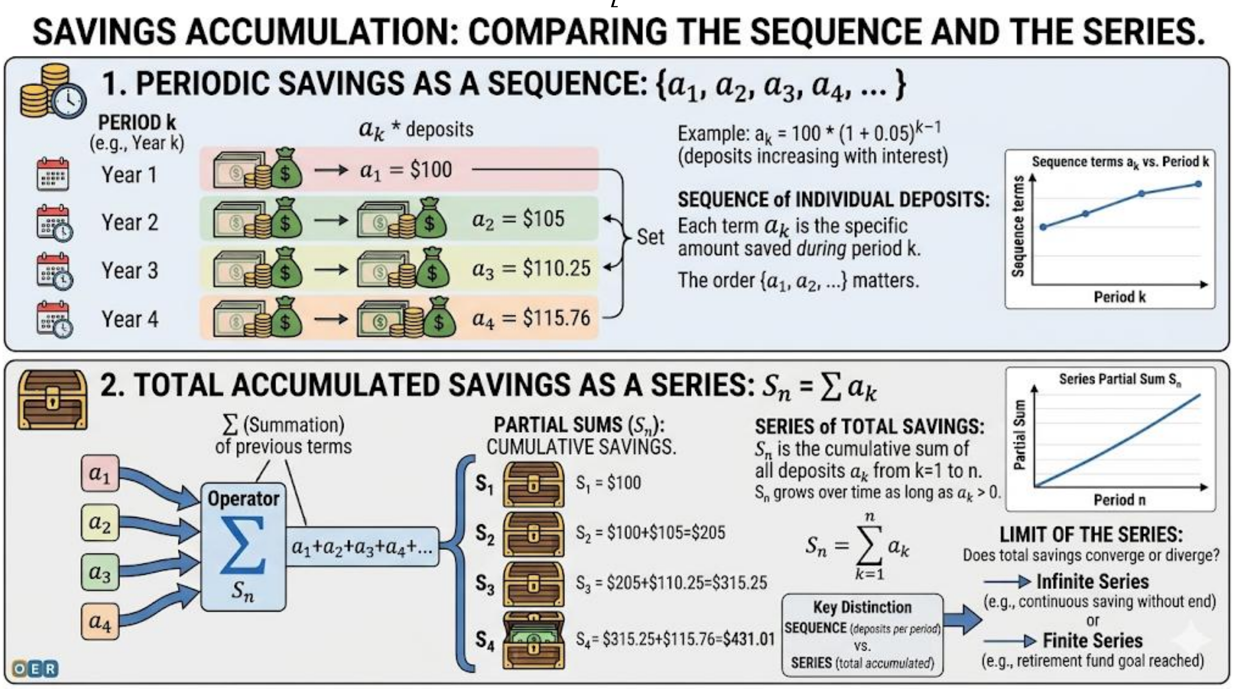

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Diagram showing savings accumulation as a sequence and total savings as a series.

Figure 2.2: Savings accumulation represented by sequences and series

Pilot questions 2.1

1. Define a sequence and a series.
2. Give one example of a finite series and one example of an infinite series.
3. What does the notation a_n represent in a sequence?
4. Write the sum of the first four natural numbers using sigma notation.
5. Explain one way sequences and series are applied in economics.

2.2 Arithmetic and Geometric Progressions

2.2.1 Arithmetic progression (AP)

An **arithmetic progression (AP)** is a sequence of numbers in which each term differs from the previous one by a constant difference, called the **common difference**. For example, 5, 10, 15, 20 is an AP with a common difference of 5.

The formula for the nth term of an AP is:
[a_n = a + (n-1)d]
where (a) is the first term, (d) is the common difference, and (n) is the position of the term.

The sum of the first (n) terms of an AP is:
[S_n = \frac{n}{2}[2a + (n-1)d]]

Economic Application: APs can represent steady increases in wages, fixed savings deposits, or regular increments in production.

Fill in the blank: The constant difference between consecutive terms in an AP is called the __________ difference.

[

Example Name / Description	First Few Terms of AP	Initial Term (a1)	Common Difference (d)	nth Term Formula	Simplified Formula	Example Value (e.g., n=10)
Simple Counting (Odds)	$1, 3, 5, 7, \dots$	1	2	$a_n = 1 + (n-1)(2)$	$a_n = 2n - 1$	$a_{10} = 2(10) - 1 = 19$
Simple Counting (Evens)	$2, 4, 6, 8, \dots$	2	2	$a_n = 2 + (n-1)(2)$	$a_n = 2n$	$a_{10} = 2(10) = 20$
Positive, Integer d	$5, 8, 11, 14, \dots$	5	3	$a_n = 5 + (n-1)(3)$	$a_n = 3n + 2$	$a_{10} = 3(10) + 2 = 32$
Negative d (Decrement)	$10, 7, 4, 1, \dots$	10	-3	$a_n = 10 + (n-1)(-3)$	$a_n = 13 - 3n$	$a_{10} = 13 - 3(10) = -17$
Large, Positive d	$100, 150, 200, \dots$	100	50	$a_n = 100 + (n-1)(50)$	$a_n = 50n + 50$	$a_{10} = 50(10) + 50 = 550$
Non-Integer d (Decimals)	$0.5, 1.0, 1.5, 2.0, \dots$	0.5	0.5	$a_n = 0.5 + (n-1)(0.5)$	$a_n = 0.5n$	$a_{10} = 0.5(10) = 5$

Example Name / Description	First Few Terms of AP	Initial Term (a1)	Common Difference (d)	nth Term Formula	Simplified Formula	Example Value (e.g., n=10)
Non-Integer d (Fractions)	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \dots$	$\frac{1}{4}$	$\frac{1}{4}$	$a_n = \frac{1}{4} + (n-1)(\frac{1}{4})$	$a_n = \frac{1}{4} n$	$a_{10} = \frac{1}{4}(10) = 2.5$
Constant Progression	$7, 7, 7, 7, \dots$	7	0	$a_n = 7 + (n-1)(0)$	$a_n = 7$	$a_{10} = 7$

]

Table 2.2: Examples of arithmetic progressions

Sequence | Common difference | nth term formula

2, 4, 6, 8 | 2 | (a_n = 2 + (n-1)2)

5, 10, 15, 20 | 5 | (a_n = 5 + (n-1)5)

2.2.2 Geometric progression (GP)

A **geometric progression (GP)** is a sequence of numbers in which each term is obtained by multiplying the previous term by a constant ratio, called the **common ratio**. For example, 3, 6, 12, 24 is a GP with a common ratio of 2.

The formula for the nth term of a GP is:
[a_n = ar^{n-1}]
where (a) is the first term, (r) is the common ratio, and (n) is the position of the term.

The sum of the first (n) terms of a GP is:
[S_n = a \frac{1-r^n}{1-r}, \quad \text{if } r \neq 1]

Economic Application: GPs are used to model compound interest, population growth, and investment returns, where values grow exponentially over time.

Fill in the blank: The constant multiplier between consecutive terms in a GP is called the __________ ratio.

[

Example Name / Description	First Few Terms of GP	Initial Term (a1)	Common Ratio (r)	nth Term Formula (an =a1·rn−1)	Example Value (e.g., n=6)
Simple Doubling	$1, 2, 4, 8, 16, \dots$	1	2	$a_n = 1 \cdot 2^{(n-1)} = 2^{n-1}$	$a_6 = 2^{6-1} = 2^5 = 32$
Simple Tripling	$3, 9, 27, 81, 243, \dots$	3	3	$a_n = 3 \cdot 3^{(n-1)} = 3^n$	$a_6 = 3^6 = 729$
Decay (Half-life)	$100, 50, 25, 12.5, \dots$	100	$1/2$ (or 0.5)	$a_n = 100 \cdot (0.5)^{(n-1)}$	$a_6 = 100 \cdot (0.5)^5 = 3.125$
Alternating Sign ($r<0$)	$5, -10, 20, -40, 80, \dots$	5	-2	$a_n = 5 \cdot (-2)^{(n-1)}$	$a_6 = 5 \cdot (-2)^5 = -160$
Fractional a_1 and Integer r	$\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \dots$	$\frac{1}{4}$	2	$a_n = \frac{1}{4} \cdot 2^{(n-1)} = 2^{n-3}$	$a_6 = 2^{6-3} = 2^3 = 8$
Decimals (Interest/Growth)	$1000, 1050, 1102.5, \dots$	1000	1.05 (5% growth)	$a_n = 1000 \cdot (1.05)^{(n-1)}$	$a_6 = 1000 \cdot (1.05)^5 \approx 1276.28$
Negative a_1 and Positive r	$-2, -6, -18, -54, \dots$	-2	3	$a_n = -2 \cdot 3^{(n-1)}$	$a_6 = -2 \cdot 3^5 = -486$

Example Name / Description	First Few Terms of GP	Initial Term (a_1)	Common Ratio (r)	nth Term Formula ($a_n = a_1 \cdot r^{n-1}$)	Example Value (e.g., $n=6$)
Constant Progression	$7, 7, 7, 7, \dots$	7	1	$a_n = 7 \cdot 1^{(n-1)} = 7$	$a_6 = 7$

]

Table 2.3: Examples of geometric progressions

Sequence | Common ratio | nth term formula

3, 6, 12, 24 | 2 | ($a_n = 3 \cdot 2^{n-1}$)

100, 50, 25, 12.5 | 0.5 | ($a_n = 100 \cdot (0.5)^{n-1}$)

2.2.3 Comparing arithmetic and geometric progressions

Although both APs and GPs are sequences, they differ in how terms are generated:

- APs grow by **addition** of a constant difference.
- GPs grow by **multiplication** of a constant ratio.

This distinction is crucial in economics. APs are suitable for modelling steady, linear growth such as fixed wage increments, while GPs are used for exponential growth such as compound interest or population expansion.

Worked Example:

- AP: A worker's salary increases by ₦5,000 each year. This forms an AP.
- GP: An investment doubles every year. This forms a GP.

Fill in the blank: APs grow by __________, while GPs grow by __________.

[

Progressive Structure	Arithmetic Progression (AP)	Geometric Progression (GP)
Mathematical Definition	A sequence where the difference between any two consecutive terms is a constant (**common difference**, d).	A sequence where the ratio of any term after the first to its preceding term is a constant (**common ratio**, r).

Progressive Structure	Arithmetic Progression (AP)	Geometric Progression (GP)
Recursive Rule	$a_n = a_{n-1} + d$	$a_n = a_{n-1} \cdot r$
General (n-th) Term	$a_n = a_1 + (n-1)d$	$a_n = a_1 \cdot r^{(n-1)}$
Sum of n Terms (S_n)	$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d]$	$S_n = \frac{a_1(r^n - 1)}{r-1}$ (if $r > 1$) or $S_n = \frac{a_1(1 - r^n)}{1-r}$ (if $r < 1$).
Infinite Sum (S_∞)	Diverges to $\pm\infty$ (unless $d=0$ and $a_1=0$).	Converges to $S_\infty = \frac{a_1}{1-r}$ if $
Graphical Representation	A straight line (linear growth or decline).	An exponential curve (j-curve or decaying curve).
Primary Economic Use	Modeling fixed, linear, or proportional changes.	Modeling exponential growth, compound decay, or time-discounting.
Economic Variable Type	**Flows** and simple accumulation.	**Stocks**, compounded flows, and future values.
Growth/Decline Rate	Constant absolute amount.	Constant percentage rate (compound growth).
Case Study: Investment	**Simple Interest Accumulation:** Total savings, $I_t = P_0(1+rt)$. Each year, interest earned is the same absolute amount ($d = P_0r$).	**Compound Interest/Growth:** Future value, $FV_t = PV_0(1+r)^t$. Each year, interest earned increases as it compounds on the previous interest ($r = 1+i$).

Progressive Structure	Arithmetic Progression (AP)	Geometric Progression (GP)
Case Study: Multiplier	**Proportional Ripple Effects (Simultaneous):** An investment (I) that causes other variables (like C) to change *synchronously* with no time lag, creating a simultaneous multiplier.	**Sequences of Spending Ripples (Iterative):** The Keynesian Multiplier. Total impact $\Delta Y = \Delta G + c\Delta G + c^2\Delta G + ...$ is an infinite GP sum: $\Delta Y = \frac{\Delta G}{1-c}$.
Case Study: Finance	**Straight-Line Depreciation:** A fixed asset loses a constant absolute value each year ($d < 0$).	**Present Value of Annuities:** The sum of a series of fixed future payments discounted to today: $PV = \sum \frac{C}{(1+i)^t}$. An infinite, decaying GP sum.
Case Study: Microeconomics	**Linear Production/Supply:** Output increases by a fixed amount (d) for every unit of input. Fixed marginal product.	**Returns to Scale/Diminishing Returns:** If outputs (Q) are a constant factor (r) of inputs (K, L). For diminishing returns, $0 < r < 1$ for additions to input.

]

Box 2.2: Key differences between AP and GP

Feature | Arithmetic progression | Geometric progression
Rule | Add constant difference | Multiply by constant ratio
Growth | Linear | Exponential
Economic use | Wage increments | Compound interest:

2.2.4 Economic applications of APs and GPs

Both APs and GPs have practical applications in economics:

- **APs**: Represent steady savings deposits, fixed wage increments, and regular production increases.
- **GPs**: Model compound interest, investment growth, and population expansion.

Worked Example:

- AP: Saving ₦1,000 monthly for 12 months results in a finite series with a total of ₦12,000.
- GP: Investing ₦1,000 at 10% annual compound interest grows according to a GP, with each year's value multiplied by 1.1.

Fill in the blank: Compound interest is best represented by a __________ progression.

[

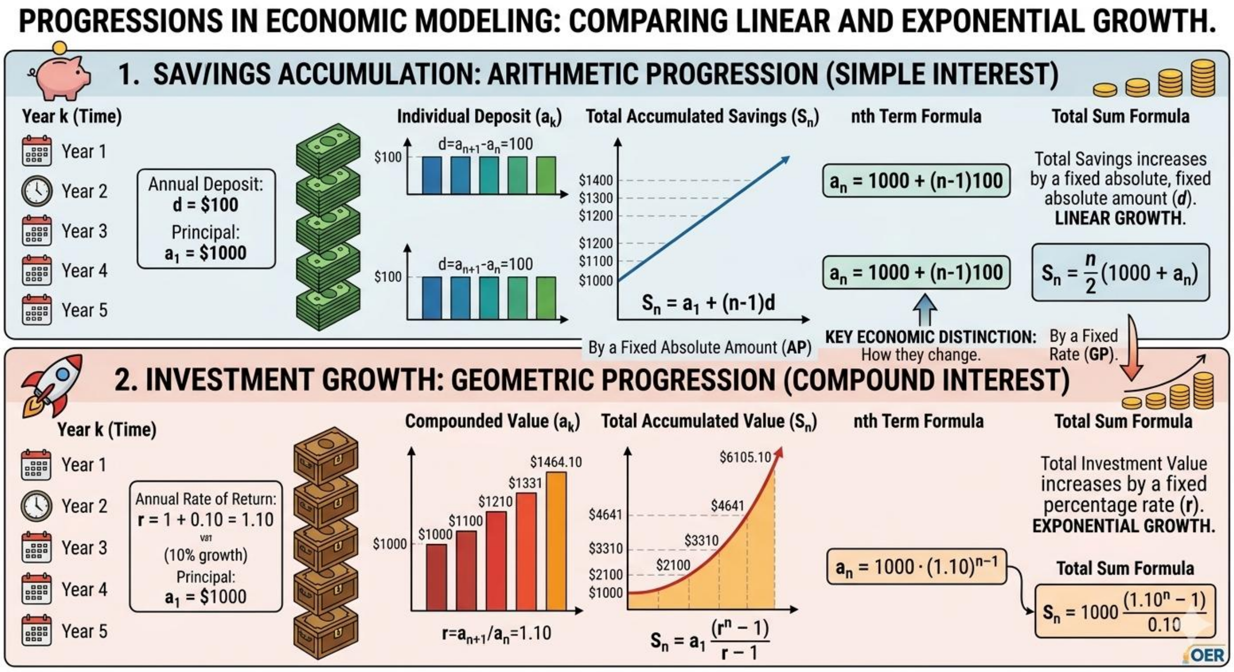

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Diagram showing savings accumulation as an AP and investment growth as a GP.

Figure 2.3: Economic applications of APs and GPs

Pilot questions 2.2

1. Define an arithmetic progression and give one example.
2. State the formula for the nth term of an AP.
3. Define a geometric progression and give one example.
4. State the formula for the sum of the first n terms of a GP.
5. Explain one economic application of APs and one of GPs.

2.3 Convergence and Divergence of Series

2.3.1 Concept of convergence

A **convergent series** is one in which the sum of its terms approaches a finite value as more terms are added. For example, the series

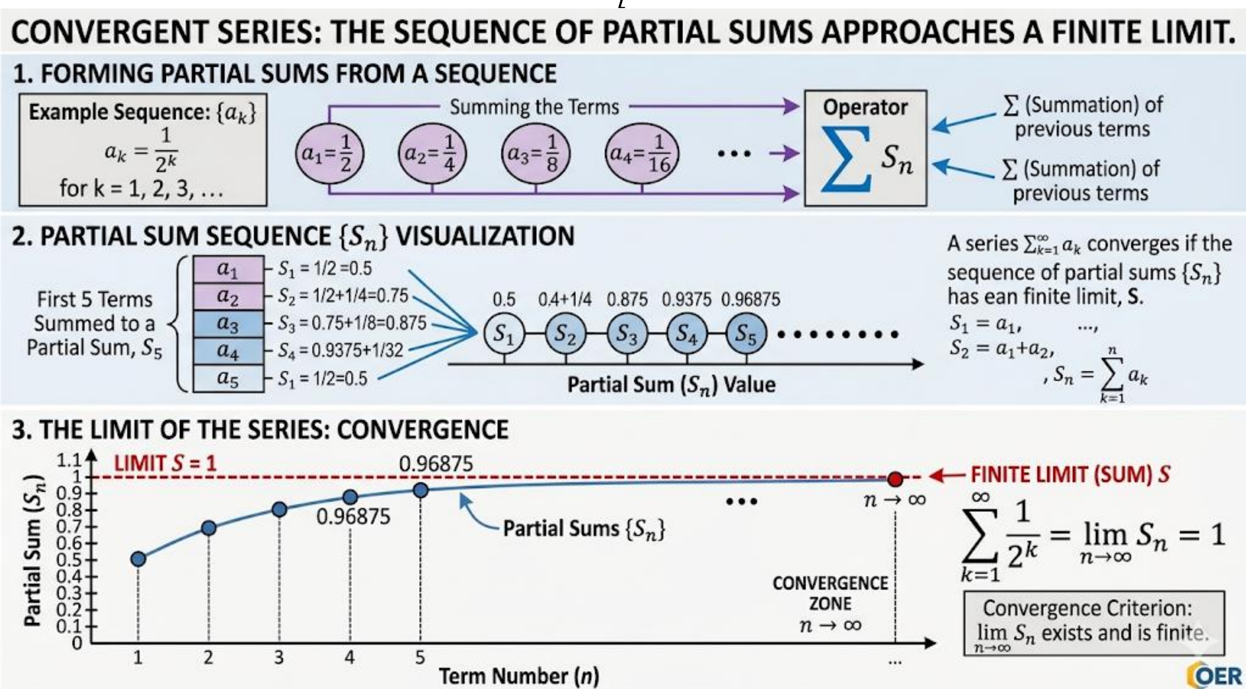
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges to 2. This means that although the series has infinitely many terms, the total sum stabilises at a particular value.

In economics, convergence is important when modelling scenarios where growth or accumulation eventually levels off, such as depreciation of assets or diminishing returns in production.

Fill in the blank: A convergent series approaches a __________ value as more terms are added.

[



]

Diagram showing a convergent series approaching a finite limit.

Figure 2.4: Convergence of a series

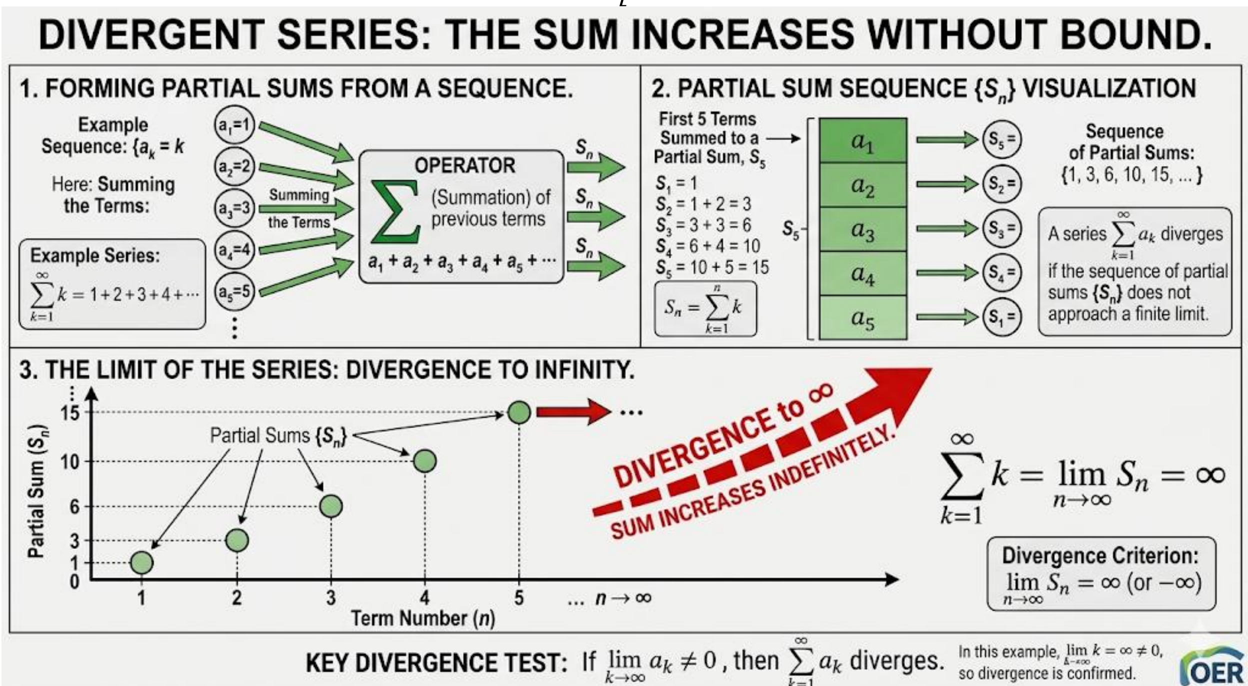
2.3.2 Concept of divergence

A **divergent series** is one in which the sum of its terms grows without bound as more terms are added. For example, the series
[1 + 2 + 3 + 4 + \cdots]
diverges because the sum increases indefinitely.

In economics, divergence may represent unsustainable growth, such as debt accumulation without repayment, or inflation spiralling upwards.

Fill in the blank: A divergent series grows __________ as more terms are added.

[



]

Diagram showing a divergent series increasing indefinitely.

Figure 2.5: Divergence of a series

2.3.3 Basic tests for convergence and divergence

There are several simple ways to test whether a series converges or diverges:

- **Ratio test**: If the ratio of successive terms tends to a value less than 1, the series converges.
- **Comparison test**: If a series has terms smaller than those of a known convergent series, it also converges.
- **Limit test**: If the terms of a series do not approach zero, the series diverges.

Worked Example: Consider the series (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots). Using the ratio test, the ratio of successive terms is (\frac{1}{2}), which is less than 1, so the series converges.

Fill in the blank: If the terms of a series do not approach zero, the series __________.

[

Test for Series (∑an)	Statement of the Test	Result for Convergence / Divergence	Common Examples & Use
Divergence Test (nth Term Test)	Calculate the limit of the individual terms: $\lim_{n \to \infty} a_n = L$	If $L \neq 0$ or the limit DNE, the series **diverges**. If $L = 0$, the test is **inconclusive**.	**Divergence:** $\sum n$, $\sum (-1)^n$, $\sum \frac{2n}{3n+1}$. Used first to quickly spot obvious divergence. *Warning: $\sum \frac{1}{n}$ has limit 0 but diverges.*
Geometric Series Test ($\sum a \cdot r^n$)	A series with a constant ratio 'r': $\sum a \cdot r^n = a + ar + ar^2 + \dots$	**Converges** if $	r
p-Series Test ($\sum \frac{1}{n^p}$)	A series of the form: $\sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$	**Converges** if $p > 1$. **Diverges** if $p \leq 1$.	**Convergence:** $\sum \frac{1}{n^2}$ (Basel problem, sum is $\pi^2/6$). **Divergence:** $\sum \frac{1}{n}$ (Harmonic series, $p=1$), $\sum \frac{1}{\sqrt{n}}$ ($p=0.5$).
Integral Test	If $f(n) = a_n$ is positive, continuous, and decreasing, compare the series sum to the improper integral $\int_1^\infty f(x)dx$.	Both the series and the integral **either converge or diverge together**.	Used when the terms can be integrated, such as $\sum \frac{1}{n^2+1}$. (integral is $\arctan(x)$, which converges).
Alternating Series Test ($\sum (-1)^n \cdot b_n$)	If $b_n \geq 0$, decreasing, and $\lim_{n \to \infty} b_n = 0$.	The series **converges**.	**Convergence:** $\sum \frac{(-1)^n}{n}$ (Alternating Harmonic series). *Note: The regular Harmonic series diverges.*

Test for Series (∑an)	Statement of the Test	Result for Convergence / Divergence	Common Examples & Use
Direct Comparison Test (D.C.T.)	Given series $\sum a_n$, and a known series $\sum c_n$ (converges) or $\sum d_n$ (diverges), with $0 \leq a_n \leq c_n$ or $0 \leq d_n \leq a_n$.	If $0 \leq a_n \leq c_n$, then $\sum a_n$ **converges**. If $0 \leq d_n \leq a_n$, then $\sum a_n$ **diverges**.	**Convergence:** $\sum \frac{1}{n^2+1}$ (since $\frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges). **Divergence:** $\sum \frac{n}{n^2-1}$ (since $\frac{n}{n^2-1} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges).

]

Box 2.3: Basic tests for convergence and divergence

Test | Description | Example

Ratio test | Successive ratio < 1 → converges | (\frac{1}{2} + \frac{1}{4} + \cdots)

Comparison test | Smaller than convergent series → converges | Comparing with (\frac{1}{n^2})

Limit test | Terms not → 0 → diverges | (1 + 2 + 3 + \cdots)

2.3.4 Economic implications of convergence and divergence

Convergence and divergence have direct economic interpretations:

- **Convergence**: Represents stabilisation, such as savings reaching a maximum, depreciation levelling off, or investment returns stabilising.
- **Divergence**: Represents unsustainable growth, such as debt increasing without repayment, inflation rising uncontrollably, or unchecked population growth.

Worked Example: A firm invests in machinery that depreciates over time. The depreciation can be modelled as a convergent series, since the value approaches zero but never becomes negative. Conversely, if a government borrows continuously without repayment, the debt accumulation can be modelled as a divergent series.

Fill in the blank: Continuous borrowing without repayment can be modelled as a __________ series.

[

Test for Series (∑an)	Statement of the Test	Result for Convergence / Divergence	Common Examples & Use
Divergence Test (nth Term Test)	Calculate the limit of the individual terms: $\lim_{n \to \infty} a_n = L$	If $L \neq 0$ or the limit DNE, the series **diverges**. If $L = 0$, the test is **inconclusive**.	**Divergence:** $\sum n$, $\sum (-1)^n$, $\sum \frac{2n}{3n+1}$. Used first to quickly spot obvious divergence. *Warning: $\sum \frac{1}{n}$ has limit 0 but diverges.*
Geometric Series Test ($\sum a \cdot r^n$)	A series with a constant ratio 'r': $\sum a \cdot r^n = a + ar + ar^2 + \dots$	**Converges** if $	r
p-Series Test ($\sum \frac{1}{n^p}$)	A series of the form: $\sum_{n=1}^\infty \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$	**Converges** if $p > 1$. **Diverges** if $p \leq 1$.	**Convergence:** $\sum \frac{1}{n^2}$ (Basel problem, sum is $\pi^2/6$). **Divergence:** $\sum \frac{1}{n}$ (Harmonic series, $p=1$), $\sum \frac{1}{\sqrt{n}}$ ($p=0.5$).
Integral Test	If $f(n) = a_n$ is positive, continuous, and decreasing, compare the series sum to the improper integral $\int_1^\infty f(x)dx$.	Both the series and the integral **either converge or diverge together**.	Used when the terms can be integrated, such as $\sum \frac{1}{n^2+1}$. (integral is $\arctan(x)$, which converges).
Alternating Series Test ($\sum (-1)^n \cdot b_n$)	If $b_n \geq 0$, decreasing, and $\lim_{n \to \infty} b_n = 0$.	The series **converges**.	**Convergence:** $\sum \frac{(-1)^n}{n}$ (Alternating Harmonic series). *Note: The regular Harmonic series diverges.*

Test for Series (∑an)	Statement of the Test	Result for Convergence / Divergence	Common Examples & Use
Direct Comparison Test (D.C.T.)	Given series $\sum a_n$, and a known series $\sum c_n$ (converges) or $\sum d_n$ (diverges), with $0 \leq a_n \leq c_n$ or $0 \leq d_n \leq a_n$.	If $0 \leq a_n \leq c_n$, then $\sum a_n$ **converges**. If $0 \leq d_n \leq a_n$, then $\sum a_n$ **diverges**.	**Convergence:** $\sum \frac{1}{n^2+1}$ (since $\frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges). **Divergence:** $\sum \frac{n}{n^2-1}$ (since $\frac{n}{n^2-1} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges).

]

Table 2.4: Economic implications of convergence and divergence

Type | Mathematical behaviour | Economic interpretation

Convergence | Sum approaches finite value | Stabilisation (e.g., depreciation, savings cap)

Divergence | Sum grows without bound | Unsustainable growth (e.g., debt, inflation)

Pilot questions 2.3

1. Define convergence in the context of series.
2. Give one example of a divergent series.
3. State the ratio test for convergence.
4. What happens if the terms of a series do not approach zero?
5. Explain one economic implication of a convergent series and one of a divergent series.

2.4 Applications of Sequences and Series in Economics

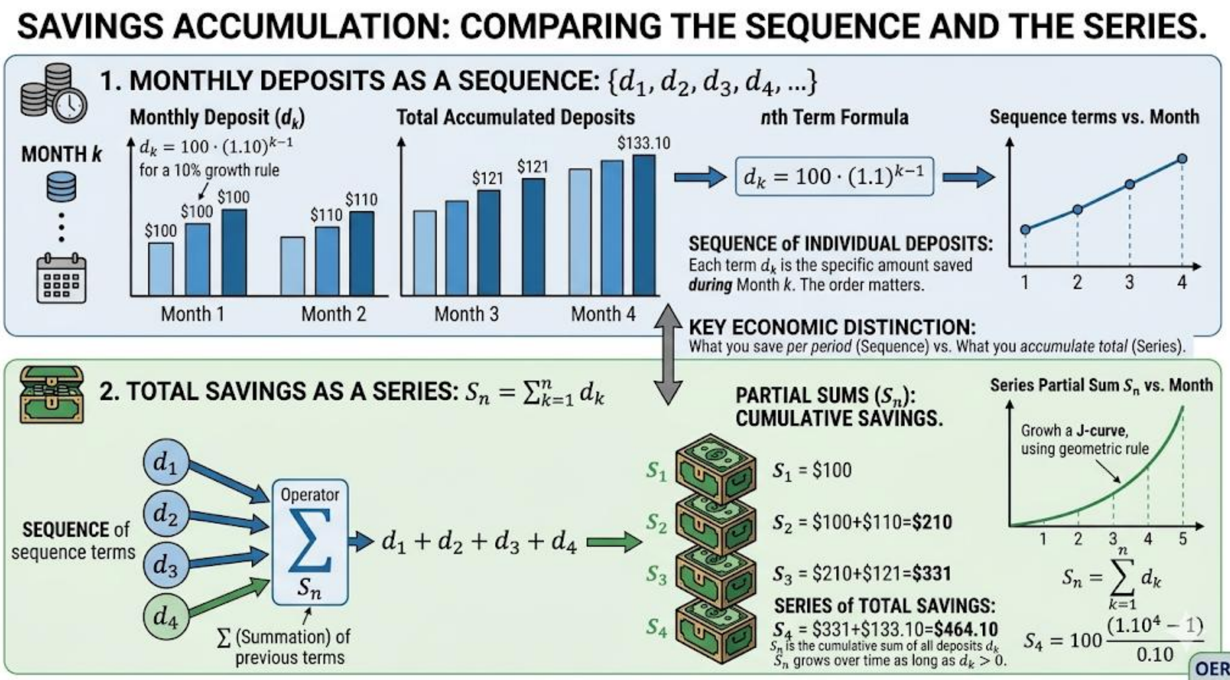
2.4.1 Savings accumulation

Savings plans often involve depositing a fixed amount regularly. This forms an **arithmetic progression (AP)** if deposits are constant, and a **series** when these deposits are summed over time. For example, depositing ₦5,000 monthly for 12 months creates a sequence of equal terms, and the total savings is the sum of this finite series.

Worked Example: A worker saves ₦10,000 monthly for one year. The sequence is 10,000, 10,000, … (12 terms). The series is the sum, giving ₦120,000.

Fill in the blank: Regular deposits of equal amounts form an __________ progression.

[



]

Diagram showing monthly deposits as a sequence and total savings as a series.

Figure 2.6: Savings accumulation represented by sequences and series

2.4.2 Investment returns

Investments that earn compound interest grow according to a **geometric progression (GP)**. Each term represents the value of the investment at a given time, and the series represents cumulative growth.

Worked Example: An investment of ₦100,000 earns 10% interest annually. The sequence of values is 100,000, 110,000, 121,000, … This is a GP with a common ratio of 1.1. The series shows the total value accumulated over several years.

Fill in the blank: Compound interest is modelled using a __________ progression.

[

End of Year (t)	Calculation Method (Geometric Progression n-th term formula)	Formula Applied	Total Investment Value
0	**Principal (Initial Term, a_1)**	$\$10,000 \cdot (1.08)^0$	**$\$10,000.00$**

End of Year (t)	Calculation Method (Geometric Progression n-th term formula)	Formula Applied	Total Investment Value
1	$Principal \cdot (r)^1$	$\$10,000 \cdot (1.08)^1$	**$\$10,800.00$**
2	$Principal \cdot (r)^2$	$\$10,000 \cdot (1.08)^2$	**$\$11,664.00$**
3	$Principal \cdot (r)^3$	$\$10,000 \cdot (1.08)^3$	**$\$12,597.12$**
4	$Principal \cdot (r)^4$	$\$10,000 \cdot (1.08)^4$	**$\$13,604.89$**
5	$Principal \cdot (r)^5$	$\$10,000 \cdot (1.08)^5$	**$\$14,693.28$**
6	$Principal \cdot (r)^6$	$\$10,000 \cdot (1.08)^6$	**$\$15,868.74$**
7	$Principal \cdot (r)^7$	$\$10,000 \cdot (1.08)^7$	**$\$17,138.24$**
8	$Principal \cdot (r)^8$	$\$10,000 \cdot (1.08)^8$	**$\$18,509.30$**
9	$Principal \cdot (r)^9$	$\$10,000 \cdot (1.08)^9$	**$\$19,990.05$**
10	$Principal \cdot (r)^{10}$	$\$10,000 \cdot (1.08)^{10}$	**$\$21,589.25$**

]

Table 2.5: Investment returns modelled by GP

Year | Value | Formula

1 | 100,000 | (a = 100,000)

2 | 110,000 | (100,000 \cdot 1.1)

3 | 121,000 | (100,000 \cdot 1.1^2)

2.4.3 Debt repayment schedules

Debt repayment often involves fixed instalments over a period, forming a **finite series**. Each payment is a term in the sequence, and the total repayment is the sum of the series.

Worked Example: A loan of ₦600,000 is repaid in 12 monthly instalments of ₦50,000. The sequence is 50,000, 50,000, … (12 terms). The series is the sum, giving ₦600,000.

In more complex cases, repayment schedules may include interest, which can be modelled using a combination of arithmetic and geometric progressions.

Fill in the blank: Fixed instalments over a set period form a __________ series.

[

Period (t)	Total Payment (Pt)	Interest Payment (It) (Decreasing)	Principal Repayment (PRt) (Increasing)	Remaining Balance
Year 1	$\$2,200$	$\$800$	$\$1,400$	$\$8,600$
Year 2	$\$2,200$	$\$688$	$\$1,512$	$\$7,088$
Year 3	$\$2,200$	$\$567$	$\$1,633$	$\$5,455$
Year 4	$\$2,200$	$\$436$	$\$1,764$	$\$3,691$
Year 5	$\$2,200$	$\$295$	$\$1,905$	$\$1,786$
Total Sum	**$\$11,000$**	**$\$2,786$**	**$\$8,214$**	**-**

]

Box 2.4: Debt repayment and series modelling

Scenario | Mathematical model | Economic interpretation

Fixed instalments | Finite series | Loan repayment without interest

Instalments with interest | Combination of AP and GP | Loan repayment with interest charges

2.4.4 Growth modelling in economics

Sequences and series can also represent growth in production, consumption, or population.

- **APs**: Model steady, linear growth such as annual wage increments.
- **GPs**: Model exponential growth such as population expansion or technological adoption.
- **Convergent series**: Represent stabilisation, such as diminishing returns.
- **Divergent series**: Represent unsustainable growth, such as unchecked inflation.

Worked Example: A population doubles every 20 years. This can be modelled as a GP with a common ratio of 2. The series shows cumulative population growth across generations.

Fill in the blank: Population expansion is best modelled using a __________ progression.

[

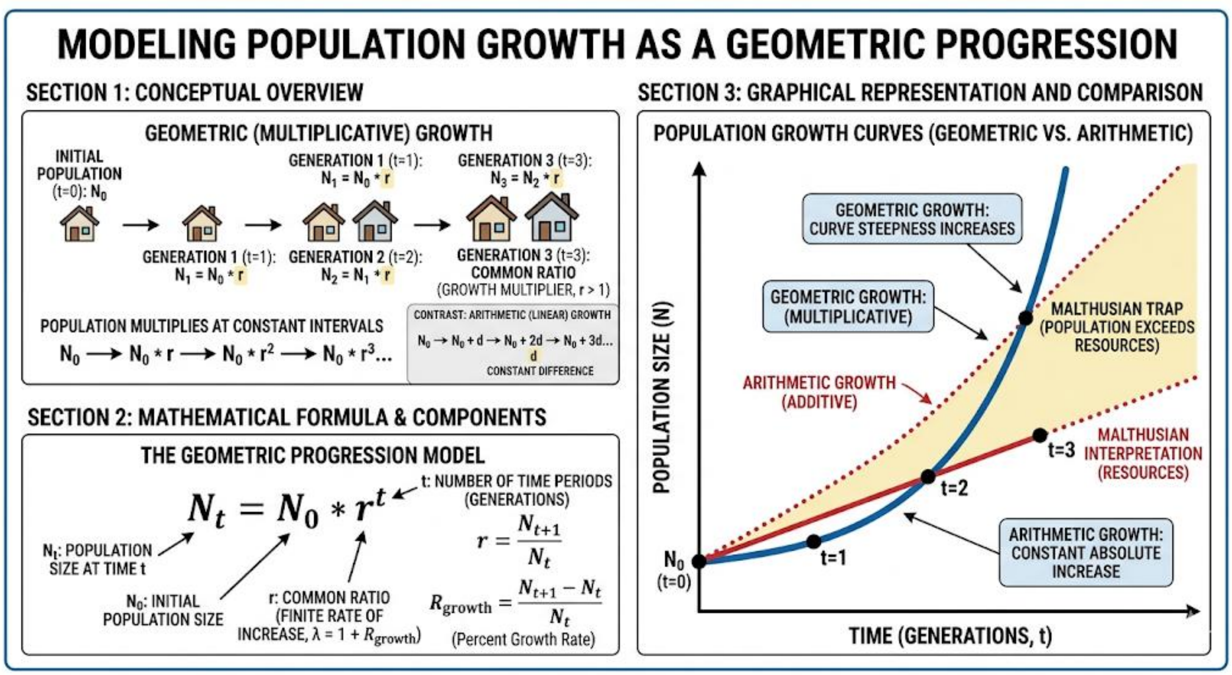


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Diagram showing population growth as a geometric progression.

Figure 2.7: Population growth modelled by GP

Pilot questions 2.4

1. Explain how savings accumulation can be represented by sequences and series.
2. Give one example of investment returns modelled by a geometric progression.
3. How are fixed debt repayments represented mathematically?
4. State one economic scenario where a convergent series is useful.
5. State one economic scenario where a divergent series may occur.

Series Summary

This Series has focused on sequences and series, showing how these mathematical tools can be applied to model savings, investment returns, debt repayment, and growth in economics. We began by clarifying the fundamentals of sequences and series, distinguishing between finite and infinite forms, and introducing the notation used to represent them. We then explored arithmetic and geometric progressions, highlighting their formulas and economic relevance. Following that, we examined convergence and divergence, learning how to test for these behaviours and interpret their implications. Finally, we applied these concepts to practical economic scenarios, including savings accumulation, compound interest, repayment schedules, and population growth.

By completing this Series, you should now be able to define and explain sequences and series, demonstrate how to calculate terms and sums of arithmetic and geometric progressions, analyse convergence and divergence, evaluate their economic implications, and design simple models that apply these concepts to real-world economic problems. In line with the Series Learning Outcomes, you are now equipped to use sequences and series confidently as tools for economic analysis.

Glossary of Terms

Arithmetic progression (AP): A sequence of numbers in which each term differs from the previous one by a constant difference.

Common difference: The fixed amount added to each term in an arithmetic progression to obtain the next term.

Common ratio: The fixed number multiplied by each term in a geometric progression to obtain the next term.

Convergent series: A series whose sum approaches a finite value as more terms are added.

Debt repayment schedule: A plan for paying back borrowed money in instalments, often represented as a finite series.

Divergent series: A series whose sum grows without bound as more terms are added.

Finite series: A series with a limited number of terms.

Geometric progression (GP): A sequence of numbers in which each term is obtained by multiplying the previous term by a constant ratio.

Infinite series: A series that continues indefinitely without end.

Investment returns: The gains or losses from an investment, often modelled using geometric progressions to represent compound interest.

Sequence: An ordered list of numbers that follow a specific rule.

Series: The sum of the terms of a sequence.

Sigma notation (Σ): A mathematical symbol used to represent the sum of terms in a series.

Savings accumulation: The process of building up savings over time, often represented by sequences and series.

Concept Check Questions [Series 2]

Objective Questions

1. Which of the following is a finite series?
   a) (2 + 4 + 6 + 8)
   b) (1 + \frac{1}{2} + \frac{1}{4} + \cdots)
   c) (3 + 6 + 12 + 24 + \cdots)
   d) (1 + 2 + 3 + 4 + \cdots)
   (Linked to SLO 2.1)

2. The nth term of an arithmetic progression is given by:
   a) (a_n = ar^{n-1})
   b) (a_n = a + (n-1)d)
   c) (S_n = \frac{n}{2}[2a + (n-1)d])
   d) (S_n = a \frac{1-r^n}{1-r})
   (Linked to SLO 2.3)

3. Which progression is most suitable for modelling compound interest?
   a) Arithmetic progression
   b) Geometric progression
   c) Finite series
   d) Divergent series
   (Linked to SLO 2.4)

4. A series whose sum grows without bound is called:
   a) Convergent
   b) Divergent
   c) Finite
   d) Infinite
   (Linked to SLO 2.5)

5. The symbol Σ in mathematics represents:
   a) Multiplication
   b) Division
   c) Summation

d) Subtraction
(Linked to SLO 2.2)

Short-Answer Questions

1. Define a sequence and a series. (Linked to SLO 2.1)
2. State the formula for the sum of the first n terms of an arithmetic progression. (Linked to SLO 2.3)
3. Write the nth term of a geometric progression. (Linked to SLO 2.4)
4. Explain what it means for a series to converge. (Linked to SLO 2.5)
5. Give one economic example of a divergent series. (Linked to SLO 2.6)

Long-Answer Questions

1. Analyse the differences between arithmetic and geometric progressions, giving one economic application of each. (Linked to SLO 2.3, SLO 2.4)
   - **Basic response**: Describe the rules for AP and GP, provide formulas, and give simple examples.
   - **Value-added response**: Extend by applying AP to wage increments and GP to compound interest, explaining why each model is appropriate in its economic context.
2. Evaluate the role of convergence and divergence in economic modelling, with examples. (Linked to SLO 2.5, SLO 2.6)
   - **Basic response**: Define convergence and divergence, explain their mathematical behaviour, and give one example of each.
   - **Value-added response**: Apply convergence to depreciation of assets and divergence to debt accumulation, discussing the implications for long-term economic sustainability.
3. Design a simple economic model using sequences and series to represent savings accumulation and investment returns. (Linked to SLO 2.7)
   - **Basic response**: Outline how savings can be represented by an AP and investment returns by a GP.
   - **Value-added response**: Provide a worked example showing monthly deposits as an AP and compound interest growth as a GP, then combine them to illustrate total wealth accumulation over time.

Answers to Concept Check Questions [Series 2]

Objective Questions

1. a) (2 + 4 + 6 + 8)
2. b) ($a_n = a + (n-1)d$)
3. b) Geometric progression
4. b) Divergent
5. c) Summation

Short-Answer Questions

1. A sequence is an ordered list of numbers following a rule, while a series is the sum of those numbers.
2. ($S_n = \frac{n}{2}[2a + (n-1)d]$).
3. ($a_n = ar^{n-1}$).
4. A convergent series approaches a finite value as more terms are added.
5. Continuous borrowing without repayment, leading to debt accumulation.

Long-Answer Questions

1. **Basic response**: APs add a constant difference, while GPs multiply by a constant ratio. AP example: wages increasing by a fixed amount yearly. GP example: investment doubling annually.
   Value-added response: APs model linear growth such as wage increments, while GPs model exponential growth such as compound interest. AP is suitable for predictable, steady increases, while GP captures compounding effects in finance.

2. **Basic response**: Convergence means the sum stabilises, divergence means it grows indefinitely. Example: convergent series in depreciation, divergent series in inflation.
   Value-added response: Convergence models stabilisation in economics, such as diminishing returns or depreciation. Divergence models unsustainable growth, such as unchecked debt or inflation. These behaviours help economists evaluate long-term viability of policies.

3. **Basic response**: Savings accumulation can be represented by an AP, while investment returns can be represented by a GP.
   Value-added response: Example: Saving ₦10,000 monthly for 12 months forms an AP with a total of ₦120,000. Investing ₦100,000 at 10% annual compound interest forms a GP with values 100,000, 110,000, 121,000, etc. Combining both shows how savings and investments together build wealth.

Answers to Pilot Questions [Series 2]

Part 2.1: Fundamentals of Sequences and Series

1. A sequence is an ordered list of numbers, and a series is the sum of those numbers.
2. Example of a finite series: ($2 + 4 + 6 + 8$). Example of an infinite series: ($1 + \frac{1}{2} + \frac{1}{4} + \cdots$).
3. (a_n) represents the nth term of a sequence.
4. ($\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$).
5. Infinite series are useful in economics for modelling processes such as compound interest or depreciation.

Part 2.2: Arithmetic and Geometric Progressions

1. An arithmetic progression is a sequence where each term differs from the previous one by a constant difference. Example: 5, 10, 15, 20.

2. The formula for the nth term of an AP is ($a_n = a + (n-1)d$).
3. A geometric progression is a sequence where each term is obtained by multiplying the previous term by a constant ratio. Example: 3, 6, 12, 24.
4. The formula for the sum of the first n terms of a GP is ($S_n = a \frac{1-r^n}{1-r}$), if ($r \neq 1$).
5. APs are applied to steady wage increments, while GPs are applied to compound interest.

Part 2.3: Convergence and Divergence of Series

1. Convergence means the sum of a series approaches a finite value as more terms are added.
2. Example of a divergent series: ($1 + 2 + 3 + 4 + \cdots$).
3. The ratio test states that if the ratio of successive terms tends to a value less than 1, the series converges.
4. If the terms of a series do not approach zero, the series diverges.
5. A convergent series can represent depreciation of assets, while a divergent series can represent unsustainable debt accumulation.

Part 2.4: Applications of Sequences and Series in Economics

1. Savings accumulation can be represented by a sequence of deposits and a series showing the total savings.
2. Example: An investment of ₦100,000 at 10% annual compound interest forms a GP with values 100,000, 110,000, 121,000, etc.
3. Fixed debt repayments are represented mathematically as a finite series.
4. A convergent series is useful in modelling diminishing returns or depreciation.
5. A divergent series may occur in scenarios such as unchecked inflation or continuous borrowing without repayment.

References and Attributions [Series 2]

Figures

- *Figure 2.1: Relationship between sequences and series*
  AI-generated suggestion: “Generate a diagram showing a sequence of numbers and the sum forming a series.”
  Suggested OER search string: “sequence and series diagram OER mathematics.”
- *Figure 2.2: Savings accumulation represented by sequences and series*
  AI-generated suggestion: “Generate a diagram showing savings accumulation as a sequence and total savings as a series.”
  Suggested OER search string: “savings accumulation sequence series diagram OER economics.”
- *Figure 2.3: Economic applications of APs and GPs*
  AI-generated suggestion: “Generate a diagram showing savings accumulation as an arithmetic progression and investment growth as a geometric progression.”
  Suggested OER search string: “economic applications arithmetic geometric progression diagram OER economics.”
- *Figure 2.4: Convergence of a series*
  AI-generated suggestion: “Generate a diagram showing a convergent series approaching

a finite limit."
Suggested OER search string: "convergent series diagram OER mathematics."

- *Figure 2.5: Divergence of a series*
AI-generated suggestion: "Generate a diagram showing a divergent series increasing indefinitely."
Suggested OER search string: "divergent series diagram OER mathematics."
- *Figure 2.6: Savings accumulation represented by sequences and series*
AI-generated suggestion: "Generate a diagram showing monthly deposits as a sequence and total savings as a series."
Suggested OER search string: "savings accumulation sequence series diagram OER economics."
- *Figure 2.7: Population growth modelled by GP*
AI-generated suggestion: "Generate a diagram showing population growth modelled as a geometric progression."
Suggested OER search string: "population growth geometric progression diagram OER economics."

Tables

- *Table 2.1: Examples of finite and infinite series*
AI-generated suggestion: "Create a table showing examples of finite and infinite series with descriptions."
Suggested OER search string: "finite infinite series examples OER mathematics."
- *Table 2.2: Examples of arithmetic progressions*
AI-generated suggestion: "Create a table showing examples of arithmetic progressions with common differences and nth term formulas."
Suggested OER search string: "arithmetic progression examples nth term formula OER mathematics."
- *Table 2.3: Examples of geometric progressions*
AI-generated suggestion: "Create a table showing examples of geometric progressions with common ratios and nth term formulas."
Suggested OER search string: "geometric progression examples nth term formula OER mathematics."
- *Table 2.4: Economic implications of convergence and divergence*
AI-generated suggestion: "Create a table showing economic implications of convergence and divergence with examples."
Suggested OER search string: "economic implications convergence divergence series OER economics."
- *Table 2.5: Investment returns modelled by GP*
AI-generated suggestion: "Create a table showing investment returns modelled by a geometric progression."
Suggested OER search string: "compound interest geometric progression example OER economics."

Boxes

- *Box 2.1: Key notation in sequences and series*
  AI-generated suggestion: "Create a text box summarising key notation in sequences and series."
  Suggested OER search string: "notation sequences series sigma OER mathematics."
- *Box 2.2: Key differences between AP and GP*
  AI-generated suggestion: "Create a text box comparing arithmetic and geometric progressions with economic applications."
  Suggested OER search string: "difference between arithmetic and geometric progression OER mathematics economics."
- *Box 2.3: Basic tests for convergence and divergence*
  AI-generated suggestion: "Create a text box summarising basic tests for convergence and divergence with examples."
  Suggested OER search string: "tests for convergence divergence series OER mathematics."
- *Box 2.4: Debt repayment and series modelling*
  AI-generated suggestion: "Create a text box summarising debt repayment schedules using series."
  Suggested OER search string: "debt repayment schedules series OER economics."

General References

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- Khan Academy. (n.d.). *Sequences, Series, and Convergence*. Retrieved from https://www.khanacademy.org

Course code: ECO 104

Course title: Introductory Mathematics II

Course credit load: 2 Units

Series 1: Derivatives of Trigonometric Functions and Expansions

Series 2: Sequences and Series in Economic Applications

Series 3: Partial and Total Derivatives with Optimisation

Series 4: Linear and Matrix Algebra in Economics

Series 5: Mathematical Relationships and Graphical Analysis of Economic Concepts

Suggested Outline

Part 3.1: Fundamentals of partial derivatives

- Definition of partial derivatives
- Notation and interpretation
- Worked examples with two-variable functions

Part 3.2: Total derivatives and their applications

- Definition of total derivatives
- Relationship between partial and total derivatives
- Economic applications such as cost and production functions

Part 3.3: Optimisation with constraints

- Concept of optimisation in economics
- Unconstrained optimisation using first and second-order conditions
- Constrained optimisation using Lagrange multipliers
- Case studies in utility maximisation and cost minimisation

Part 3.4: Economic applications of optimisation

- Optimisation in production functions
- Utility maximisation in consumer theory
- Profit maximisation in firm behaviour

- Policy implications of optimisation models

Introduction

In the last Series, we examined sequences and series, learning how they can be applied to model savings, investment returns, debt repayment, and growth in economics. Building on that foundation, this Series introduces you to **partial and total derivatives with optimisation**, which are essential tools for analysing functions of several variables and for determining the best possible outcomes in economic decision-making. These concepts allow economists to study how changes in one variable affect another and to identify conditions under which resources are used most efficiently.

The aim of this Series is to equip you with the ability to calculate partial and total derivatives, apply them to economic functions, and use optimisation techniques to solve both constrained and unconstrained problems in economics.

We shall commence this Series by exploring the fundamentals of partial derivatives, focusing on their definition, notation, and interpretation. Next, we will move on to total derivatives, examining their relationship with partial derivatives and their applications in economics. Following that, we will study optimisation with constraints, including first and second-order conditions and the use of Lagrange multipliers. Finally, we will conclude by applying optimisation techniques to economic scenarios such as production, utility, and profit maximisation, thereby rounding off the Series with practical relevance.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 3.1 **define** partial derivatives and explain their notation and interpretation,

SLO 3.2 **demonstrate** how to calculate partial derivatives for functions of two or more variables,

SLO 3.3 **define** total derivatives and explain their relationship with partial derivatives,

SLO 3.4 **apply** total derivatives to economic functions such as cost and production functions,

SLO 3.5 **analyse** optimisation problems using first and second-order conditions in unconstrained settings,

SLO 3.6 **evaluate** constrained optimisation problems using Lagrange multipliers,

SLO 3.7 **apply** optimisation techniques to economic scenarios such as production, utility, and profit maximisation, and

SLO 3.8 **design** simple optimisation models that illustrate efficient resource allocation in economics.

3.1 Fundamentals of Partial Derivatives

3.1.1 Definition of partial derivatives

A **partial derivative** is the derivative of a function of two or more variables with respect to one variable, while keeping the other variables constant. For example, if we have a function (f(x,y) = x^2y + 3y), the partial derivative with respect to (x) treats (y) as a constant.

This concept is important because many economic functions depend on multiple variables. For instance, a production function may depend on both labour and capital. By taking a partial derivative, we can measure how output changes when labour increases, while capital remains fixed.

Fill in the blank: A partial derivative measures the rate of change of a function with respect to one variable while keeping __________ constant.

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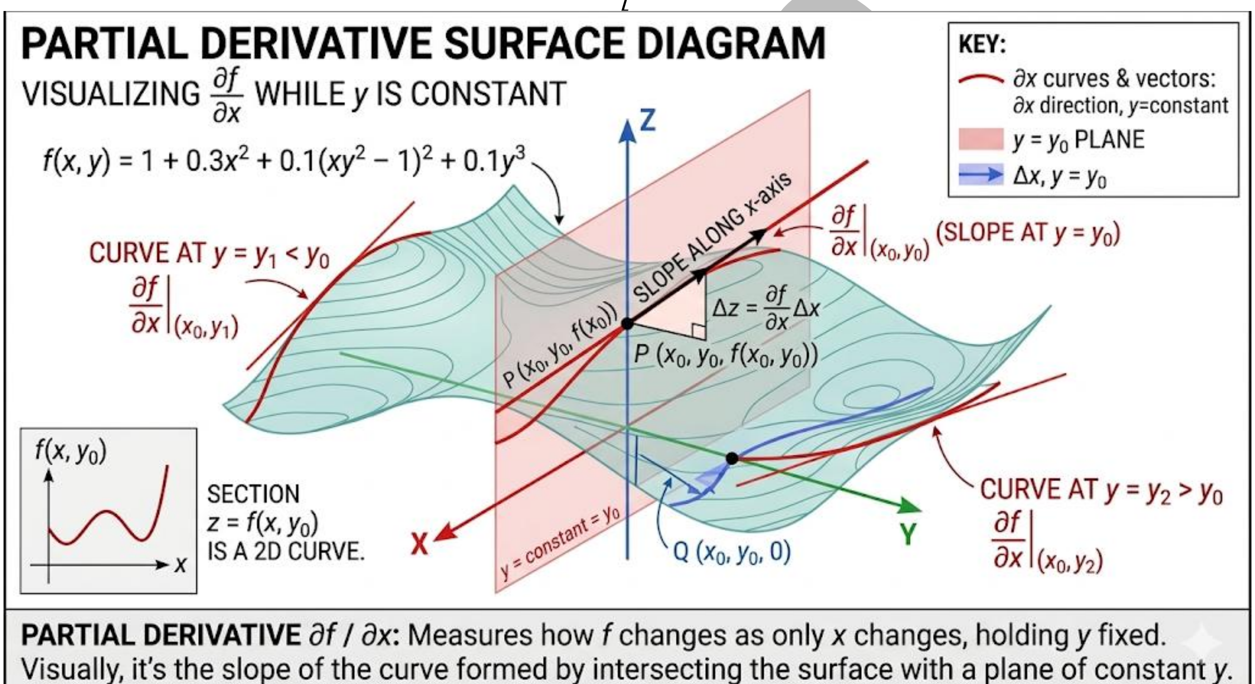

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Diagram showing a surface (z = f(x,y)) and the slope along the (x)-direction while holding (y) constant.

Figure 3.1: Visualising a partial derivative along one axis

3.1.2 Notation and interpretation

The notation for partial derivatives is:

- (\frac{\partial f}{\partial x}): partial derivative of (f) with respect to (x).
- (\frac{\partial f}{\partial y}): partial derivative of (f) with respect to (y).

The symbol ∂ is used instead of the usual (d) to indicate that the derivative is taken with respect to one variable, while others are held constant.

Interpretation: Partial derivatives show how sensitive a function is to changes in one variable. In economics, this can mean how output changes when labour increases, while capital is held constant.

Fill in the blank: The symbol ∂ is used to denote a __________ derivative.

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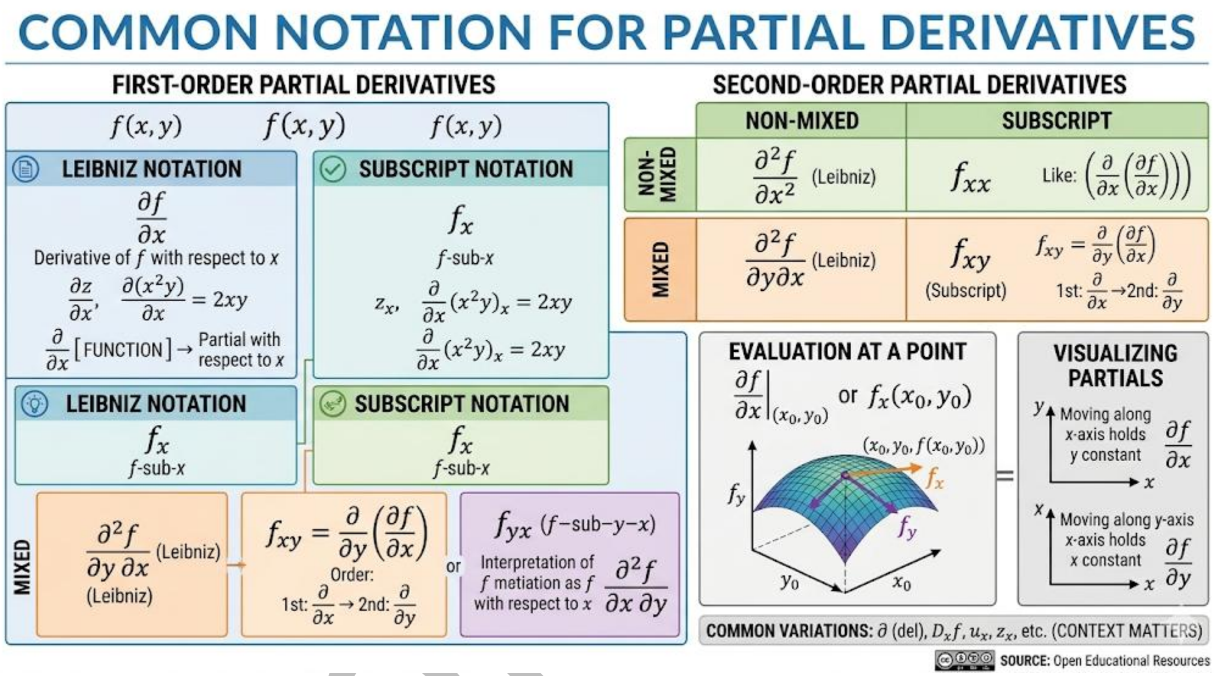

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Box 3.1: Common notation for partial derivatives

Notation | Meaning

(\frac{\partial f}{\partial x}) | Partial derivative of (f) with respect to (x)

(\frac{\partial f}{\partial y}) | Partial derivative of (f) with respect to (y)

3.1.3 Worked examples with two-variable functions

Let us consider a function (f(x,y) = x^2y + 3y).

- Partial derivative with respect to (x):
  [\frac{\partial f}{\partial x} = 2xy]
- Partial derivative with respect to (y):
  [\frac{\partial f}{\partial y} = x^2 + 3]

Another example: If (f(x,y) = xy + y^2), then

- (\frac{\partial f}{\partial x} = y)
- (\frac{\partial f}{\partial y} = x + 2y)

Economic Application: Suppose (f(x,y)) represents output depending on labour ((x)) and capital ((y)). The partial derivative with respect to labour shows how output changes when labour increases, holding capital constant.

Fill in the blank: If (f(x,y) = x^2y + 3y), then (\frac{\partial f}{\partial y} = __________).

[

EXAMPLES OF FUNCTIONS WITH THEIR PARTIAL DERIVATIVES

FUNCTION TYPE	FUNCTION: $f(x,y)$	PARTIAL DERIVATIVE W.R.T. x: $f_x = \frac{\partial f}{\partial x}$	PARTIAL DERIVATIVE W.R.T. y: $f_y = \frac{\partial f}{\partial y}$
Polynomial	$f(x,y) = x^2 + 3xy + y^3$	$2x + 3y$	$3x + 3y^2$
Exponential	$f(x,y) = e^{2x}\sin(y)$	$2e^{2x}\sin(y)$	$e^{2x}\cos(y)$
Logarithmic	$f(x,y) = \ln(x^2 + y^2)$	$\frac{2x}{x^2 + y^2}$	$\frac{2y}{x^2 + y^2}$
Product	$f(x,y) = x^3y^4$	$3x^2y^4$	$4x^3y^3$
Quotient	$f(x,y) = \frac{x+y}{x^2}$	$\frac{x^2 - 2x(x+y)}{x^4} = \frac{-x-2y}{x^3}$	$\frac{1}{x^2}$
Trigonometric	$f(x,y) = \cos(xy) + \sin(x)$	$-y\sin(xy) + \cos(x)$	$-x\sin(xy)$
Square Root (Chain Rule)	$f(x,y) = \sqrt{x^2 - y^2}$	$\frac{x}{\sqrt{x^2 - y^2}}$	$\frac{-y}{\sqrt{x^2 - y^2}}$

Green: Treat y as a constant, differentiate w.r.t. x. Blue: Treat x as a constant, differentiate w.r.t. y.

]

Table 3.1: Examples of partial derivatives of functions
Function | (\frac{\partial f}{\partial x}) | (\frac{\partial f}{\partial y})
(f(x,y) = x^2y + 3y) | (2xy) | (x^2 + 3)
(f(x,y) = xy + y^2) | (y) | (x + 2y)

3.1.4 Economic interpretation of partial derivatives

Partial derivatives are widely used in economics to analyse marginal changes. For example:

- In a **production function**, the partial derivative with respect to labour shows the **marginal product of labour** (the additional output from one more unit of labour, holding capital constant).
- In a **utility function**, the partial derivative with respect to a good shows the **marginal utility** of that good (the additional satisfaction from consuming one more unit, holding other goods constant).

- In a **cost function**, the partial derivative with respect to output shows the **marginal cost** (the additional cost of producing one more unit, holding other factors constant).

Worked Example: If a production function is (Q(L,K) = L^{0.5}K^{0.5}), then
[\frac{\partial Q}{\partial L} = 0.5L^{-0.5}K^{0.5}]
This represents the marginal product of labour.

Fill in the blank: In a production function, the partial derivative with respect to labour represents the __________ product of labour.

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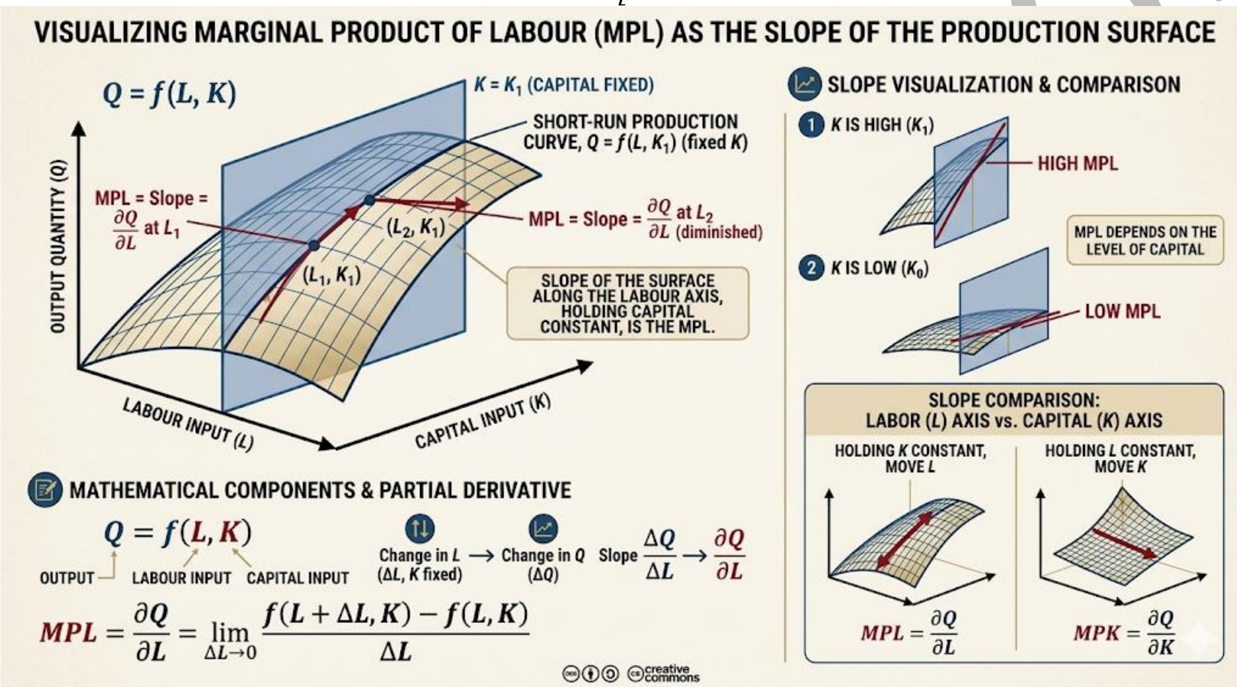

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Diagram showing a production function surface with slope along the labour axis.

Figure 3.2: Marginal product of labour as a partial derivative

Pilot questions 3.1

1. Define a partial derivative.
2. State the notation for the partial derivative of a function (f(x,y)) with respect to (x).
3. Calculate (\frac{\partial f}{\partial x}) if (f(x,y) = x^2y + 3y).
4. Explain the economic interpretation of a partial derivative in a production function.
5. What symbol is used to denote a partial derivative?

3.2 Total Derivatives and Their Applications

3.2.1 Definition of total derivatives

A **total derivative** measures how a function changes when all of its variables change simultaneously. Unlike a partial derivative, which focuses on one variable at a time, the total derivative accounts for the combined effect of changes in multiple variables.

For example, if (z = f(x,y)) and both (x) and (y) depend on another variable (t), the total derivative of (z) with respect to (t) is:
[\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}]

This formula shows how changes in (x) and (y) together affect (z).

Fill in the blank: A total derivative accounts for the combined effect of changes in __________ variables.

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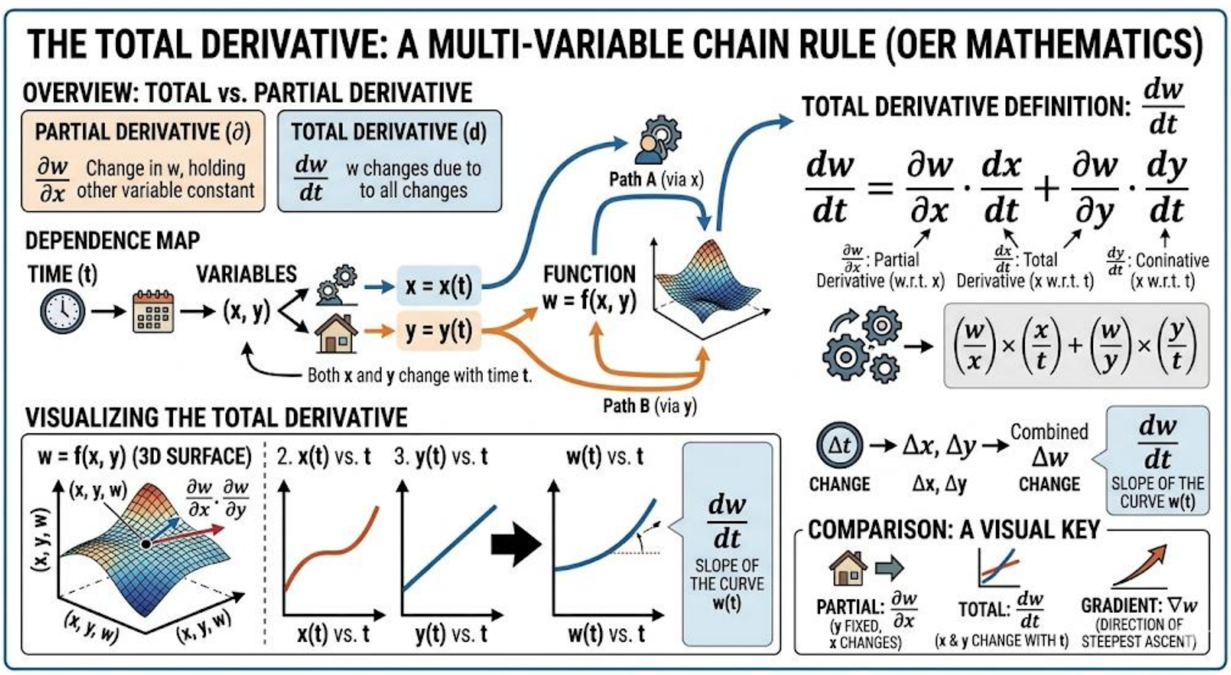

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Diagram showing a function (z = f(x,y)) with both (x) and (y) changing over time.

Figure 3.3: Visualising a total derivative with multiple variables changing

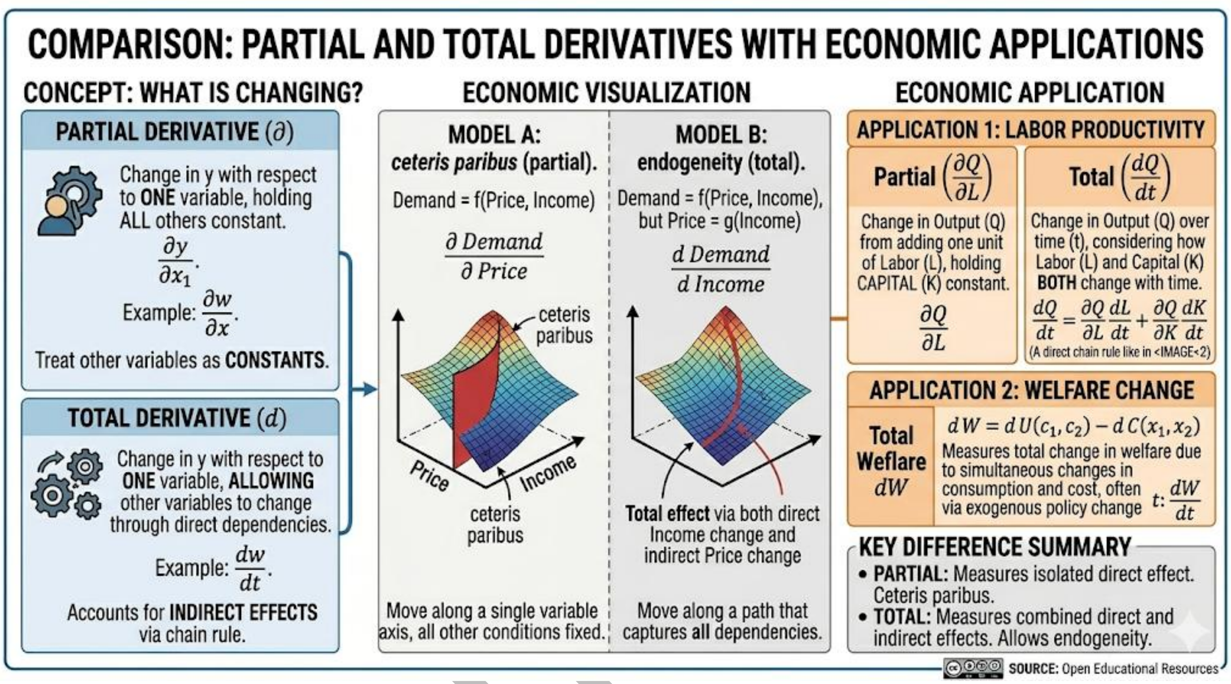
3.2.2 Relationship between partial and total derivatives

Partial derivatives measure the effect of changing one variable while holding others constant. Total derivatives extend this idea by combining the effects of all variables when they change together.

For example, in a production function (Q(L,K)), the partial derivative with respect to labour ((L)) shows the marginal product of labour. However, if both labour and capital change with time, the total derivative shows how output changes overall.

Fill in the blank: Partial derivatives hold other variables constant, while total derivatives allow __________ variables to change.

[



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Box 3.2: Comparing partial and total derivatives
Feature | Partial derivative | Total derivative
Focus | One variable at a time | All variables together
Notation | (\frac{\partial f}{\partial x}) | (\frac{df}{dt})
Economic use | Marginal product of one input | Overall change in output when all inputs vary

3.2.3 Economic applications of total derivatives

Total derivatives are especially useful in economics when analysing functions where multiple inputs change simultaneously. Examples include:

- **Production functions**: Measuring how output changes when both labour and capital vary.
- **Cost functions**: Determining how total cost changes when both input prices and quantities change.
- **Utility functions**: Analysing how overall utility changes when consumption of multiple goods varies together.

Worked Example: Suppose a production function is (Q(L,K) = L^{0.5}K^{0.5}), and both (L) and (K) depend on time (t). The total derivative of (Q) with respect to (t) is:
[\frac{dQ}{dt} = \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial K}\frac{dK}{dt}]
This shows how output changes when both labour and capital change over time.

Fill in the blank: In a cost function, the total derivative shows how total cost changes when both __________ and __________ vary.

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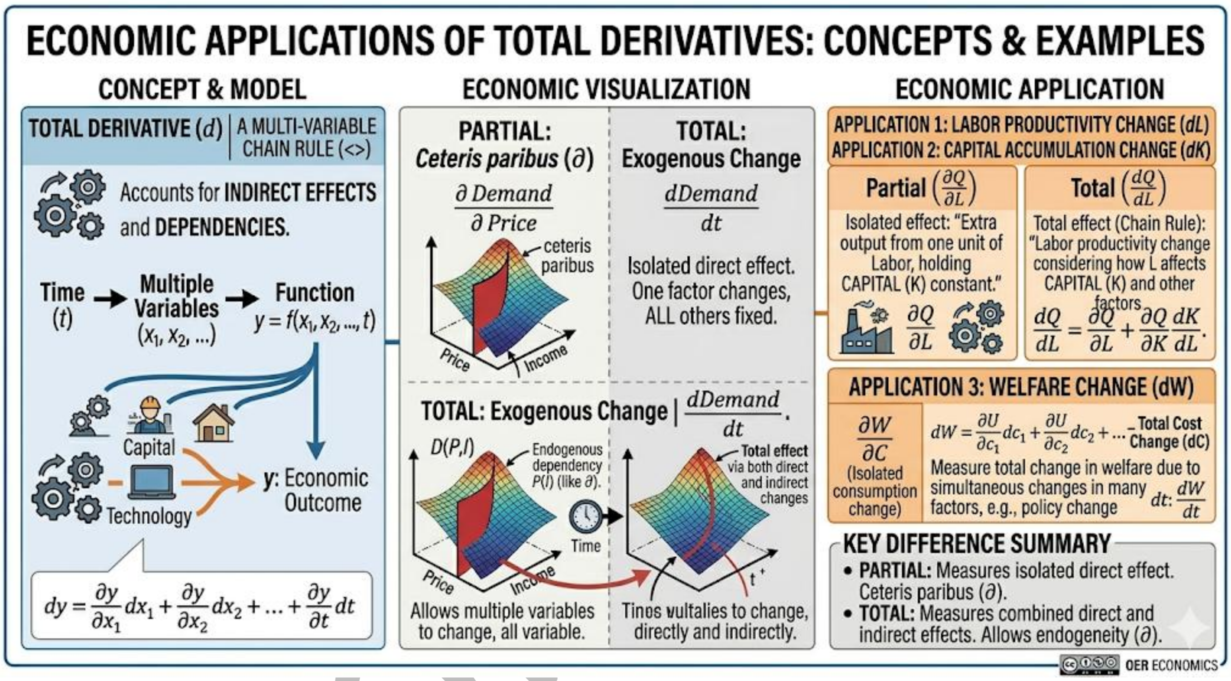

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Table 3.2: Economic applications of total derivatives

Function | Variables changing | Interpretation
Production function | Labour and capital | Overall change in output
Cost function | Input prices and quantities | Overall change in cost
Utility function | Consumption of goods | Overall change in satisfaction

Pilot questions 3.2

1. Define a total derivative.
2. State the formula for the total derivative of (z = f(x,y)) when both (x) and (y) depend on (t).
3. Explain the difference between partial and total derivatives.
4. Give one economic application of total derivatives.
5. In a production function, what does the total derivative represent when both labour and capital change?

3.3 Optimisation With Constraints

3.3.1 Concept of optimisation in economics

Optimisation refers to the process of finding the maximum or minimum value of a function, subject to certain conditions. In economics, optimisation is central to decision-making: firms aim to maximise profit, consumers aim to maximise utility, and governments often aim to minimise costs or maximise welfare.

Optimisation problems can be classified into two types:

- **Unconstrained optimisation**, where there are no restrictions on the variables.
- **Constrained optimisation**, where variables must satisfy certain conditions, such as budget limits or resource constraints.

Fill in the blank: Optimisation in economics often involves finding the __________ or __________ value of a function.

[

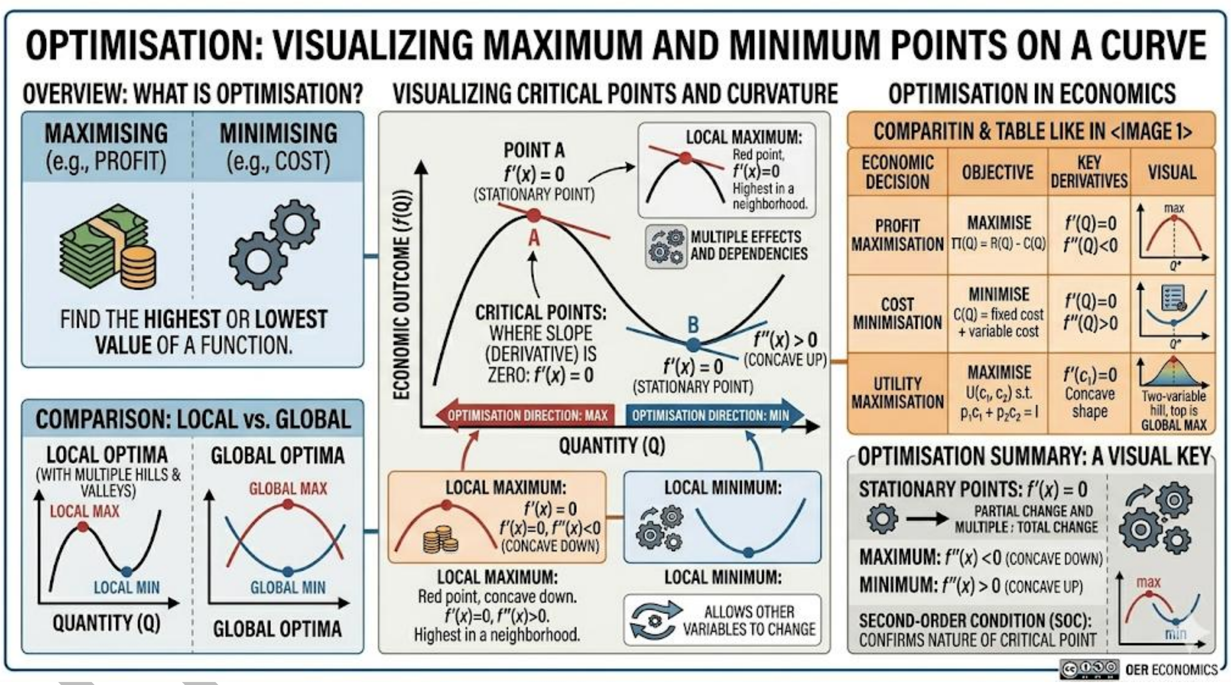

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Diagram showing a curve with a maximum and minimum point.

Figure 3.4: Visualising optimisation in a function

3.3.2 Unconstrained optimisation

In **unconstrained optimisation**, we use derivatives to find critical points of a function. The steps are:

1. Take the first derivative of the function and set it equal to zero to find critical points.
2. Use the second derivative to determine whether the critical point is a maximum, minimum, or saddle point.

Worked Example: Suppose profit is given by (\pi(x) = -x^2 + 10x).

- First derivative: (\pi'(x) = -2x + 10). Setting (\pi'(x) = 0) gives (x = 5).
- Second derivative: (\pi''(x) = -2), which is negative, so (x = 5) is a maximum.

Economic interpretation: The firm maximises profit when output is 5 units.

Fill in the blank: If the second derivative is negative, the critical point is a __________.

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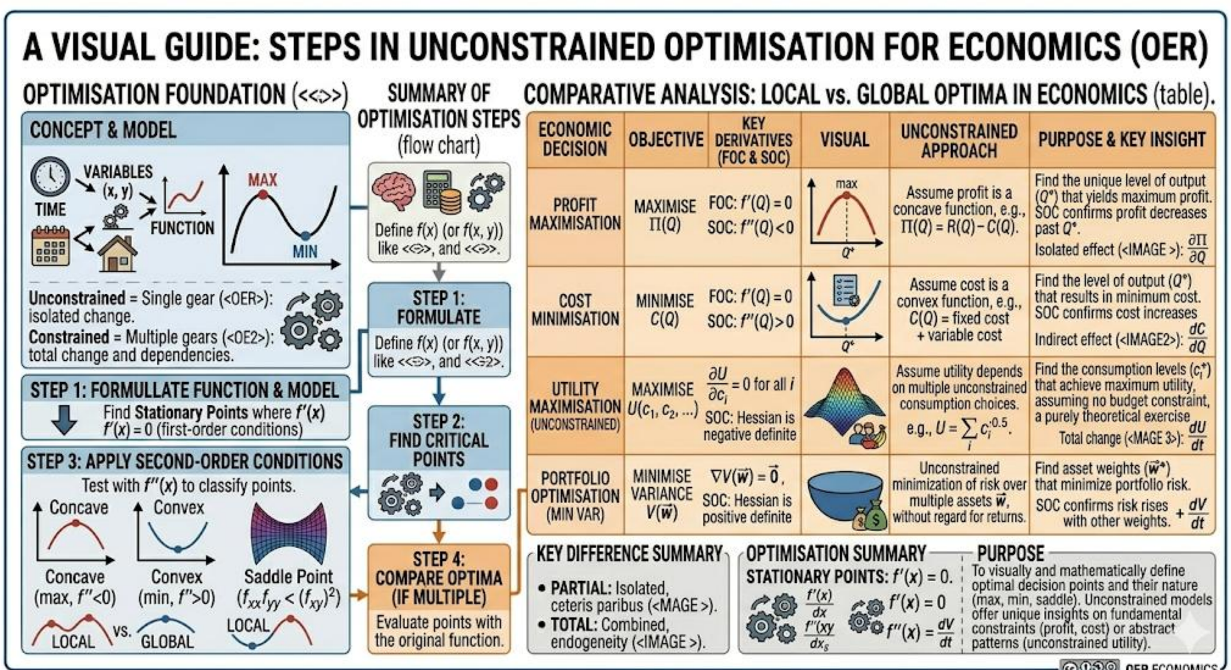

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Table 3.3: Steps in unconstrained optimisation

Step | Action | Purpose

1 | First derivative = 0 | Find critical points

2 | Second derivative test | Identify maximum or minimum

3.3.3 Constrained optimisation using Lagrange multipliers

In many economic problems, optimisation must respect constraints. For example, a consumer maximises utility subject to a budget constraint.

The **Lagrangian method** is used here. If we want to maximise (f(x,y)) subject to (g(x,y) = c), we form the Lagrangian:
[\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda(c - g(x,y))]
We then take partial derivatives with respect to (x), (y), and (\lambda), and solve the system of equations.

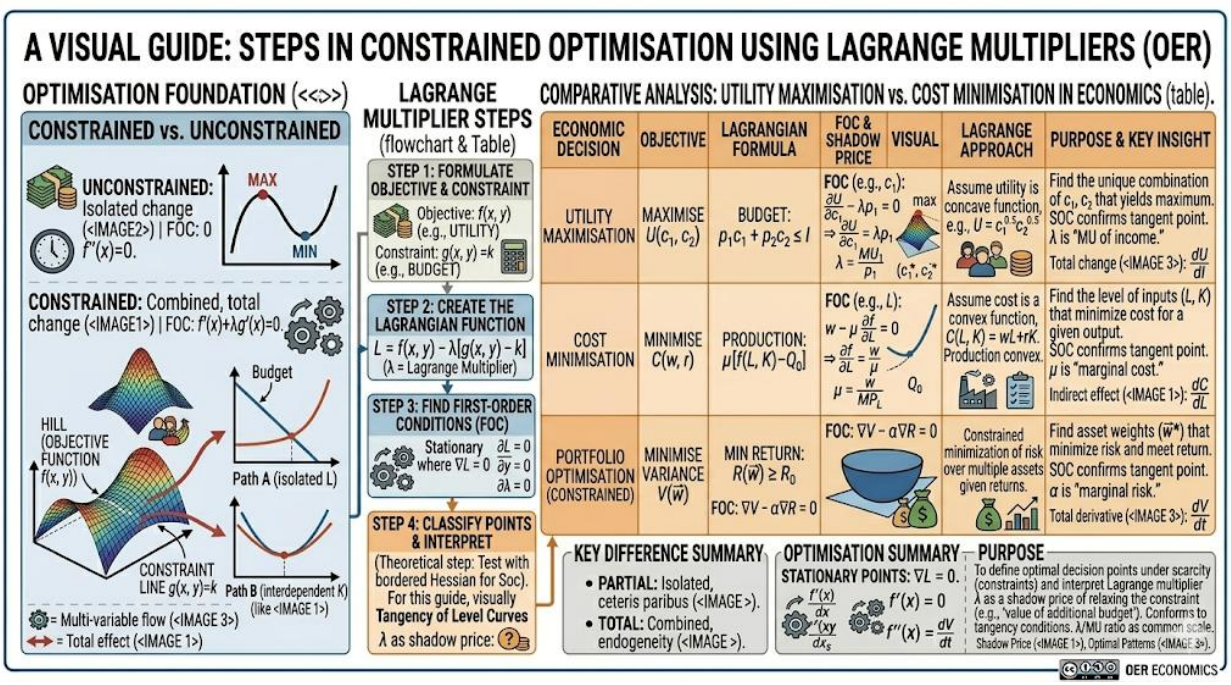
Worked Example: Maximise (U(x,y) = xy) subject to (x + y = 10).

- Lagrangian: (\mathcal{L}(x,y,\lambda) = xy + \lambda(10 - x - y)).
- Partial derivatives:
  (\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0),
  (\frac{\partial \mathcal{L}}{\partial y} = x - \lambda = 0),
  (\frac{\partial \mathcal{L}}{\partial \lambda} = 10 - x - y = 0).
- Solving gives (x = y = 5).

Economic interpretation: The consumer maximises utility by consuming equal amounts of both goods.

Fill in the blank: The Lagrangian method introduces a new variable called __________.

[



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Box 3.3: Steps in constrained optimisation using Lagrange multipliers
Step | Action

1 | Form the Lagrangian function
2 | Take partial derivatives with respect to all variables and the multiplier
3 | Solve the system of equations

3.3.4 Case studies in economic optimisation

Optimisation techniques are applied in various economic contexts:

- **Utility maximisation**: Consumers allocate income across goods to maximise satisfaction.
- **Cost minimisation**: Firms choose input combinations to minimise cost for a given output.
- **Profit maximisation**: Firms determine output levels that maximise profit.

Worked Example: A firm's cost function is (C(L,K) = 10L + 5K), subject to producing (Q(L,K) = L^{0.5}K^{0.5} = 100). Using Lagrange multipliers, the firm finds the optimal combination of labour and capital that minimises cost while meeting the production target.

Fill in the blank: In consumer theory, optimisation is used to maximise __________ subject to a budget constraint.

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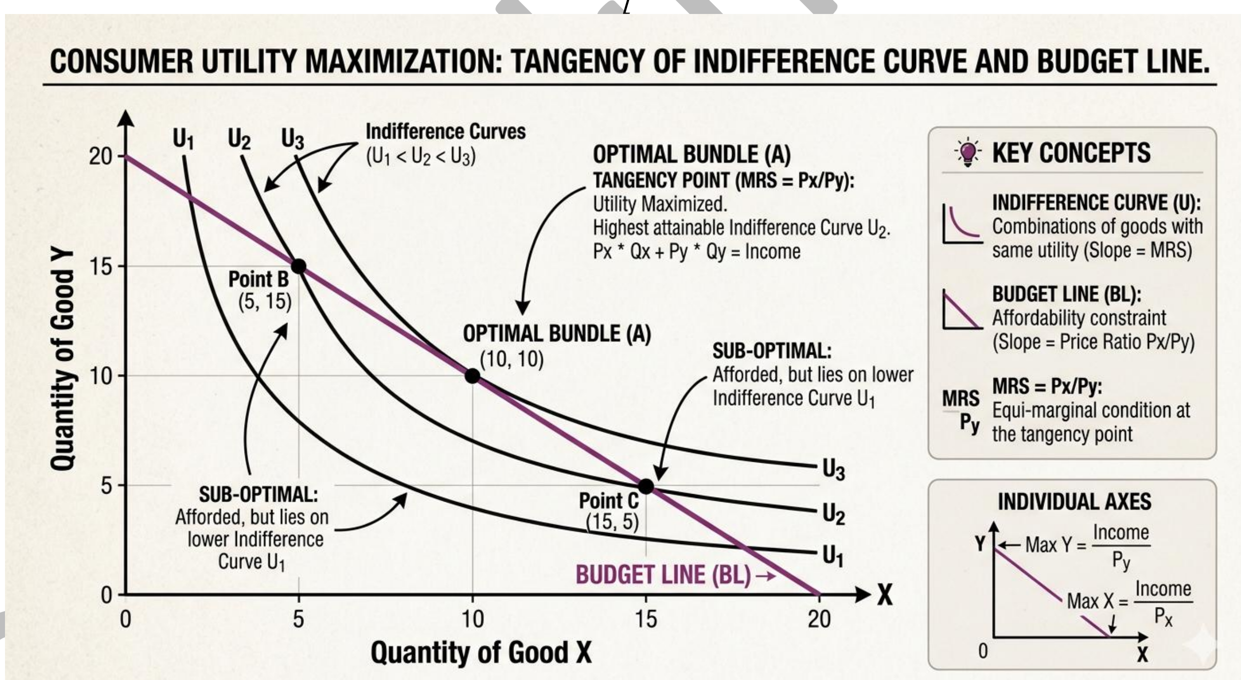

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Diagram showing a consumer's indifference curve tangent to a budget line.
Figure 3.5: Utility maximisation with a budget constraint

Pilot questions 3.3

1. Define optimisation in economics.
2. State the steps in unconstrained optimisation.
3. What does a negative second derivative indicate about a critical point?
4. Write the general form of the Lagrangian function.
5. Explain one economic application of constrained optimisation.

3.4 Economic Applications of Optimisation

3.4.1 Optimisation in production functions

In economics, firms often seek to maximise output given limited resources. A **production function** describes the relationship between inputs such as labour and capital and the resulting output. Optimisation helps firms determine the most efficient combination of inputs.

Worked Example: Suppose a production function is (Q(L,K) = L^{0.5}K^{0.5}). Using constrained optimisation, a firm can minimise cost while achieving a target output. The solution identifies the optimal balance between labour and capital.

Fill in the blank: Optimisation in production functions helps firms determine the most __________ combination of inputs.

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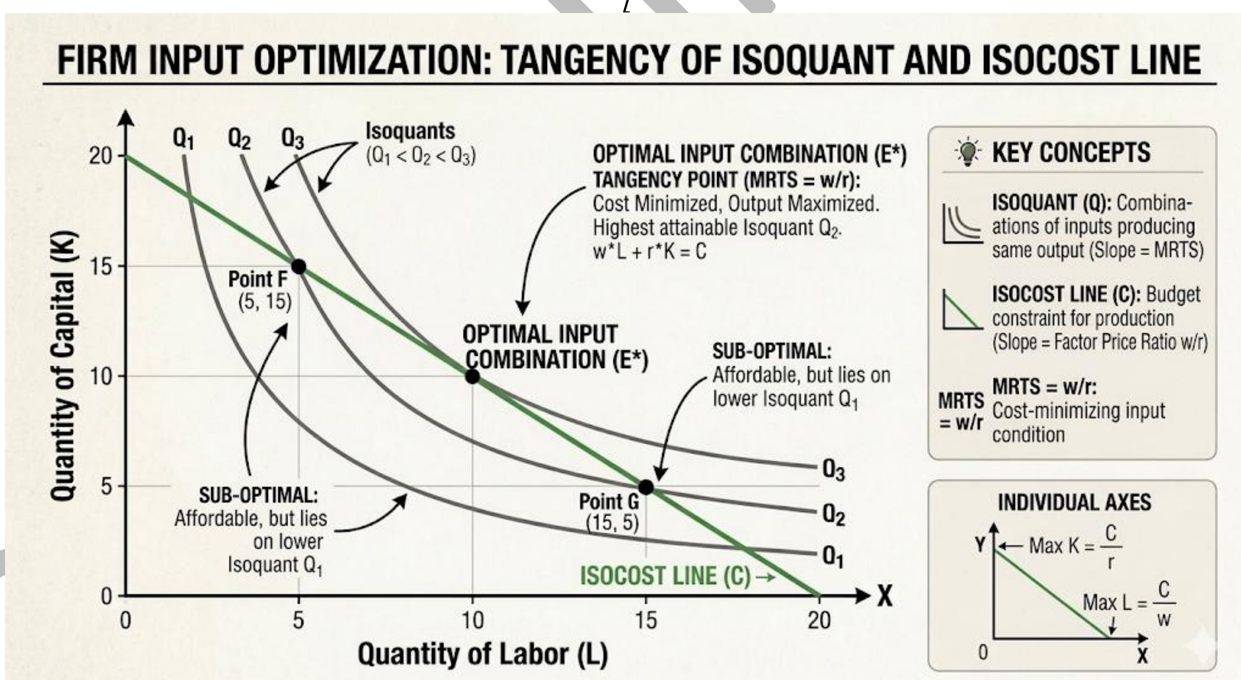

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Diagram showing isoquants and isocost lines intersecting at the optimal input combination.

Figure 3.6: Optimal input combination in production

3.4.2 Utility maximisation in consumer theory

Consumers aim to maximise **utility**, or satisfaction, subject to a budget constraint. Optimisation techniques, particularly Lagrange multipliers, are used to find the point where the consumer's indifference curve is tangent to the budget line.

Worked Example: A consumer has utility function ($U(x,y) = xy$) and budget constraint ($x + y = 10$). Optimisation shows that the consumer maximises utility by consuming equal amounts of both goods.

Fill in the blank: In consumer theory, the optimal point occurs where the indifference curve is __________ to the budget line.

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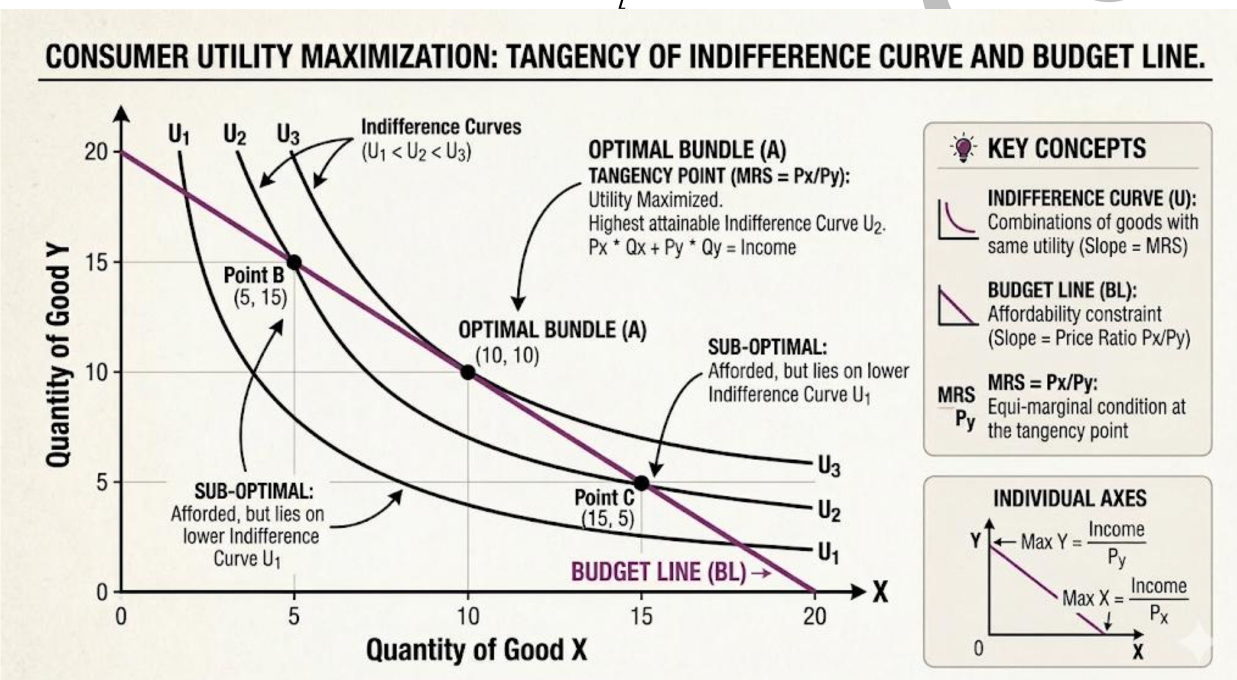

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Diagram showing indifference curve tangent to budget line.

Figure 3.7: Utility maximisation in consumer theory

3.4.3 Profit maximisation in firm behaviour

Firms aim to maximise **profit**, which is the difference between revenue and cost. Optimisation involves determining the level of output where marginal revenue equals marginal cost.

Worked Example: If revenue is ($R(Q) = 20Q$) and cost is ($C(Q) = Q^2 + 10Q$), then profit is ($\pi(Q) = 20Q - (Q^2 + 10Q)$).

- First derivative: ($\pi'(Q) = 10 - 2Q$). Setting ($\pi'(Q) = 0$) gives ($Q = 5$).
- Second derivative: ($\pi''(Q) = -2$), which is negative, so ($Q = 5$) is a maximum.

Economic interpretation: The firm maximises profit by producing 5 units of output.

Fill in the blank: Profit is maximised when marginal revenue equals __________.

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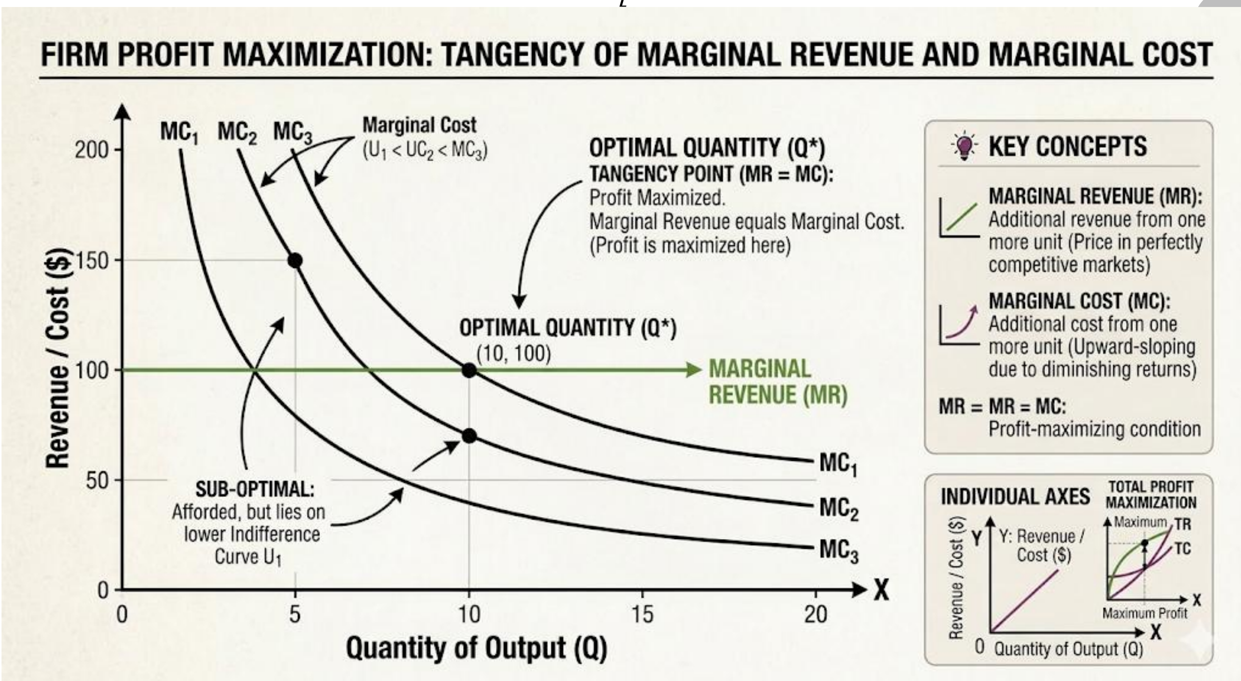

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Table 3.4: Profit maximisation using optimisation

Function | Step | Result

Revenue: (R(Q) = 20Q) | Derive marginal revenue | MR = 20

Cost: (C(Q) = Q^2 + 10Q) | Derive marginal cost | MC = 2Q + 10

Profit: (\pi(Q) = R(Q) - C(Q)) | Optimise | Maximum at Q = 5

3.4.4 Policy implications of optimisation models

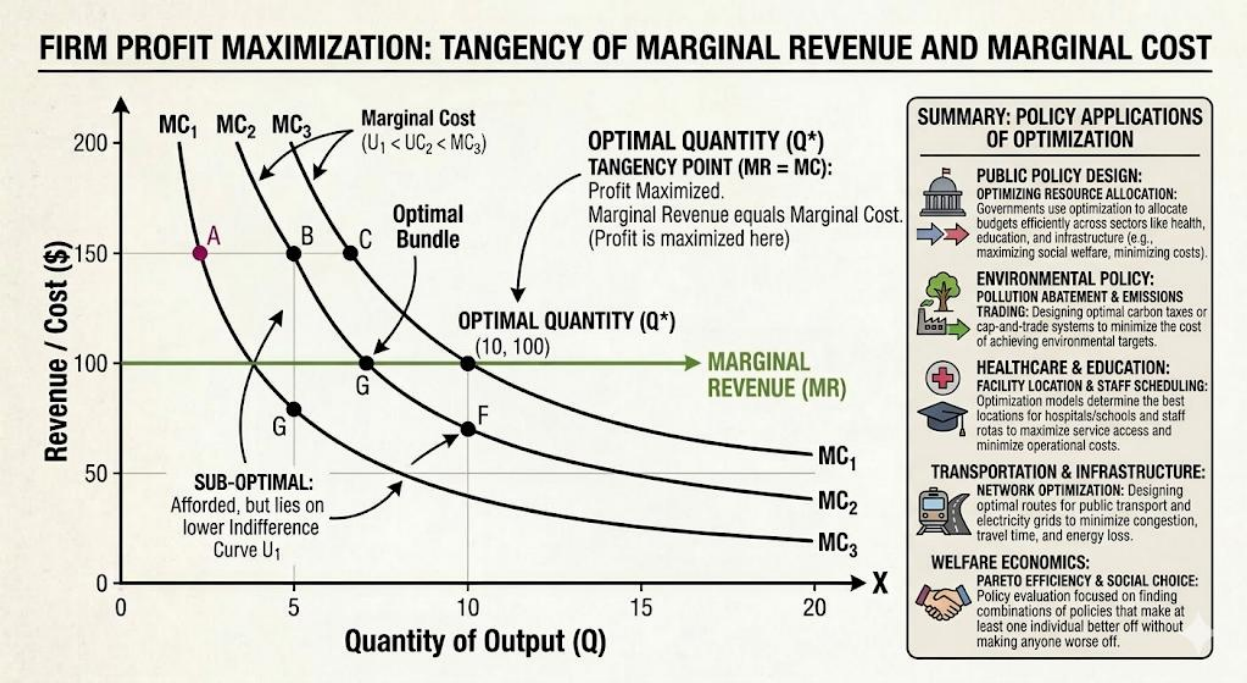
Optimisation is not limited to firms and consumers. Governments and policymakers also use optimisation models to allocate resources efficiently, design tax policies, and manage public spending. For example:

- **Cost minimisation** in infrastructure projects.
- **Welfare maximisation** in social programmes.
- **Resource allocation** in health and education sectors.

Worked Example: A government aims to maximise social welfare subject to a fixed budget. Using optimisation, it determines the best allocation of funds between healthcare and education to achieve the highest welfare outcome.

Fill in the blank: Policymakers use optimisation models to allocate __________ efficiently across competing needs.

[



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Box 3.4: Policy applications of optimisation

Application | Optimisation goal

Infrastructure projects | Minimise cost

Social programmes | Maximise welfare

Resource allocation | Efficient distribution of funds

Pilot questions 3.4

1. Explain how optimisation is applied in production functions.
2. What condition defines the optimal point in consumer utility maximisation?
3. State the rule for profit maximisation in firm behaviour.
4. Give one example of how governments use optimisation in policy-making.
5. In a cost minimisation problem, what is the main objective?

Series Summary

This Series has introduced you to partial and total derivatives with optimisation, showing how these mathematical tools are applied to economic analysis and decision-making. We began by defining partial derivatives, explaining their notation, and working through examples to highlight their role in measuring marginal changes. We then moved on to total derivatives, examining how they combine the effects of multiple variables changing together and applying them to production, cost, and utility functions. Following that, we explored optimisation techniques, first in unconstrained settings using first and second-order conditions, and then in constrained problems with Lagrange multipliers. Finally, we applied these optimisation methods to real economic contexts such as production efficiency, consumer utility maximisation, profit maximisation, and policy design.

By completing this Series, you should now be able to define and calculate partial and total derivatives, apply them to economic functions, analyse optimisation problems both with and without constraints, and evaluate their applications in key areas of economics. In line with the Series Learning Outcomes, you are equipped to design and apply optimisation models that illustrate efficient resource allocation, profit maximisation, and welfare improvement, thereby strengthening your ability to use mathematics as a tool for economic reasoning.

Glossary of Terms

Constraint: A condition or restriction that limits the values variables can take in an optimisation problem, such as a budget limit or resource availability.

Critical point: A point where the first derivative of a function is zero, used to identify possible maxima, minima, or saddle points.

Indifference curve: A curve showing combinations of goods that give a consumer the same level of satisfaction.

Isoquant: A curve showing different combinations of inputs that produce the same level of output.

Isocost line: A line showing all combinations of inputs that can be purchased for a given total cost.

Lagrangian function: A mathematical expression used in constrained optimisation, combining the objective function and the constraint with a multiplier.

Lagrange multiplier: A variable introduced in constrained optimisation to measure how the objective function changes when the constraint changes.

Marginal cost: The additional cost of producing one more unit of output, found using derivatives of the cost function.

Marginal product: The additional output produced by using one more unit of an input, holding other inputs constant.

Marginal utility: The additional satisfaction gained from consuming one more unit of a good, holding other goods constant.

Optimisation: The process of finding the maximum or minimum value of a function, often used in economics to maximise profit or utility, or minimise cost.

Partial derivative: The derivative of a function with respect to one variable, while keeping other variables constant.

Profit maximisation: The process of determining the level of output where profit is highest, usually where marginal revenue equals marginal cost.

Total derivative: The derivative of a function that accounts for simultaneous changes in all variables.

Utility function: A mathematical representation of a consumer's preferences, showing the satisfaction derived from different combinations of goods.

Concept Check Questions [Series 3]

Objective Questions

1. Which of the following best describes a partial derivative?
   a) The derivative of a function with respect to all variables at once
   b) The derivative of a function with respect to one variable, holding others constant
   c) The slope of a tangent line to a curve in one dimension
   d) The maximum value of a function
   (Linked to SLO 3.1)

2. The total derivative of $z = f(x,y)$, where both x and y depend on t, is:
   a) $\frac{dz}{dt} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$
   b) $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
   c) $\frac{dz}{dt} = \frac{df}{dt}$
   d) $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dy}{dt}$
   (Linked to SLO 3.3)

3. In unconstrained optimisation, if the second derivative of a function at a critical point is negative, the point is:
   a) A minimum
   b) A maximum
   c) A saddle point
   d) Undefined
   (Linked to SLO 3.5)

4. The Lagrangian function introduces which additional variable?
   a) Constraint
   b) Multiplier
   c) Marginal cost
   d) Isoquant
   (Linked to SLO 3.6)

5. Profit is maximised when:
   a) Marginal cost equals average cost
   b) Marginal revenue equals marginal cost
   c) Total revenue equals total cost
   d) Marginal utility equals marginal product
   (Linked to SLO 3.7)

Short-Answer Questions

1. Define a partial derivative. (Linked to SLO 3.1)
2. State the formula for the total derivative of ($z = f(x,y)$) when both (x) and (y) depend on (t). (Linked to SLO 3.3)
3. Explain the difference between partial and total derivatives. (Linked to SLO 3.3)
4. Write the general form of the Lagrangian function. (Linked to SLO 3.6)
5. Give one economic application of optimisation in consumer theory. (Linked to SLO 3.7)

Long-Answer Questions

1. Analyse the role of partial derivatives in economics, giving examples from production and utility functions. (Linked to SLO 3.2, SLO 3.4)
    - **Basic response**: Define partial derivatives and show how they measure marginal changes in output or utility.
    - **Value-added response**: Extend by applying partial derivatives to marginal product of labour and marginal utility of goods, explaining their significance in resource allocation.
2. Evaluate the difference between unconstrained and constrained optimisation, with examples. (Linked to SLO 3.5, SLO 3.6)
    - **Basic response**: Describe the steps in unconstrained optimisation and introduce the Lagrangian method for constrained problems.
    - **Value-added response**: Apply unconstrained optimisation to profit maximisation and constrained optimisation to utility maximisation, discussing why constraints are crucial in real-world economics.
3. Design a simple optimisation model for a firm aiming to minimise cost while meeting a production target. (Linked to SLO 3.8)
    - **Basic response**: Outline the cost function and production constraint, and form the Lagrangian.
    - **Value-added response**: Provide a worked example with labour and capital inputs, solve for the optimal combination, and interpret the result in terms of efficiency.

Answers to Concept Check Questions [Series 3]

Objective Questions

1. b) The derivative of a function with respect to one variable, holding others constant
2. b) ($\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$)
3. b) A maximum
4. b) Multiplier
5. b) Marginal revenue equals marginal cost

Short-Answer Questions

1. A partial derivative is the derivative of a function with respect to one variable, keeping other variables constant.
2. $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
3. Partial derivatives measure change in one variable while holding others constant; total derivatives measure combined change when all variables vary.
4. $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda(c - g(x,y))$.
5. Utility maximisation subject to a budget constraint.

Long-Answer Questions

1. **Basic response**: Partial derivatives measure marginal changes. In production, they show marginal product of labour or capital. In utility, they show marginal utility of goods. **Value-added response**: They help firms allocate resources efficiently and consumers decide optimal consumption bundles, forming the basis of marginal analysis in economics.

2. **Basic response**: Unconstrained optimisation uses first and second derivatives to find maxima or minima. Constrained optimisation uses Lagrange multipliers to respect restrictions.

Answers to Pilot Questions [Series 3]

Part 3.1: Fundamentals of Partial Derivatives

1. A partial derivative is the derivative of a function with respect to one variable, keeping other variables constant.
2. The notation is $\frac{\partial f}{\partial x}$ for the partial derivative of $f(x,y)$ with respect to x.
3. $\frac{\partial f}{\partial x} = 2xy$ when $f(x,y) = x^2y + 3y$.
4. In a production function, a partial derivative shows the marginal product of an input, such as labour, holding other inputs constant.
5. The symbol ∂ is used to denote a partial derivative.

Part 3.2: Total Derivatives and Their Applications

1. A total derivative measures how a function changes when all of its variables change simultaneously.
2. $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.
3. Partial derivatives hold other variables constant, while total derivatives account for changes in all variables together.
4. One economic application is analysing how total cost changes when both input prices and quantities vary.
5. In a production function, the total derivative represents the overall change in output when both labour and capital change.

Part 3.3: Optimisation With Constraints

1. Optimisation in economics is the process of finding the maximum or minimum value of a function, often profit, utility, or cost.
2. Steps: (i) Take the first derivative and set it equal to zero to find critical points; (ii) Use the second derivative to determine whether the point is a maximum or minimum.
3. A negative second derivative indicates a maximum.
4. General form: (\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda(c - g(x,y))).
5. One application is utility maximisation, where consumers maximise satisfaction subject to a budget constraint.

Part 3.4: Economic Applications of Optimisation

1. Optimisation in production functions helps firms determine the most efficient combination of inputs to maximise output or minimise cost.
2. The optimal point in consumer utility maximisation occurs where the indifference curve is tangent to the budget line.
3. Profit maximisation occurs when marginal revenue equals marginal cost.
4. Governments use optimisation to allocate resources efficiently, for example between healthcare and education.
5. In cost minimisation, the main objective is to achieve a given level of output at the lowest possible cost.

References and Attributions [Series 3]

Figures

- *Figure 3.1: Visualising a partial derivative along one axis*
  AI-generated suggestion: "Generate a 3D surface diagram showing a function of two variables with slope along x-axis while y is constant."
  Suggested OER search string: "partial derivative surface diagram OER mathematics."
- *Figure 3.2: Marginal product of labour as a partial derivative*
  AI-generated suggestion: "Generate a 3D diagram of a production function surface showing slope along labour axis to represent marginal product of labour."
  Suggested OER search string: "production function marginal product labour partial derivative OER economics."
- *Figure 3.3: Visualising a total derivative with multiple variables changing*
  AI-generated suggestion: "Generate a diagram showing a function of two variables where both variables change with respect to time, illustrating total derivative."
  Suggested OER search string: "total derivative diagram OER mathematics."
- *Figure 3.4: Visualising optimisation in a function*
  AI-generated suggestion: "Generate a diagram showing a curve with maximum and minimum points to illustrate optimisation."
  Suggested OER search string: "optimisation maximum minimum function diagram OER economics."
- *Figure 3.5: Utility maximisation with a budget constraint*
  AI-generated suggestion: "Generate a diagram showing indifference curve tangent to

budget line to illustrate utility maximisation."
Suggested OER search string: "utility maximisation indifference curve budget line OER economics."

- *Figure 3.6: Optimal input combination in production*
  AI-generated suggestion: "Generate a diagram showing isoquants and isocost lines intersecting at the optimal input combination."
  Suggested OER search string: "production function optimisation isoquant isocost diagram OER economics."
- *Figure 3.7: Utility maximisation in consumer theory*
  AI-generated suggestion: "Generate a diagram showing indifference curve tangent to budget line to illustrate utility maximisation."
  Suggested OER search string: "utility maximisation indifference curve budget line OER economics."

Tables

- *Table 3.1: Examples of partial derivatives of functions*
  AI-generated suggestion: "Create a table showing examples of functions with their partial derivatives with respect to x and y."
  Suggested OER search string: "examples of partial derivatives functions OER mathematics."
- *Table 3.2: Economic applications of total derivatives*
  AI-generated suggestion: "Create a table showing economic applications of total derivatives with examples."
  Suggested OER search string: "economic applications total derivatives OER economics."
- *Table 3.3: Steps in unconstrained optimisation*
  AI-generated suggestion: "Create a table summarising steps in unconstrained optimisation with purpose."
  Suggested OER search string: "unconstrained optimisation steps OER economics."
- *Table 3.4: Profit maximisation using optimisation*
  AI-generated suggestion: "Create a table showing steps in profit maximisation using optimisation."
  Suggested OER search string: "profit maximisation marginal revenue marginal cost OER economics.

Boxes

- *Box 3.1: Common notation for partial derivatives*
  AI-generated suggestion: "Create a text box summarising common notation for partial derivatives."
  Suggested OER search string: "partial derivative notation OER mathematics."
- *Box 3.2: Comparing partial and total derivatives*
  AI-generated suggestion: "Create a text box comparing partial and total derivatives with economic applications."
  Suggested OER search string: "difference between partial and total derivatives OER economics."

- *Box 3.3: Steps in constrained optimisation using Lagrange multipliers*
  AI-generated suggestion: “Create a text box summarising steps in constrained optimisation using Lagrange multipliers.”
  Suggested OER search string: “Lagrange multipliers constrained optimisation steps OER economics.”
- *Box 3.4: Policy applications of optimisation*
  AI-generated suggestion: “Create a text box summarising policy applications of optimisation.”
  Suggested OER search string: “policy applications optimisation economics OER.”

General References

- Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). New York: McGraw-Hill.
- Stewart, J. (2016). *Calculus: Early Transcendentals* (8th ed.). Boston: Cengage Learning.
- OpenStax. (2016). *Calculus Volume 1–3*. Rice University. Retrieved from https://openstax.org/subjects/math
- Khan Academy. (n.d.). *Partial derivatives, total derivatives, and optimisation*. Retrieved from https://www.khanacademy.org

Course code: ECO 104

Course title: Introductory Mathematics II

Course credit load: 2 Units

Series 1: Derivatives of Trigonometric Functions and Expansions

Series 2: Sequences and Series in Economic Applications

Series 3: Partial and Total Derivatives with Optimisation

Series 4: Linear and Matrix Algebra in Economics

Series 5: Mathematical Relationships and Graphical Analysis of Economic Concepts

Suggested Outline

Part 4.1: Fundamentals of linear algebra

- Definition of linear equations and systems
- Methods of solving linear equations (substitution, elimination)
- Economic relevance of linear systems (e.g., market equilibrium)

Part 4.2: Introduction to matrices

- Definition of a matrix and types of matrices (square, rectangular, identity, zero)
- Basic operations: addition, subtraction, scalar multiplication
- Matrix multiplication and its interpretation in economics

Part 4.3: Determinants and inverses of matrices

- Definition and calculation of determinants
- Properties of determinants
- Inverse of a matrix and conditions for existence
- Applications in solving systems of linear equations

Part 4.4: Applications of matrix algebra in economics

- Input-output analysis in production
- Leontief model of inter-industry relationships
- Use of matrices in demand-supply analysis

- Policy and planning applications

Introduction

In the last Series, we explored partial and total derivatives with optimisation, learning how these tools help economists analyse marginal changes and determine efficient outcomes. Building on that foundation, this Series introduces you to **linear and matrix algebra in economics**, which provides powerful methods for handling systems of equations and modelling complex relationships between economic variables. These techniques are widely used in areas such as market equilibrium analysis, production planning, and inter-industry studies.

The aim of this Series is to equip you with the ability to apply linear and matrix algebra to economic problems, enabling you to solve systems of equations, work with matrices, and interpret their applications in real-world economic contexts.

We shall commence this Series by examining the fundamentals of linear algebra, focusing on linear equations and systems, and their economic relevance. Next, we will move on to matrices, exploring their types, operations, and multiplication. Following that, we will study determinants and inverses of matrices, highlighting their role in solving systems of equations. Finally, we will conclude by applying matrix algebra to economic models such as input-output analysis, the Leontief model, and demand-supply frameworks, thereby rounding off the Series with practical applications.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 4.1 **define** linear equations and explain how systems of linear equations are represented in economics,

SLO 4.2 **demonstrate** methods of solving systems of linear equations using substitution and elimination,

SLO 4.3 **define** a matrix and identify different types of matrices such as square, rectangular, identity, and zero,

SLO 4.4 **perform** basic matrix operations including addition, subtraction, scalar multiplication, and matrix multiplication,

SLO 4.5 **calculate** determinants of matrices and explain their properties,

SLO 4.6 **determine** the inverse of a matrix and apply it to solving systems of linear equations,

SLO 4.7 **apply** matrix algebra to economic models such as input-output analysis and the Leontief model, and

SLO 4.8 **analyse** economic problems involving demand-supply relationships using matrix methods.

4.1 Fundamentals of Linear Algebra

4.1.1 Definition of linear equations and systems

A **linear equation** is an equation in which each variable appears to the power of one, and terms are either constants or products of constants and variables. For example, (2x + 3y = 6) is a linear equation in two variables.

A **system of linear equations** is a collection of two or more linear equations involving the same set of variables. Solving such systems means finding values of the variables that satisfy all equations simultaneously. In economics, systems of linear equations are used to represent situations such as market equilibrium, where supply and demand equations must hold at the same time.

Fill in the blank: A system of linear equations involves two or more __________ equations with the same variables.

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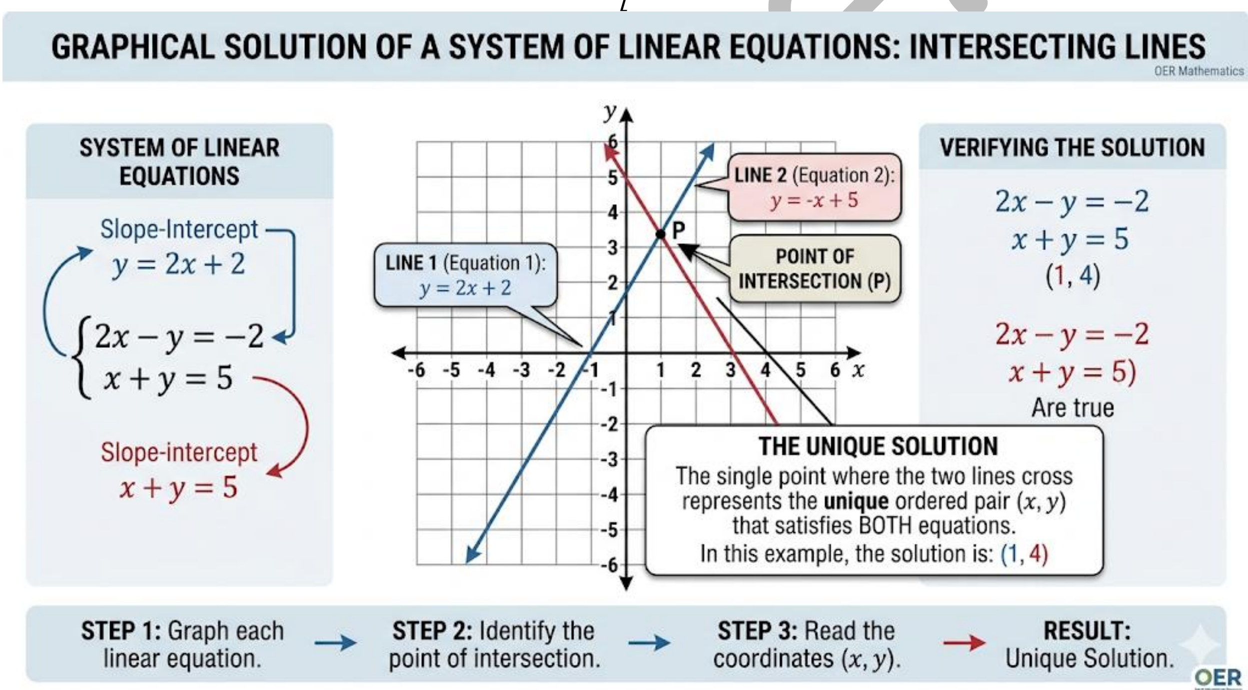

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Diagram showing two straight lines intersecting at a point, representing the solution to a system of two linear equations.

Figure 4.1: Graphical solution of a system of linear equations

4.1.2 Methods of solving linear equations

There are several methods for solving systems of linear equations:

- **Substitution method**: Solve one equation for one variable and substitute into the other.
- **Elimination method**: Add or subtract equations to eliminate one variable, then solve for the other.
- **Graphical method**: Plot each equation on a graph and identify the point of intersection.

Worked Example: Solve the system
[x + y = 6 \quad \text{and} \quad 2x - y = 3]
Using elimination: Add the two equations to get (3x = 9), so (x = 3). Substituting into the first equation gives (y = 3).

Fill in the blank: In the elimination method, one variable is removed by __________ equations together.

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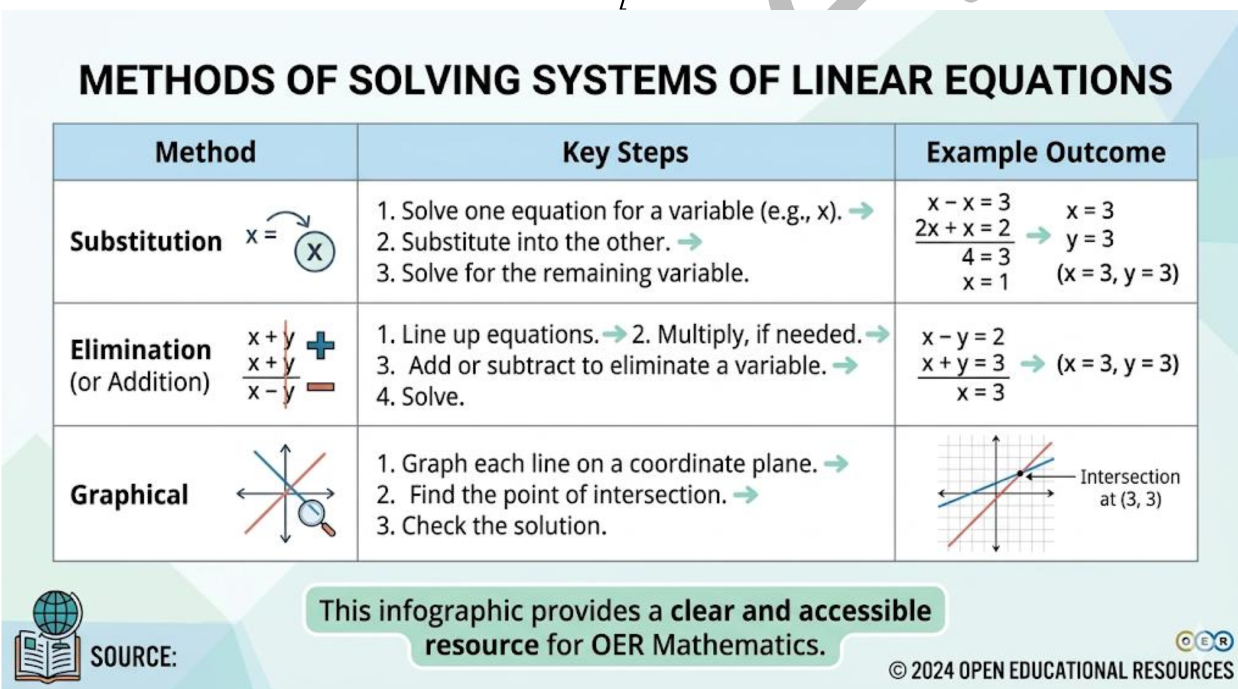

METHODS OF SOLVING SYSTEMS OF LINEAR EQUATIONS

Method	Key Steps	Example Outcome
Substitution $x =$	1. Solve one equation for a variable (e.g., x). 2. Substitute into the other. 3. Solve for the remaining variable.	$x - x = 3$, $2x + x = 2$, $4 = 3$, $x = 1$ → $x = 3$, $y = 3$, $(x = 3, y = 3)$
Elimination (or Addition) $x + y$, $x + y$, $x - y$	1. Line up equations. 2. Multiply, if needed. 3. Add or subtract to eliminate a variable. 4. Solve.	$x - y = 2$, $x + y = 3$, $x = 3$ → $(x = 3, y = 3)$
Graphical	1. Graph each line on a coordinate plane. 2. Find the point of intersection. 3. Check the solution.	Intersection at (3, 3)

]

Table 4.1: Methods of solving systems of linear equations
Method | Key steps | Example outcome
Substitution | Solve one equation for a variable, substitute into the other | (x = 3, y = 3)
Elimination | Add or subtract equations to eliminate a variable | (x = 3, y = 3)
Graphical | Plot equations and find intersection point | Intersection at (3,3)

4.1.3 Economic relevance of linear systems

Linear systems are widely used in economics to model relationships between variables. Examples include:

- **Market equilibrium**: Supply and demand equations form a system of linear equations, and their solution gives the equilibrium price and quantity.
- **Input-output analysis**: Industries are represented by linear equations showing how outputs of one sector are inputs to another.
- **Budget constraints**: Linear equations represent the relationship between income, prices, and quantities of goods consumed.

Worked Example: Suppose demand is ($Q_d = 20 - 2P$) and supply is ($Q_s = 4P - 10$). Setting ($Q_d = Q_s$) gives the equilibrium price and quantity.

Fill in the blank: In market equilibrium analysis, the solution to the system of supply and demand equations gives the __________ price and quantity.

[Insert Box here]
Box 4.1: Economic applications of linear systems
Application | Description
Market equilibrium | Solving supply and demand equations for equilibrium price and quantity
Input-output analysis | Representing inter-industry relationships using linear systems
Budget constraints | Expressing consumer choices under income and price limits
AI prompt suggestion: “Create a text box summarising economic applications of linear systems.”
Search string: “economic applications of linear systems OER economics”

4.1.4 Worked examples and practice

Let us consider a more detailed example involving three variables:

$$x + y + z = 6$$

$$2x - y + z = 3$$

$$x + 2y - z = 4$$

This system can be solved using elimination or substitution, but in practice, matrix methods (covered in later Parts) are more efficient. For now, solving step by step shows how linear systems can represent multiple relationships simultaneously.

Economic interpretation: Such systems can represent three markets interacting, where each equation reflects the balance of supply and demand in one market.

Fill in the blank: Systems with three or more variables are often solved more efficiently using __________ methods.

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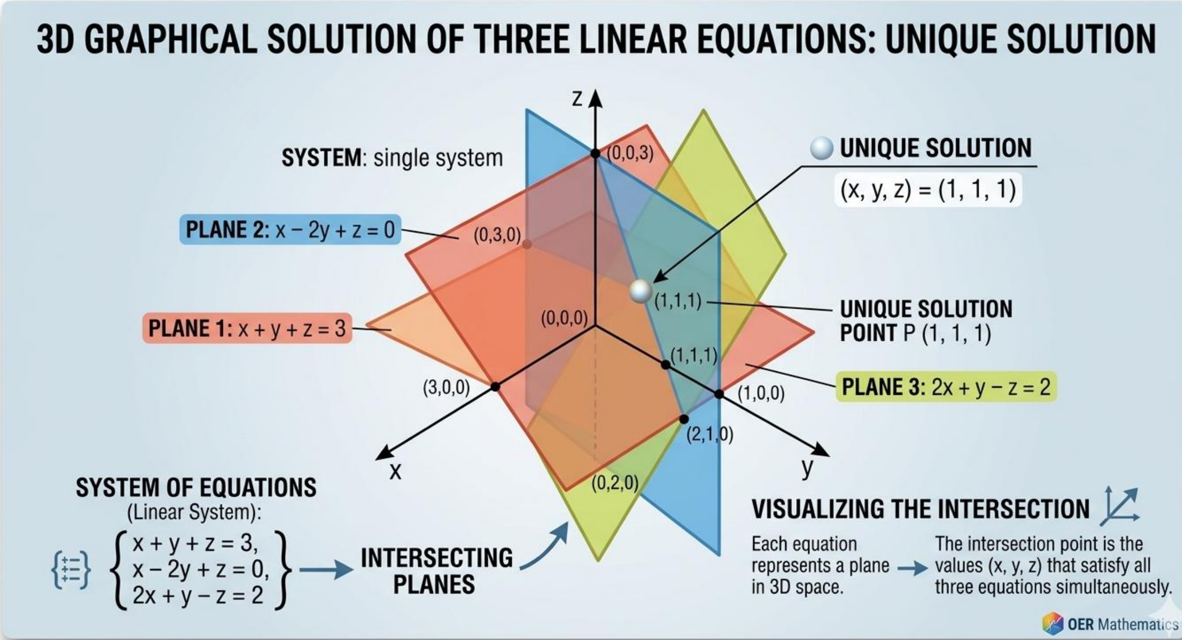

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Diagram showing three planes in 3D space intersecting at a single point, representing the solution to a system of three equations.

Figure 4.2: Graphical representation of a system of three linear equations

Pilot questions 4.1

1. Define a linear equation.
2. What is a system of linear equations?
3. State one method of solving systems of linear equations.
4. Solve the system (x + y = 6) and (2x - y = 3).
5. Give one economic application of linear systems.
6. Why are matrix methods often preferred for solving systems with three or more variables?

4.2 Introduction to Matrices

4.2.1 Definition of a matrix and types of matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. Matrices are widely used in mathematics and economics to represent and manipulate systems of equations and relationships between variables.

Types of matrices include:

- **Square matrix**: A matrix with the same number of rows and columns.
- **Rectangular matrix**: A matrix with different numbers of rows and columns.

- **Identity matrix**: A square matrix with ones on the diagonal and zeros elsewhere.
- **Zero matrix**: A matrix in which all entries are zero.

Fill in the blank: A matrix with the same number of rows and columns is called a __________ matrix.

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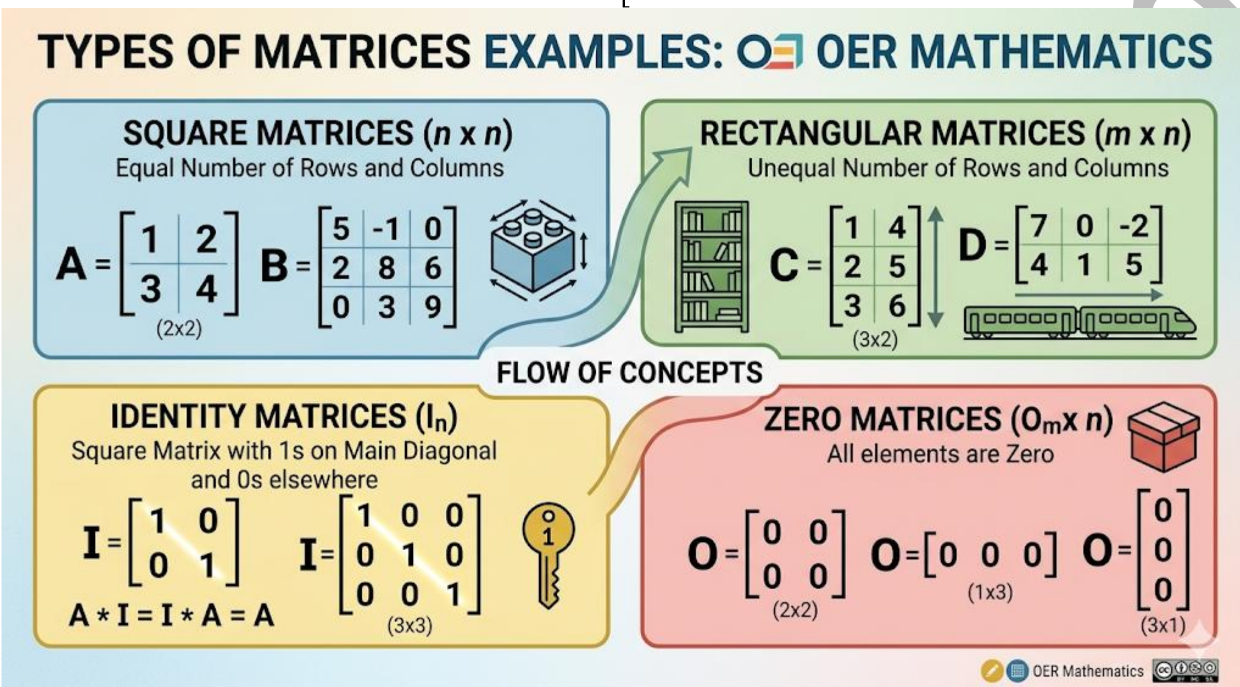

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Diagram showing examples of different types of matrices (square, rectangular, identity, zero).
Figure 4.3: Examples of different types of matrices

4.2.2 Basic operations on matrices

Matrices can be manipulated using several operations:

- **Addition and subtraction**: Matrices of the same size can be added or subtracted by combining corresponding elements.
- **Scalar multiplication**: Each element of a matrix is multiplied by a constant (scalar).
- **Matrix multiplication**: Multiplying two matrices involves taking the dot product of rows and columns. This operation is only defined when the number of columns in the first matrix equals the number of rows in the second.

Worked Example:
[A = \begin{bmatrix} 1 & 2 \ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \ 1 & 3 \end{bmatrix}]

- Addition: (A + B = \begin{bmatrix} 3 & 2 \ 4 & 7 \end{bmatrix})

- Scalar multiplication: (2A = \begin{bmatrix} 2 & 4 \ 6 & 8 \end{bmatrix})
- Multiplication: (AB = \begin{bmatrix} 1\cdot2 + 2\cdot1 & 1\cdot0 + 2\cdot3 \ 3\cdot2 + 4\cdot1 & 3\cdot0 + 4\cdot3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \ 10 & 12 \end{bmatrix})

Fill in the blank: Matrix multiplication is only defined when the number of __________ in the first matrix equals the number of rows in the second.

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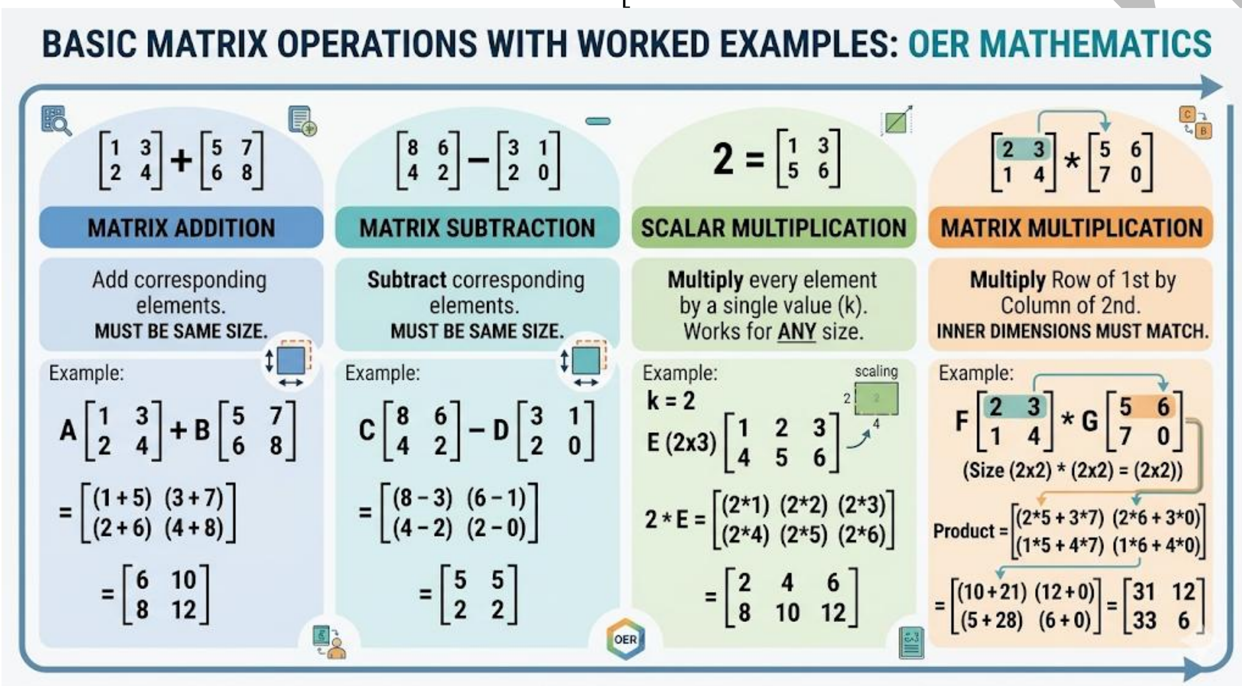


Table 4.2: Basic matrix operations with examples
Operation | Rule | Example outcome
Addition | Add corresponding elements | (\begin{bmatrix} 3 & 2 \ 4 & 7 \end{bmatrix})
Scalar multiplication | Multiply each element by constant | (\begin{bmatrix} 2 & 4 \ 6 & 8 \end{bmatrix})
Matrix multiplication | Dot product of rows and columns | (\begin{bmatrix} 4 & 6 \ 10 & 12 \end{bmatrix})

4.2.3 Interpretation of matrices in economics

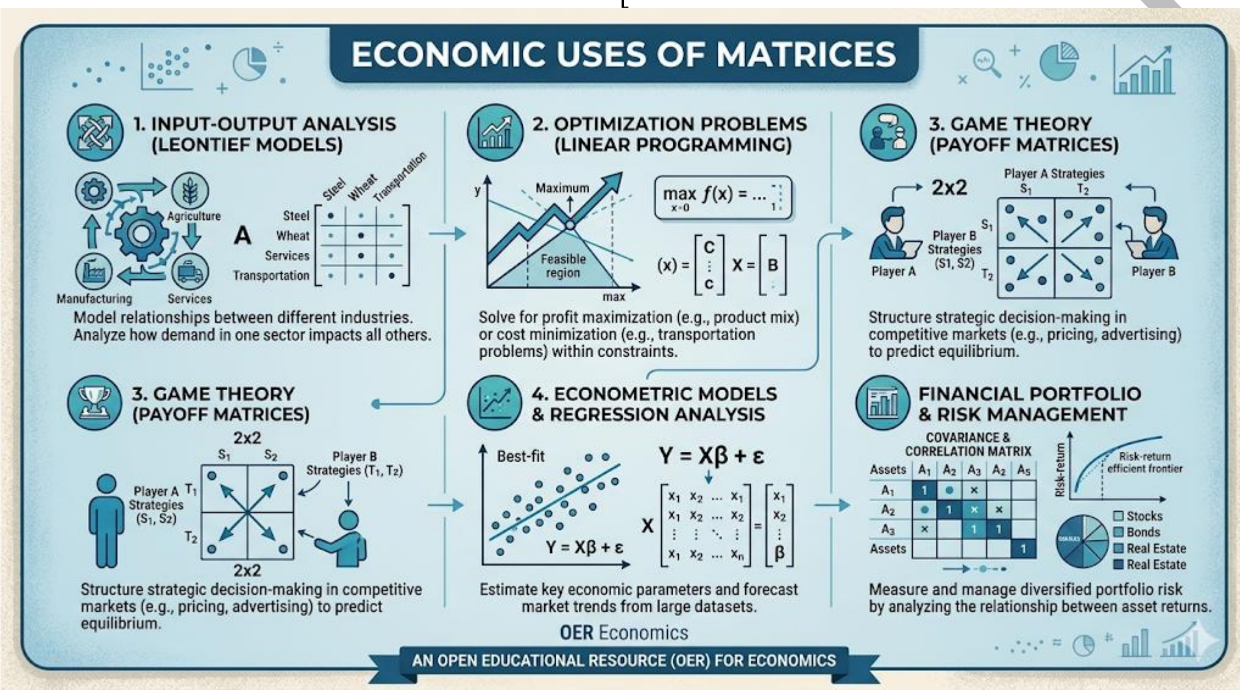
Matrices are not just abstract mathematical objects; they have practical economic interpretations. For example:

- **Input-output analysis**: A matrix can represent how outputs of one industry serve as inputs to another.
- **Demand-supply relationships**: Matrices can capture multiple equations representing demand and supply across different markets.
- **Budget allocation**: Matrices can be used to represent how resources are distributed across different sectors.

Worked Example: In a two-sector economy, the input-output matrix might look like:
[\begin{bmatrix} 0.3 & 0.2 \ 0.4 & 0.1 \end{bmatrix}]
This shows that 30% of sector 1's output is used within sector 1, and 20% is used in sector 2, while 40% of sector 2's output is used in sector 1, and 10% is used within sector 2.

Fill in the blank: In input-output analysis, a matrix shows how the output of one __________ is used as input in another.

[



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Box 4.2: Economic uses of matrices

Application | Description
Input-output analysis | Represents inter-industry relationships
Demand-supply systems | Captures multiple market equations
Budget allocation | Shows distribution of resources across sectors

Pilot questions 4.2

1. Define a matrix.
2. Name two types of matrices.
3. State the condition for matrix multiplication to be defined.
4. Perform the multiplication of (A = \begin{bmatrix} 1 & 2 \ 3 & 4 \end{bmatrix}) and (B = \begin{bmatrix} 2 & 0 \ 1 & 3 \end{bmatrix}).
5. Give one economic application of matrices.

4.4 Applications of Matrix Algebra in Economics

4.4.1 Input-output analysis in production

One of the most important applications of matrix algebra in economics is **input-output analysis**, developed by Wassily Leontief. This method uses matrices to describe how the output of one industry is used as input in another. Each entry in the matrix represents the proportion of one sector's output consumed by another sector.

Worked Example: Consider a two-sector economy with the input-output matrix:
$$A = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}$$
This means that 30% of sector 1's output is used within sector 1, 20% is used in sector 2, 40% of sector 2's output is used in sector 1, and 10% is used within sector 2.

Fill in the blank: Input-output analysis shows how the output of one __________ is used as input in another.

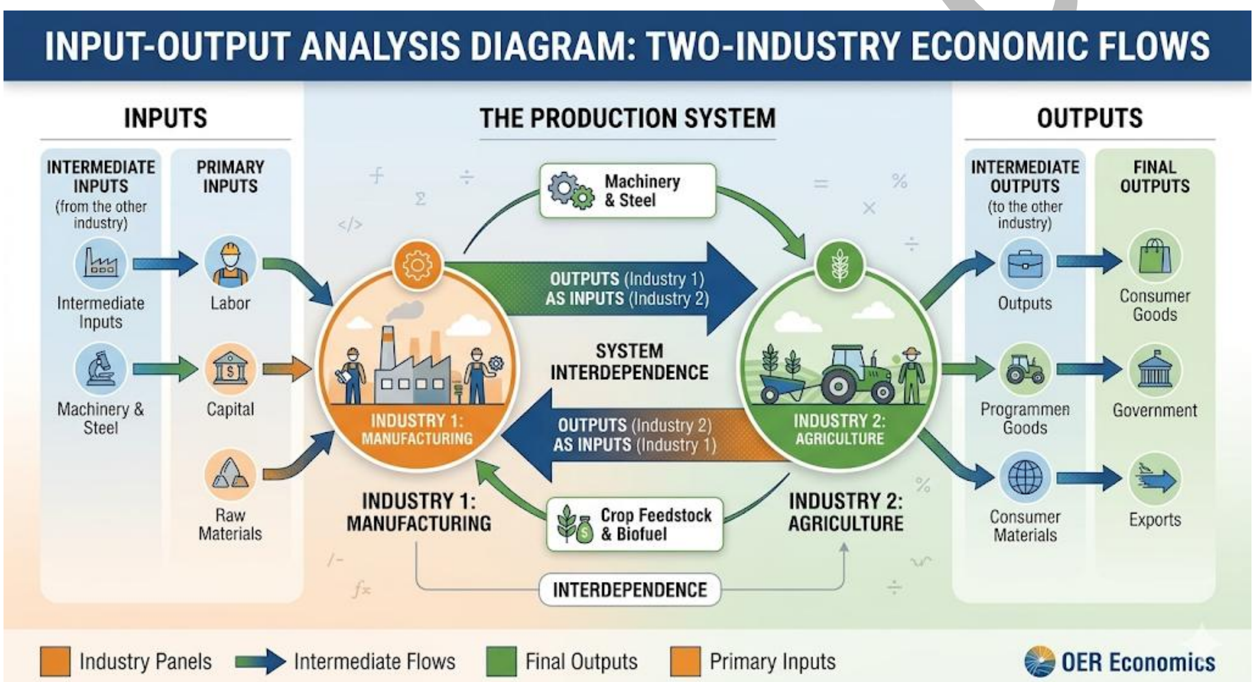

Diagram showing two industries with arrows indicating flows of outputs as inputs between them.
Figure 4.6: Input-output relationships between two sectors

4.4.2 The Leontief model of inter-industry relationships

The **Leontief model** uses matrix algebra to calculate the total output required to meet both intermediate demand (from other industries) and final demand (from consumers). The model is expressed as:
$$X = (I - A)^{-1}D$$
where:

- (X) = vector of total outputs,

- (I) = identity matrix,
- (A) = input-output matrix,
- (D) = vector of final demand.

This formula shows how matrix inverses are used to solve for total output in an economy.

Worked Example: If final demand is (D = \begin{bmatrix} 100 \ 80 \end{bmatrix}), and the input-output matrix is as above, the Leontief inverse ((I - A)^{-1}) can be calculated and multiplied by (D) to find the required outputs of each sector.

Fill in the blank: In the Leontief model, total output is given by (X = (I - A)^{-1} \times __________).

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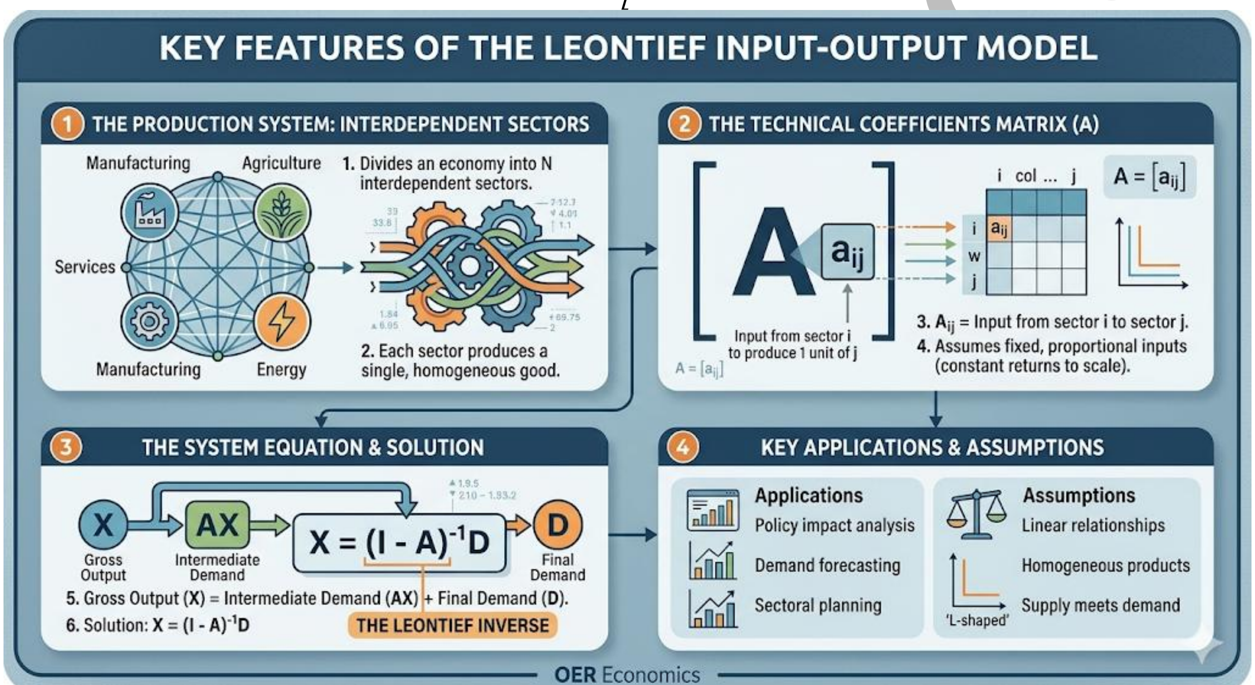

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Box 4.4: Key features of the Leontief model

Feature | Explanation

Intermediate demand | Output used by other industries

Final demand | Output consumed by households and government

Leontief inverse | Matrix used to calculate total output

4.4.3 Use of matrices in demand-supply analysis

Matrices can also be applied to systems of demand and supply equations across multiple markets. For example, in a two-market system, demand and supply equations can be expressed in matrix form, and solutions can be found using determinants and inverses.

Worked Example: Suppose demand and supply equations for two goods are represented as:
[AX = B]
where (A) is the coefficient matrix, (X) is the vector of prices, and (B) is the vector of constants. Solving for (X) using (A^{-1}B) gives the equilibrium prices in both markets.

Fill in the blank: In demand-supply analysis, equilibrium prices can be found by solving the system (AX = __________).

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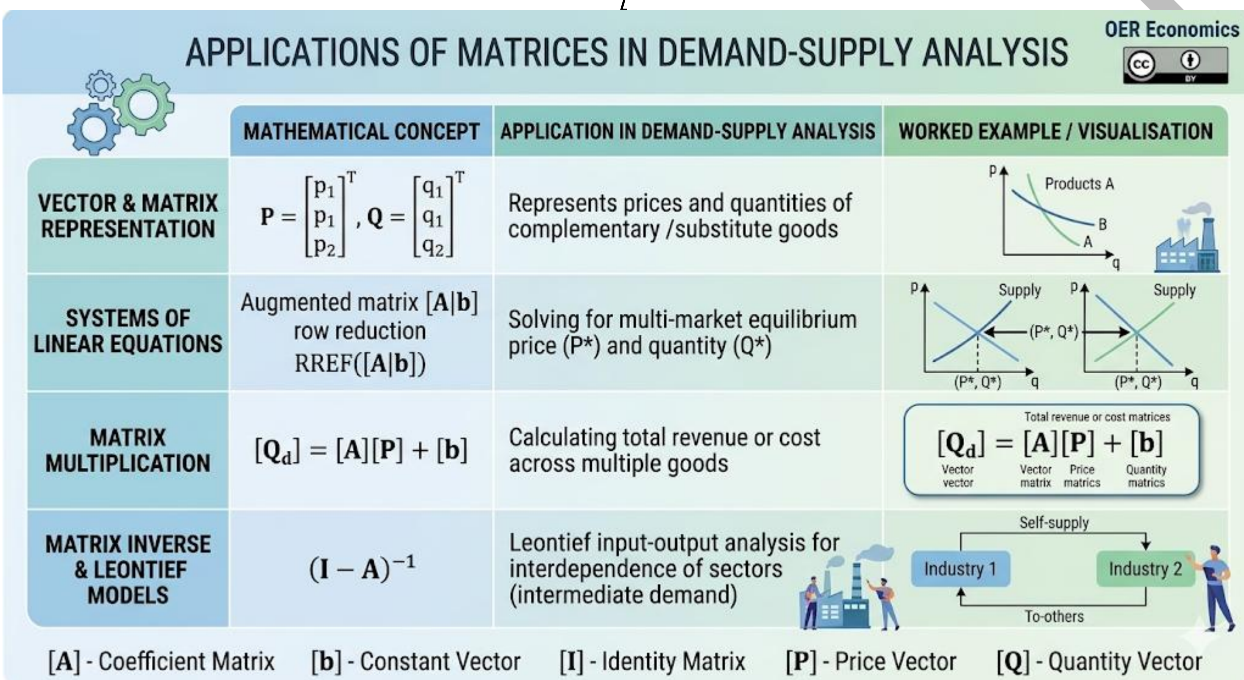

APPLICATIONS OF MATRICES IN DEMAND-SUPPLY ANALYSIS

OER Economics

	MATHEMATICAL CONCEPT	APPLICATION IN DEMAND-SUPPLY ANALYSIS	WORKED EXAMPLE / VISUALISATION
VECTOR & MATRIX REPRESENTATION	$\mathbf{P} = \begin{bmatrix} p_1 \\ p_1 \\ p_2 \end{bmatrix}^T, \mathbf{Q} = \begin{bmatrix} q_1 \\ q_1 \\ q_2 \end{bmatrix}^T$	Represents prices and quantities of complementary /substitute goods	p, Products A, B, A, q
SYSTEMS OF LINEAR EQUATIONS	Augmented matrix $[\mathbf{A}\vert\mathbf{b}]$ row reduction $\mathrm{RREF}([\mathbf{A}\vert\mathbf{b}])$	Solving for multi-market equilibrium price (P*) and quantity (Q*)	p, Supply, (P*, Q*), q; p, Supply, (P*, Q*), q
MATRIX MULTIPLICATION	$[\mathbf{Q}_d] = [\mathbf{A}][\mathbf{P}] + [\mathbf{b}]$	Calculating total revenue or cost across multiple goods	Total revenue or cost matrices: $[\mathbf{Q}_d] = [\mathbf{A}][\mathbf{P}] + [\mathbf{b}]$ — Vector vector, Vector matrix, Price matrics, Quantity matrics
MATRIX INVERSE & LEONTIEF MODELS	$(\mathbf{I} - \mathbf{A})^{-1}$	Leontief input-output analysis for interdependence of sectors (intermediate demand)	Self-supply, Industry 1, Industry 2, To-others

[A] - Coefficient Matrix [b] - Constant Vector [I] - Identity Matrix [P] - Price Vector [Q] - Quantity Vector

]

Table 4.4: Applications of matrices in demand-supply analysis
Application | Matrix role
Two-market equilibrium | Solves for prices using inverses
Multi-market systems | Captures relationships across several goods
Policy analysis | Evaluates effects of changes in demand or supply

4.4.4 Policy and planning applications

Governments and policymakers use matrix algebra to plan and allocate resources efficiently. Examples include:

- **Infrastructure planning**: Matrices can represent costs and resource flows across projects.
- **Social welfare programmes**: Matrices can model distribution of funds across sectors.
- **Macroeconomic forecasting**: Input-output matrices help predict how changes in one sector affect the whole economy.

Worked Example: A government allocates funds across healthcare, education, and transport. A matrix can represent the distribution, and adjustments can be made using matrix operations to reflect policy changes.

Fill in the blank: Policymakers use matrices to allocate __________ efficiently across competing sectors.

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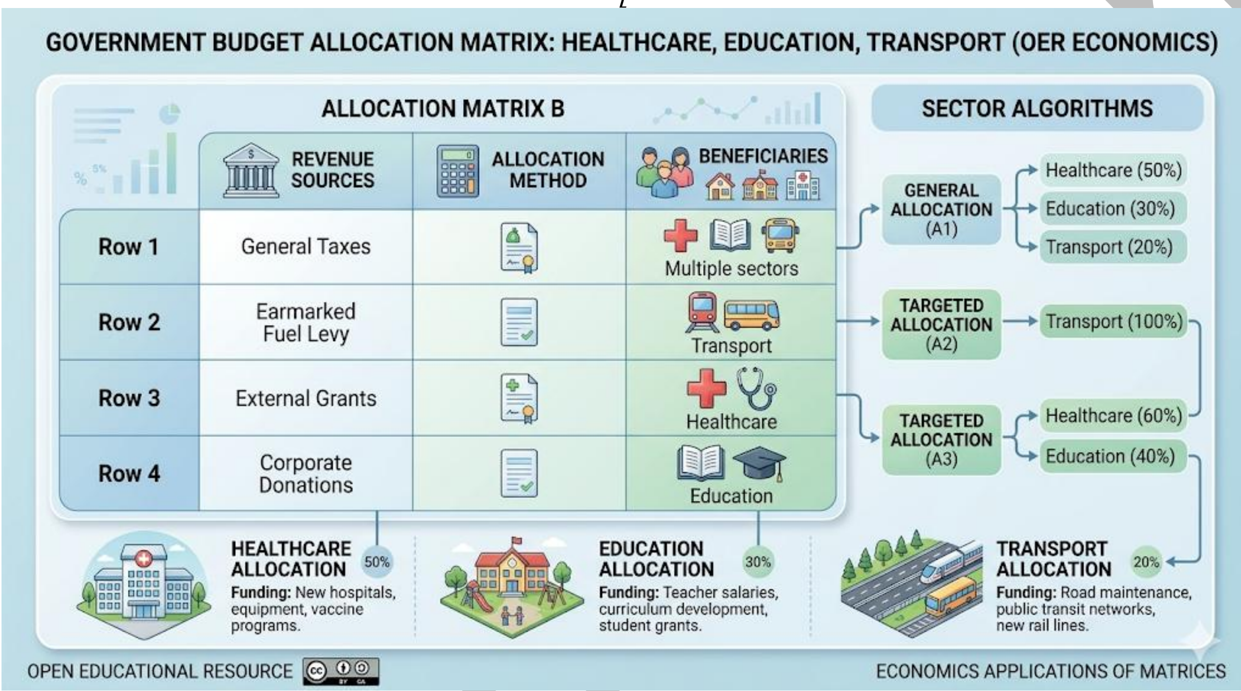

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Diagram showing a government budget allocation matrix across healthcare, education, and transport.

Figure 4.7: Policy applications of matrix algebra

Pilot questions 4.4

1. Explain input-output analysis in economics.
2. State the formula for the Leontief model.
3. How are matrices used in demand-supply analysis?
4. Give one example of policy application of matrix algebra.
5. What does the Leontief inverse represent?

Series Summary

This Series has explored linear and matrix algebra in economics, highlighting how these mathematical tools provide a structured way to analyse systems of equations and model relationships across markets and industries. We began with the fundamentals of linear equations and systems, examining methods of solution and their economic relevance. We then introduced matrices, defined their types, and practised basic operations such as addition, multiplication, and scalar manipulation. Following that, we studied determinants and inverses, focusing on their properties, conditions for existence, and their role in solving systems of equations. Finally, we applied matrix algebra to economic contexts, including input-output analysis, the Leontief model, demand-supply systems, and policy planning.

By completing this Series, you should now be able to define and solve systems of linear equations, perform and interpret matrix operations, calculate determinants and inverses, and apply these concepts to real economic models. In line with the Series Learning Outcomes, you are equipped to analyse equilibrium conditions, apply the Leontief model to inter-industry relationships, and evaluate resource allocation problems using matrix methods. This ensures you can confidently use algebraic tools to support economic reasoning and decision-making.

Glossary of Terms

Budget constraint: A linear equation showing the relationship between income, prices, and quantities of goods a consumer can purchase.

Coefficient matrix: The matrix formed from the coefficients of variables in a system of linear equations, used to represent the system compactly.

Determinant: A single number calculated from a square matrix that indicates whether the matrix has an inverse and helps in solving systems of equations.

Elimination method: A technique for solving systems of linear equations by adding or subtracting equations to remove one variable.

Final demand: The portion of output consumed by households, government, or exported, rather than used as input by other industries.

Identity matrix: A square matrix with ones on the diagonal and zeros elsewhere, which acts like the number 1 in matrix multiplication.

Input-output analysis: An economic method using matrices to show how the output of one industry is used as input in another.

Inverse of a matrix: A matrix that, when multiplied by the original matrix, gives the identity matrix. It exists only if the determinant is non-zero.

Leontief inverse: The matrix ((I - A)^{-1}) used in the Leontief model to calculate total output required to meet both intermediate and final demand.

Leontief model: A matrix-based economic model that calculates total output needed to meet both intermediate and final demand.

Linear equation: An equation in which variables appear to the power of one, often used to represent straight-line relationships.

Matrix: A rectangular arrangement of numbers in rows and columns, used to represent systems of equations and economic relationships.

Matrix multiplication: An operation where rows of one matrix are combined with columns of another, defined only when the number of columns in the first equals the number of rows in the second.

Scalar multiplication: Multiplying every entry in a matrix by a constant number.

Square matrix: A matrix with the same number of rows and columns.

Substitution method: A technique for solving systems of equations by solving one equation for a variable and substituting it into another.

System of linear equations: A set of two or more linear equations involving the same variables, solved to find values that satisfy all equations simultaneously.

Zero matrix: A matrix in which all entries are zero.

Concept Check Questions [Series 4]

Objective Questions

1. Which of the following best describes a square matrix?
   a) A matrix with more rows than columns
   b) A matrix with more columns than rows
   c) A matrix with the same number of rows and columns
   d) A matrix with all entries equal to zero
   (Linked to SLO 4.3)

2. The determinant of a 2×2 matrix (\begin{bmatrix} a & b \ c & d \end{bmatrix}) is:
   a) (ab + cd)
   b) (ad - bc)
   c) (ac - bd)
   d) (a + d)
   (Linked to SLO 4.5)

3. A matrix has an inverse if and only if:
   a) Its determinant is zero
   b) Its determinant is non-zero
   c) It is rectangular
   d) It has a row of zeros
   (Linked to SLO 4.6)

4. In the Leontief model, total output is calculated using:
   a) (X = A^{-1}D)
   b) (X = (I - A)^{-1}D)
   c) (X = AD)
   d) (X = (A - I)D)
   (Linked to SLO 4.7)

5. In demand-supply analysis, equilibrium prices are found by solving:
   a) (AX = B)

b) (AB = X)
c) (X = A + B)
d) (X = A - B)
(Linked to SLO 4.8)

Short-Answer Questions

1. Define a linear equation. (Linked to SLO 4.1)
2. State one method of solving systems of linear equations. (Linked to SLO 4.2)
3. Name two types of matrices. (Linked to SLO 4.3)
4. Write the formula for the inverse of a 2×2 matrix. (Linked to SLO 4.6)
5. Explain the role of the Leontief inverse in input-output analysis. (Linked to SLO 4.7)

Long-Answer Questions

1. Analyse the economic relevance of systems of linear equations, giving examples from market equilibrium and budget constraints. (Linked to SLO 4.1, SLO 4.2)
   - **Basic response**: Describe systems of linear equations and explain how they are used to find equilibrium in supply and demand or to represent budget constraints.
   - **Value-added response**: Extend by showing how systems of equations can represent multiple markets simultaneously, and discuss why matrix methods are preferred for larger systems.
2. Evaluate the importance of determinants and inverses in solving economic problems. (Linked to SLO 4.5, SLO 4.6)
   - **Basic response**: Define determinants and inverses, and explain how they are used to solve systems of equations.
   - **Value-added response**: Apply these concepts to input-output analysis, showing how the Leontief inverse helps calculate total output in an economy.
3. Apply matrix algebra to design a simple input-output model for a two-sector economy. (Linked to SLO 4.7, SLO 4.8)
   - **Basic response**: Outline the input-output matrix and show how it represents flows between two sectors.
   - **Value-added response**: Solve for total output using the Leontief model, interpret the results, and discuss implications for resource allocation and policy planning.

Answers to Concept Check Questions [Series 4]

Objective Questions

1. c) A matrix with the same number of rows and columns
2. b) (ad - bc)
3. b) Its determinant is non-zero
4. b) (X = (I - A)^{-1}D)
5. a) (AX = B)

Short-Answer Questions

1. A linear equation is an equation in which variables appear to the power of one, often representing straight-line relationships.
2. One method is the elimination method.
3. Examples include square matrix and identity matrix.
4. ($A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$).
5. The Leontief inverse ($(I - A)^{-1}$) is used to calculate the total output required to meet both intermediate and final demand.

Long-Answer Questions

1. **Basic response**: Systems of linear equations are used to represent relationships such as supply and demand or budget constraints, and solving them gives equilibrium values.
   Value-added response: They can also represent multiple markets simultaneously, and matrix methods provide efficient solutions for larger systems.

2. **Basic response**: Determinants and inverses are tools for solving systems of equations, ensuring unique solutions when the determinant is non-zero.
   Value-added response: In economics, they are applied in input-output analysis, where the Leontief inverse helps calculate total output across industries.

3. **Basic response**: A two-sector input-output model uses a matrix to represent flows of goods between sectors.
   Value-added response: By applying the Leontief model, total output can be calculated, showing how sectors depend on each other and guiding policy decisions on resource allocation.

Answers to Pilot Questions [Series 4]

Part 4.1: Fundamentals of Linear Algebra

1. A linear equation is an equation in which variables appear to the power of one.
2. A system of linear equations is a set of two or more linear equations involving the same variables.
3. One method is the elimination method.
4. The solution is ($x = 3, y = 3$).
5. One economic application is market equilibrium analysis.
6. Systems with three or more variables are often solved more efficiently using matrix methods.

Part 4.2: Introduction to Matrices

1. A matrix is a rectangular arrangement of numbers in rows and columns.
2. Two types are square matrix and zero matrix.
3. Matrix multiplication is defined when the number of columns in the first matrix equals the number of rows in the second.
4. The product is ($\begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$).
5. One economic application is input-output analysis.

Part 4.3: Determinants and Inverses of Matrices

1. A determinant is a scalar value associated with a square matrix.
2. If two rows of a matrix are identical, the determinant is zero.
3. The inverse of a 2×2 matrix is (A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \ -c & a \end{bmatrix}).
4. A matrix has an inverse only if its determinant is non-zero.
5. One economic application is solving systems of equations in input-output analysis.

Part 4.4: Applications of Matrix Algebra in Economics

1. Input-output analysis shows how the output of one industry is used as input in another.
2. The formula is (X = (I - A)^{-1}D).
3. Matrices are used to represent and solve systems of demand and supply equations across multiple markets.
4. One example is government budget allocation across sectors.
5. The Leontief inverse represents the matrix used to calculate total output required to meet both intermediate and final demand.

References and Attributions [Series 4]

This section compiles references, attributions, and illustration notes for **Series 4: Linear and Matrix Algebra in Economics**. It should be placed at the end of the Series during final compilation.

Figures

- *Figure 4.1: Graphical solution of a system of linear equations* AI-generated suggestion: “Generate a diagram showing two straight lines intersecting at a point to represent solution of a system of linear equations.” Suggested OER search string: “system of linear equations graphical solution OER mathematics.”

- *Figure 4.2: Graphical representation of a system of three linear equations* AI-generated suggestion: “Generate a 3D diagram showing three planes intersecting at a point to represent solution of three linear equations.” Suggested OER search string: “system of three linear equations graphical solution OER mathematics.”

- *Figure 4.3: Examples of different types of matrices* AI-generated suggestion: “Generate a diagram showing examples of square, rectangular, identity, and zero matrices.” Suggested OER search string: “types of matrices examples OER mathematics.”

- *Figure 4.4: Determinant of a 2×2 matrix* AI-generated suggestion: “Generate a diagram showing calculation of determinant for a 2×2 matrix.” Suggested OER search string: “determinant of 2x2 matrix OER mathematics.”

- *Figure 4.5: Inverse of a 2×2 matrix* AI-generated suggestion: “Generate a diagram showing calculation of inverse of a 2×2 matrix.” Suggested OER search string: “inverse of 2x2 matrix OER mathematics.”

- *Figure 4.6: Input-output relationships between two sectors* AI-generated suggestion: "Generate a diagram showing two industries with arrows indicating flows of outputs as inputs between them." Suggested OER search string: "input-output analysis diagram OER economics."
- *Figure 4.7: Policy applications of matrix algebra* AI-generated suggestion: "Generate a diagram showing a government budget allocation matrix across healthcare, education, and transport." Suggested OER search string: "matrix applications government budget allocation OER economics."

Tables

- *Table 4.1: Methods of solving systems of linear equations* AI-generated suggestion: "Create a table summarising methods of solving systems of linear equations with examples." Suggested OER search string: "methods of solving systems of linear equations OER mathematics."
- *Table 4.2: Basic matrix operations with examples* AI-generated suggestion: "Create a table summarising basic matrix operations with worked examples." Suggested OER search string: "matrix operations addition multiplication OER mathematics."
- *Table 4.3: Applications of determinants and inverses in economics* AI-generated suggestion: "Create a table showing applications of determinants and inverses in economics." Suggested OER search string: "applications of determinants and inverses OER economics."
- *Table 4.4: Applications of matrices in demand-supply analysis* AI-generated suggestion: "Create a table showing applications of matrices in demand-supply analysis." Suggested OER search string: "matrix applications demand supply equilibrium OER economics."

Boxes

- *Box 4.1: Economic applications of linear systems* AI-generated suggestion: "Create a text box summarising economic applications of linear systems." Suggested OER search string: "economic applications of linear systems OER economics."
- *Box 4.2: Economic uses of matrices* AI-generated suggestion: "Create a text box summarising economic uses of matrices." Suggested OER search string: "economic applications of matrices OER economics."
- *Box 4.3: Properties of determinants* AI-generated suggestion: "Create a text box summarising properties of determinants." Suggested OER search string: "properties of determinants OER mathematics."
- *Box 4.4: Key features of the Leontief model* AI-generated suggestion: "Create a text box summarising key features of the Leontief model." Suggested OER search string: "Leontief model input-output analysis OER economics."

General References

- Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). New York: McGraw-Hill.
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- OpenStax. (2016). *Calculus Volume 1–3*. Rice University. Retrieved from https://openstax.org/subjects/math
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Course code: ECO 104

Course title: Introductory Mathematics II

Course credit load: 2 Units

Series 1: Derivatives of Trigonometric Functions and Expansions

Series 2: Sequences and Series in Economic Applications

Series 3: Partial and Total Derivatives with Optimisation

Series 4: Linear and Matrix Algebra in Economics

Series 5: Mathematical Relationships and Graphical Analysis of Economic Concepts

Suggested Outline

Part 5.1: Mathematical relationships in economics

- Definition and types of mathematical relationships (linear, non-linear, functional forms)
- Direct and inverse relationships between economic variables
- Economic examples (price–quantity, income–consumption, production functions)

Part 5.2: Graphical representation of economic concepts

- Cartesian plane and plotting economic functions
- Graphical analysis of demand and supply curves
- Shifts versus movements along curves

Part 5.3: Advanced graphical analysis

- Elasticity and slope interpretation
- Indifference curves and budget lines
- Isoquants and isocost lines in production theory

Part 5.4: Applications of mathematical and graphical analysis

- Market equilibrium analysis using graphs
- Comparative statics and policy evaluation
- Dynamic graphical illustrations (growth trends, cycles)

Introduction

Mathematical relationships and graphical analysis are essential tools in economics, as they allow us to represent abstract concepts in clear and visual ways. While equations describe the precise link between variables, graphs provide intuitive illustrations that make economic reasoning more accessible. This Series will help you connect mathematical expressions with their graphical counterparts, enabling you to interpret and analyse economic concepts more effectively.
The aim of this Series is to equip you with the ability to describe, represent, and interpret economic relationships both mathematically and graphically, and to apply these skills in analysing equilibrium, elasticity, consumer behaviour, and production decisions.
We shall commence this Series by examining different types of mathematical relationships and their relevance to economics. Next, we will focus on graphical representation, learning how to plot and interpret demand and supply curves. Following that, we will explore advanced graphical tools such as elasticity, indifference curves, and isoquants. Finally, we will conclude with applications of mathematical and graphical analysis in market equilibrium, comparative statics, and dynamic economic illustrations.

Series Learning Outcomes

Upon completing this Series, you should be able to:
SLO 5.1 **define** different types of mathematical relationships such as linear, non-linear, direct, and inverse, and explain their relevance to economics,
SLO 5.2 **illustrate** economic concepts such as demand and supply using graphical representation on the Cartesian plane,
SLO 5.3 **distinguish** between movements along a curve and shifts of a curve in graphical analysis,
SLO 5.4 **interpret** elasticity and slope in graphical terms, linking them to consumer and producer behaviour,
SLO 5.5 **analyse** indifference curves and budget lines to explain consumer choice,
SLO 5.6 **apply** isoquants and isocost lines to production theory and resource allocation,
SLO 5.7 **demonstrate** how mathematical and graphical tools can be used to determine market equilibrium,
SLO 5.8 **evaluate** comparative statics and policy changes using graphical analysis, and
SLO 5.9 **represent** dynamic economic concepts such as growth trends and cycles through appropriate graphical illustrations.

5.1 Mathematical Relationships in Economics

5.1.1 Definition and types of mathematical relationships

A **mathematical relationship** expresses how one variable changes in response to another. In economics, these relationships are often written as functions, equations, or inequalities, and they help us describe consumer behaviour, production processes, and market interactions.
Types of relationships include:

- **Linear relationship**: A straight-line relationship where change in one variable leads to a proportional change in another. Example: (Q = a + bP), where (Q) is quantity and (P) is price.
- **Non-linear relationship**: A curved relationship where changes are not proportional. Example: (Q = aP^2).
- **Direct relationship**: Both variables move in the same direction. Example: income and consumption.
- **Inverse relationship**: Variables move in opposite directions. Example: price and demand.
- **Functional relationships**: Expressed as (Y = f(X)), meaning (Y) depends on (X).

Fill in the blank: When two variables move in opposite directions, they have an __________ relationship.

[

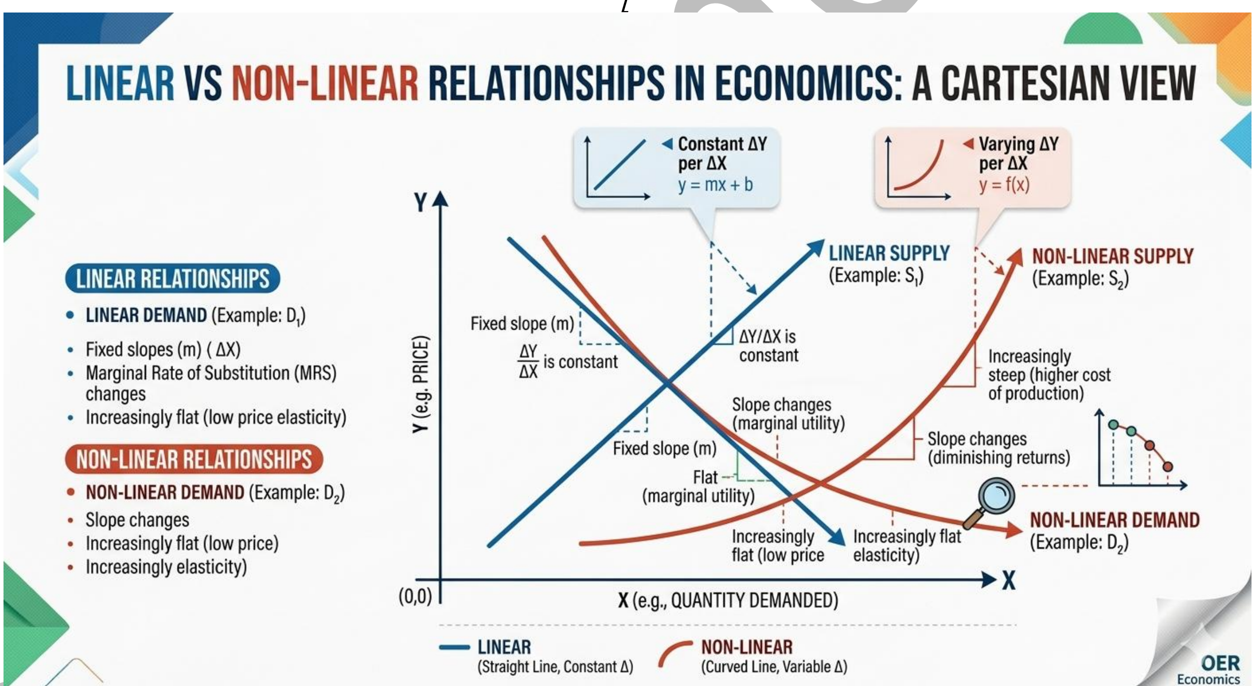

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Diagram showing linear and non-linear curves on a Cartesian plane.
Figure 5.1: Linear and non-linear relationships

5.1.2 Direct and inverse relationships between economic variables

Economic variables often exhibit predictable patterns:

- **Direct relationship**: As income increases, consumption also increases.

- **Inverse relationship**: As price rises, demand falls.

Worked Example: If the demand function is (Q = 100 - 2P), then as price (P) increases, quantity demanded (Q) decreases. This illustrates an inverse relationship.
Fill in the blank: The demand function (Q = 100 - 2P) shows an __________ relationship between price and quantity demanded.

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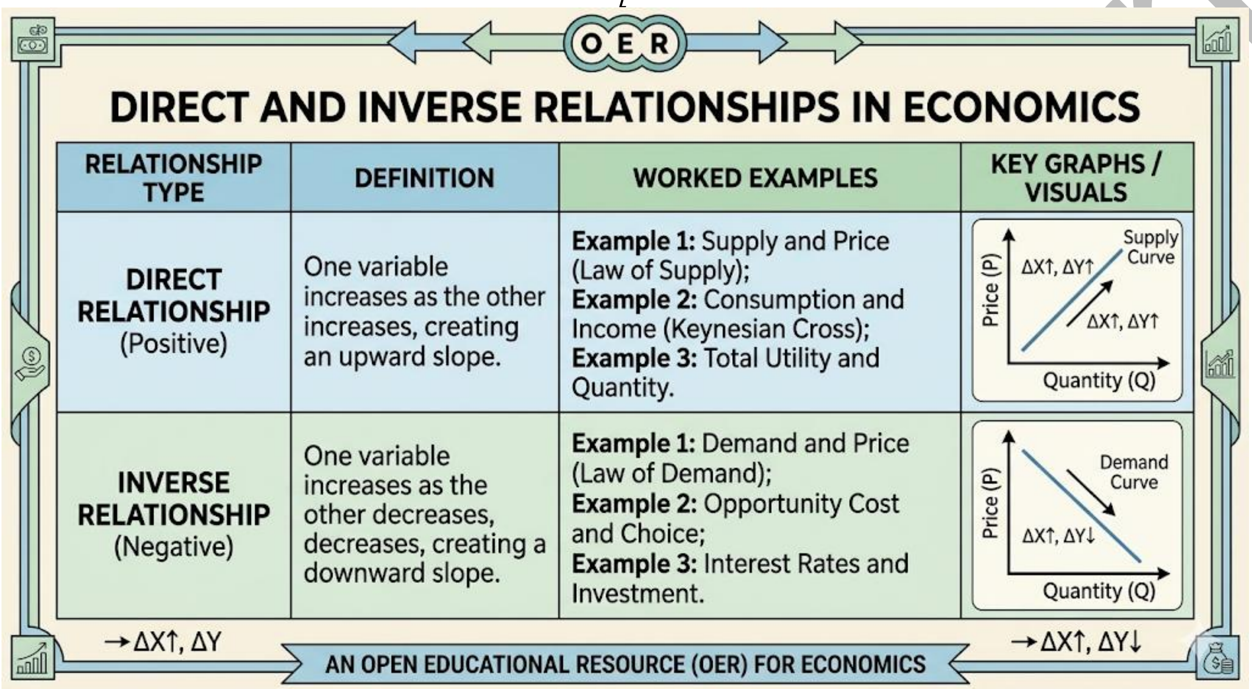
O E R

DIRECT AND INVERSE RELATIONSHIPS IN ECONOMICS

RELATIONSHIP TYPE	DEFINITION	WORKED EXAMPLES	KEY GRAPHS / VISUALS
DIRECT RELATIONSHIP (Positive)	One variable increases as the other increases, creating an upward slope.	**Example 1:** Supply and Price (Law of Supply); **Example 2:** Consumption and Income (Keynesian Cross); **Example 3:** Total Utility and Quantity.	Price (P); Supply Curve; ΔX↑, ΔY↑; ΔX↑, ΔY↑; Quantity (Q)
INVERSE RELATIONSHIP (Negative)	One variable increases as the other decreases, decreases, creating a downward slope.	**Example 1:** Demand and Price (Law of Demand); **Example 2:** Opportunity Cost and Choice; **Example 3:** Interest Rates and Investment.	Price (P); Demand Curve; ΔX↑, ΔY↓; Quantity (Q)

→ΔX↑, ΔY

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→ΔX↑, ΔY↓

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Table 5.1: Examples of direct and inverse relationships in economics

Relationship | Example | Economic interpretation

Direct | Income and consumption | Higher income leads to higher consumption

Inverse | Price and demand | Higher price reduces demand

5.1.3 Linear and non-linear functions in economics

Economists use both linear and non-linear functions to model behaviour:

- **Linear functions** are simple and easy to interpret. They are often used for demand and supply functions.
- **Non-linear functions** capture more complex relationships, such as diminishing returns in production or utility.

Worked Example:

- Linear demand: (Q = 50 - P).
- Non-linear utility: (U = \sqrt{X}), showing diminishing marginal utility.

Fill in the blank: A utility function that shows diminishing marginal utility is often __________ in shape.

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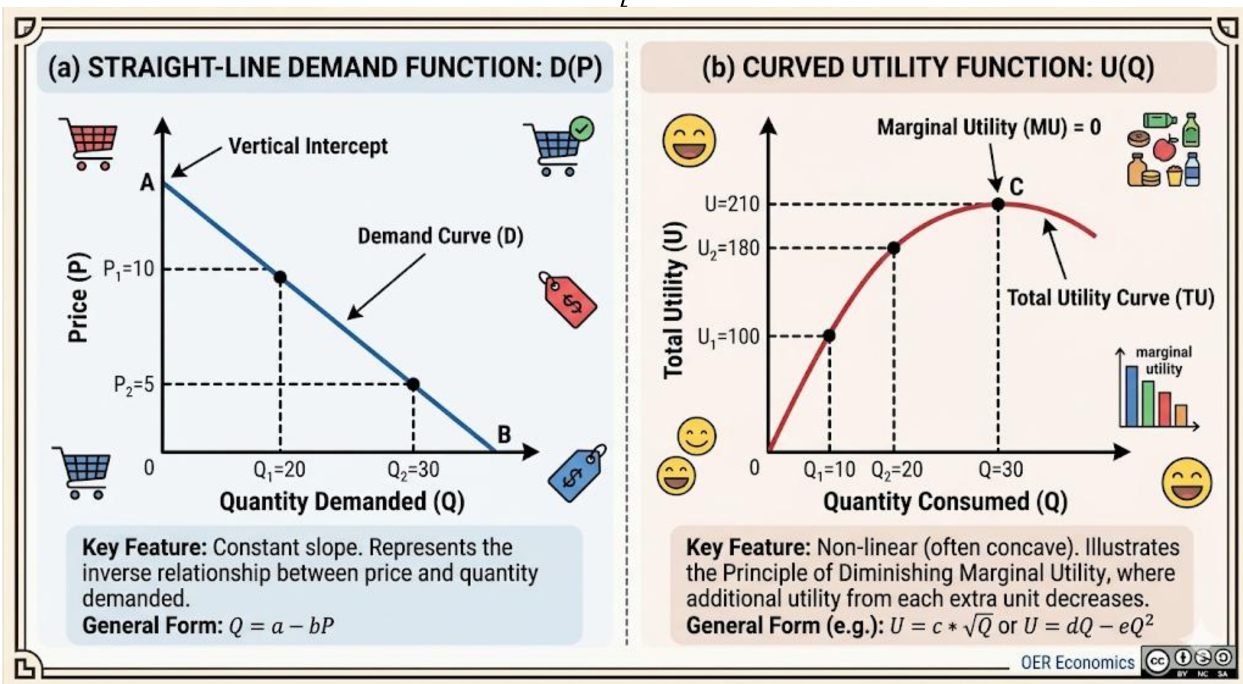

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Diagram showing a straight-line demand function and a curved utility function.

Figure 5.2: Linear demand and non-linear utility functions

5.1.4 Economic examples of mathematical relationships

Mathematical relationships are used to model key economic concepts:

- **Price–quantity relationship**: Demand and supply functions show how price affects quantity.
- **Income–consumption relationship**: Consumption functions show how income affects spending.
- **Production functions**: Show how inputs such as labour and capital affect output.
- **Cost functions**: Relate output levels to total, average, and marginal costs.

Worked Example: A simple production function might be (Q = 5L), meaning each unit of labour produces 5 units of output. This is a linear relationship.
Fill in the blank: A production function that shows output as a function of labour and capital is called a __________ function.

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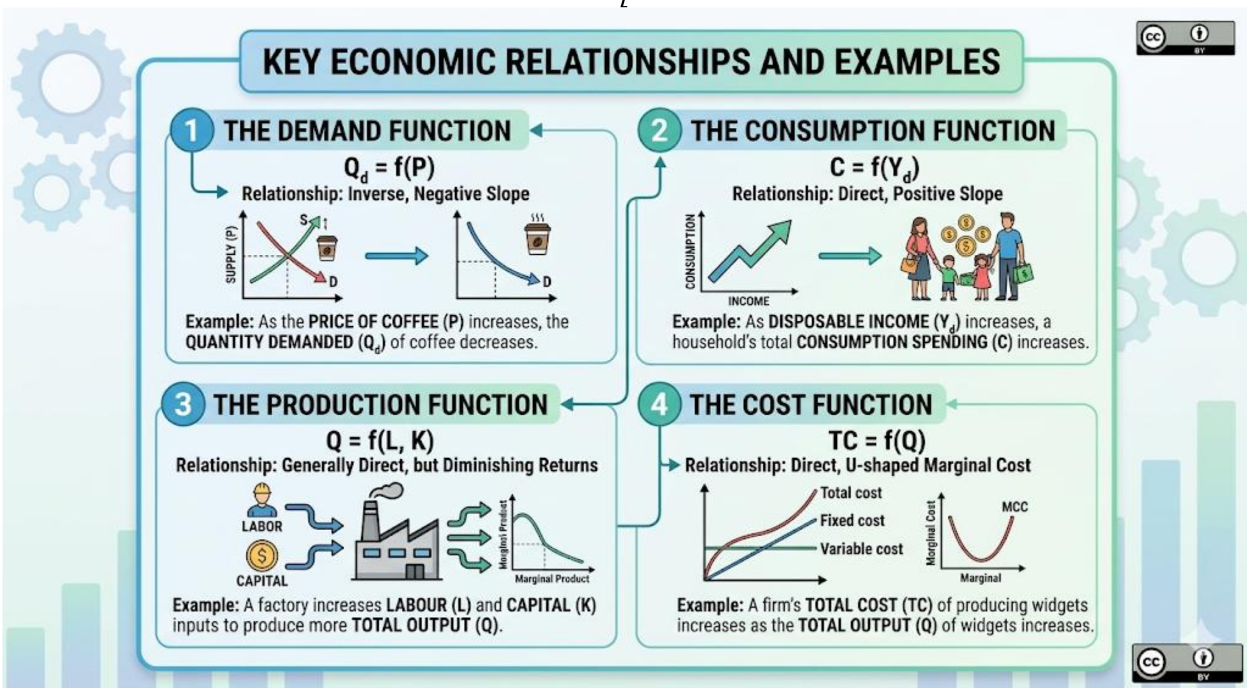

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Box 5.1: Key economic relationships

Relationship | Function form | Example

Price–quantity | Demand function | (Q = 100 - 2P)

Income–consumption | Consumption function | (C = a + bY)

Production | Production function | (Q = f(L, K))

Cost | Cost function | (C = f(Q))

5.1.5 Importance of mathematical relationships in economics

Mathematical relationships are not just abstract tools; they have practical importance:

- They allow economists to **predict outcomes** when variables change.
- They help in **policy analysis**, showing how taxes or subsidies affect demand and supply.
- They provide a **foundation for graphical analysis**, making concepts easier to visualise.
- They are essential for **modelling dynamic behaviour**, such as growth and cycles.

Worked Example: A subsidy that lowers production costs shifts the supply function, increasing output at every price level.
Fill in the blank: Mathematical relationships help economists to __________ the effects of policy changes.

Pilot questions 5.1 (Expanded)

1. Define a mathematical relationship in economics.
2. Distinguish between linear and non-linear functions with examples.
3. Give one example each of a direct and an inverse relationship.

4. Explain how a consumption function represents the relationship between income and spending.
5. Write a simple production function involving labour and capital.
6. Why are mathematical relationships important in economic analysis?

5.2 Graphical Representation of Economic Concepts

5.2.1 Cartesian plane and plotting economic functions

The **Cartesian plane** is a two-dimensional grid defined by the horizontal axis (x-axis) and vertical axis (y-axis). In economics, it is used to plot functions that represent relationships between variables such as price and quantity. Each point on the plane corresponds to a pair of values ((x, y)).
For example, the demand function (Q = 100 - 2P) can be plotted by assigning values to price (P) and calculating the corresponding quantity demanded (Q). The resulting points can then be joined to form the demand curve.
Fill in the blank: The horizontal axis in the Cartesian plane is called the __________ axis.

[

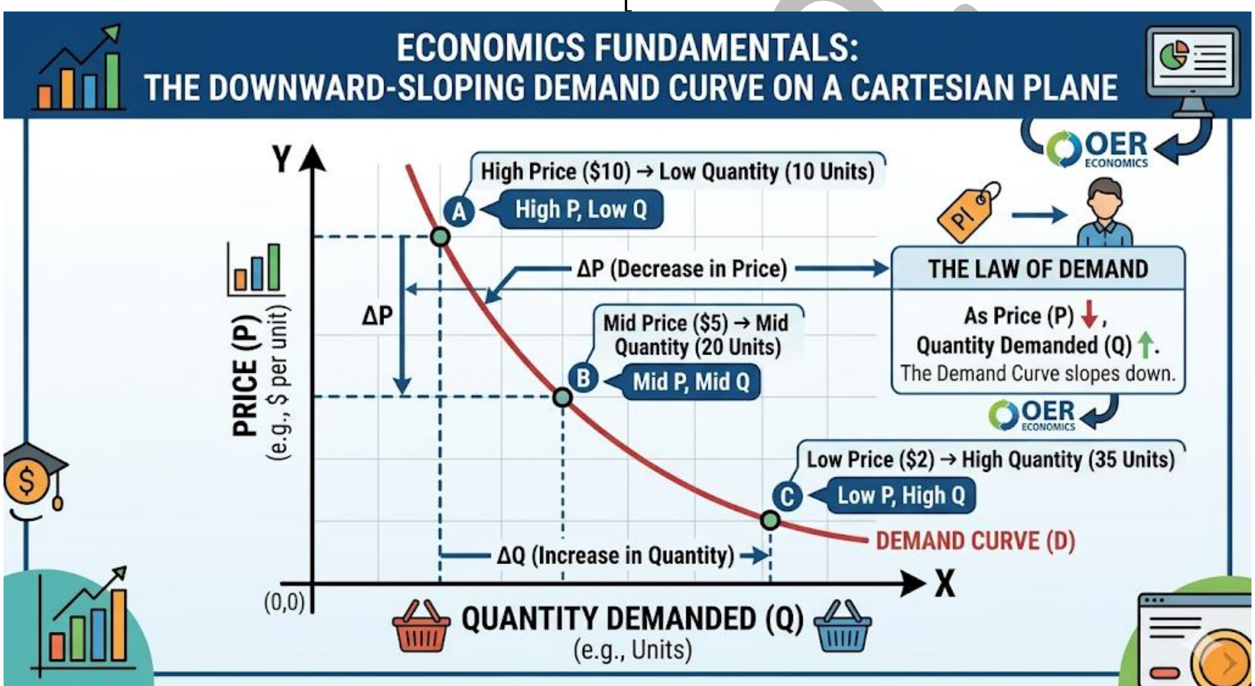

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Diagram showing a Cartesian plane with labelled axes and a plotted demand curve.

Figure 5.2: Demand curve on a Cartesian plane

5.2.2 Graphical analysis of demand and supply curves

Demand and supply curves are fundamental tools in economics:

- **Demand curve**: Shows the inverse relationship between price and quantity demanded. It slopes downward.
- **Supply curve**: Shows the direct relationship between price and quantity supplied. It slopes upward.

The intersection of demand and supply curves represents **market equilibrium**, where quantity demanded equals quantity supplied.
Worked Example: Demand (Q_d = 100 - 2P), Supply (Q_s = 20 + 3P). Plotting both curves shows equilibrium at the point where they intersect.
Fill in the blank: The point where demand and supply curves intersect is called __________.

[

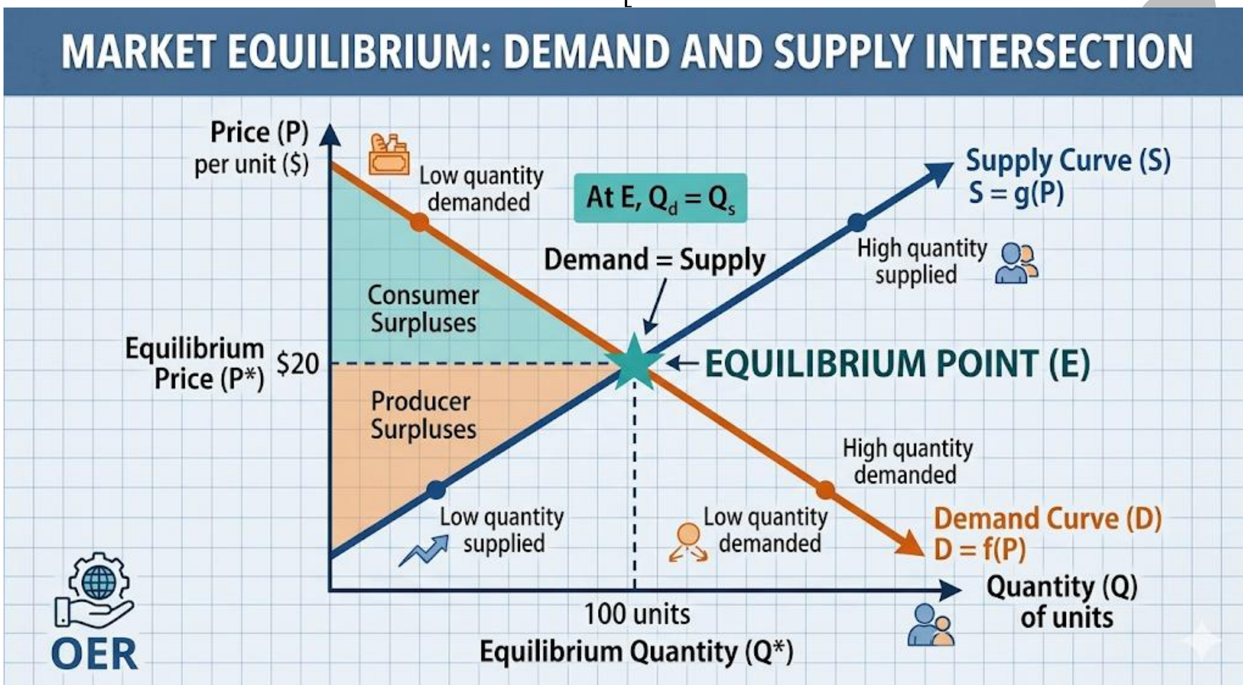

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Diagram showing demand and supply curves intersecting at equilibrium.
Figure 5.3: Market equilibrium on a demand-supply graph

5.2.3 Shifts versus movements along curves

It is important to distinguish between **movements along a curve** and **shifts of a curve**:

- **Movement along a curve**: Occurs when the variable on the axis changes. For example, a change in price causes movement along the demand curve.
- **Shift of a curve**: Occurs when other factors change, such as income, tastes, or technology. This results in the entire curve moving to a new position.

Worked Example: If consumer income increases, the demand curve shifts outward, indicating higher demand at every price level.
Fill in the blank: A change in consumer income causes a __________ of the demand curve.

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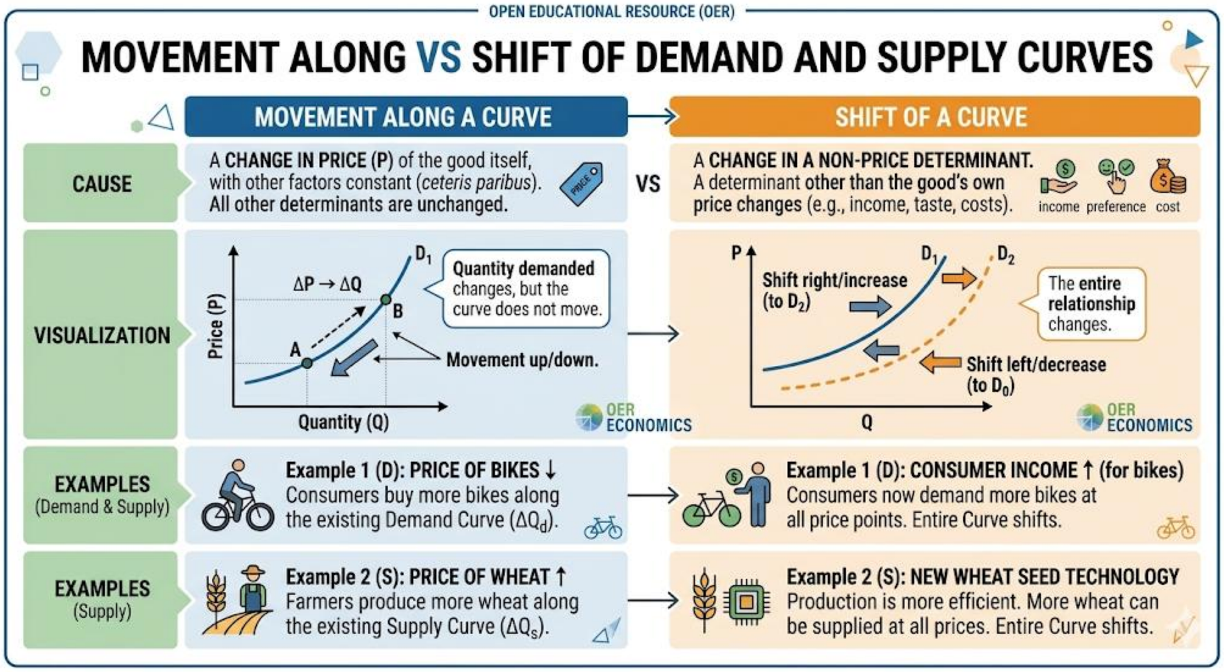

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Table 5.2: Distinction between movements and shifts

Type | Cause | Effect

Movement along curve | Change in price | Change in quantity demanded or supplied

Shift of curve | Change in income, technology, or preferences | Entire curve moves outward or inward

5.2.4 Economic interpretation of graphs

Graphs are not just visual aids; they provide meaningful insights:

- They show equilibrium conditions clearly.
- They illustrate the impact of policy changes (e.g., taxes shifting supply curves).
- They help compare static and dynamic changes in markets.

Worked Example: A tax on producers shifts the supply curve upward, leading to a new equilibrium with higher price and lower quantity.

Fill in the blank: A tax on producers causes the supply curve to __________.

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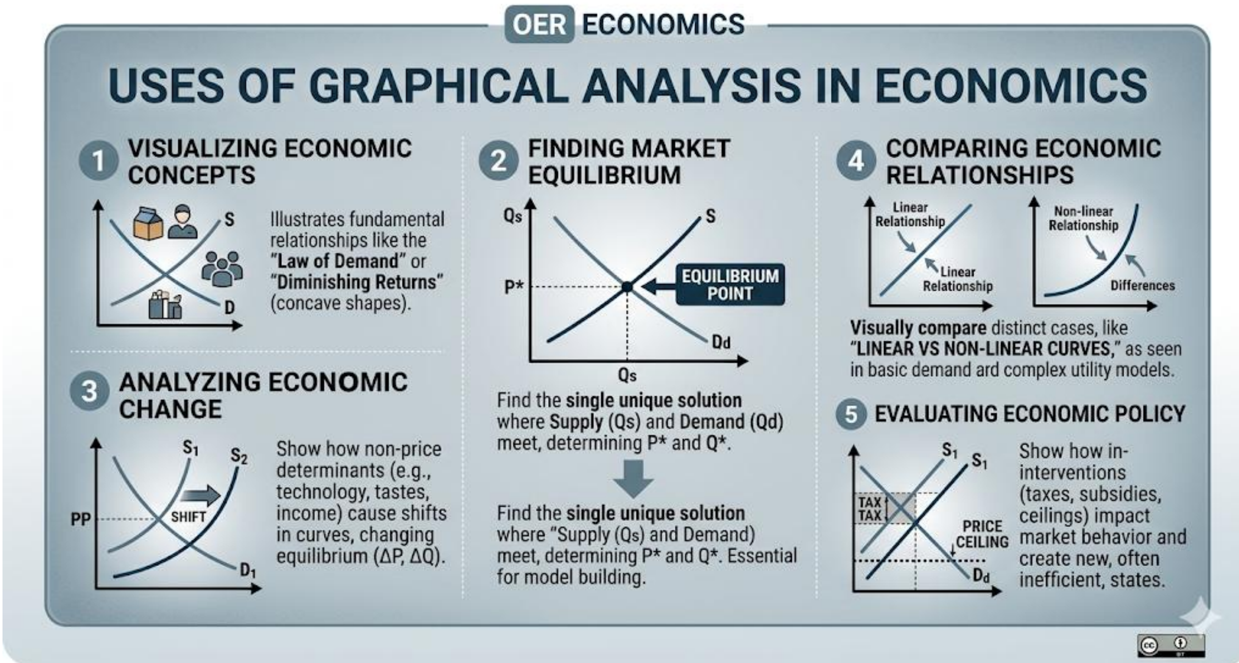

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Box 5.2: Uses of graphical analysis in economics

Application | Explanation

Equilibrium analysis | Shows where demand equals supply

Policy evaluation | Illustrates effects of taxes or subsidies

Comparative statics | Compares old and new equilibrium positions

Pilot questions 5.2

1. What is the Cartesian plane used for in economics?
2. State the difference between demand and supply curves.
3. What does the intersection of demand and supply curves represent?
4. Distinguish between a movement along a curve and a shift of a curve.
5. How does a tax on producers affect the supply curve?

5.3 Advanced Graphical Analysis

5.3.1 Elasticity and slope interpretation

Elasticity is a central concept in economics because it measures the **responsiveness** of one variable to changes in another. Graphically, elasticity is closely related to the slope of demand and supply curves, but it is not identical.

- **Slope**: Measures the absolute change in one variable relative to another. For example, in the demand function (Q = 100 - 2P), the slope is (-2), meaning that for every unit increase in price, quantity demanded falls by 2 units.

- **Elasticity**: Adjusts slope by considering proportional changes. It is calculated as:
  $$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$
  This formula shows that elasticity varies along the curve, even if the slope is constant.

Worked Example: For ($Q = 100 - 2P$), at ($P = 20$), ($Q = 60$). Elasticity is ($-2 \cdot \frac{20}{60} = -0.67$). This means demand is inelastic at this point.
Fill in the blank: A flatter demand curve indicates __________ elasticity.

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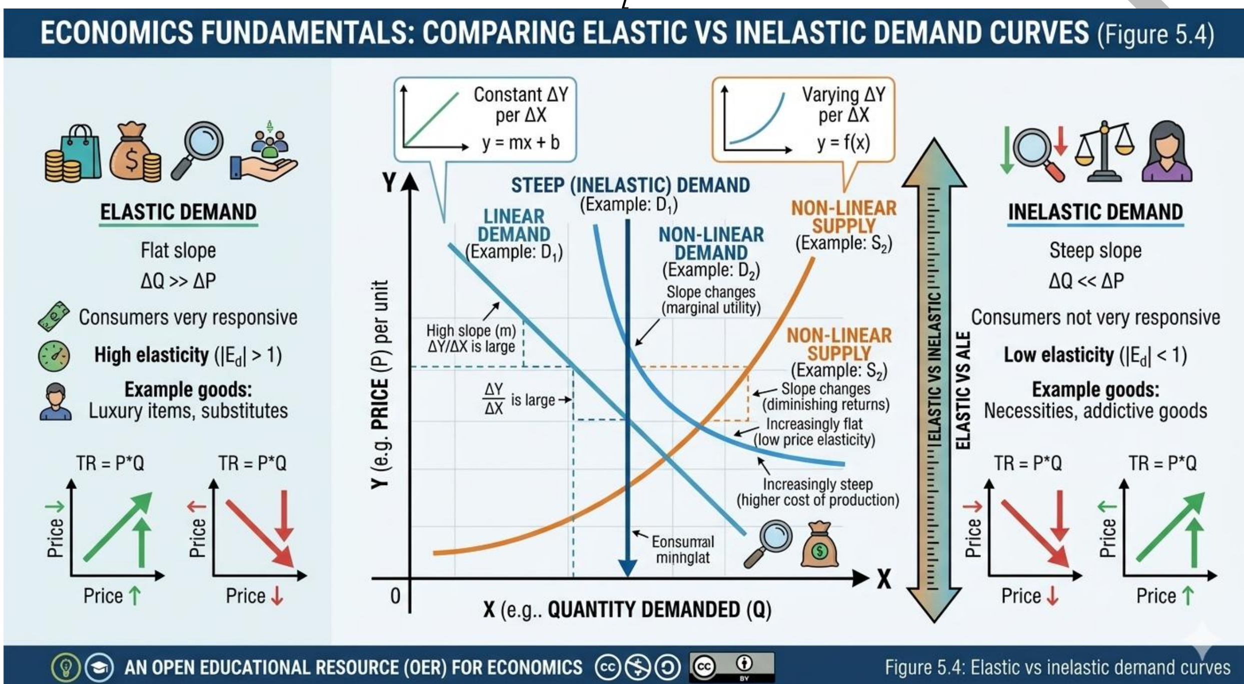

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Diagram showing two demand curves, one steep (inelastic) and one flat (elastic).
Figure 5.4: Elastic vs inelastic demand curves

5.3.2 Indifference curves and budget lines

Indifference curves and budget lines are graphical tools used to analyse consumer choice.

- **Indifference curves**: Represent combinations of two goods that yield the same level of satisfaction. They slope downward and are convex to the origin, reflecting diminishing marginal rate of substitution (MRS).
- **Budget lines**: Show all combinations of goods a consumer can afford given income and prices. The slope is the ratio of prices, ($-P_x/P_y$).

Consumer equilibrium occurs where the budget line is tangent to an indifference curve. At this point, the consumer maximises utility subject to their budget constraint.
Worked Example: If income is 100, price of good X is 10, and price of good Y is 20, the budget line is ($10X + 20Y = 100$). The equilibrium occurs where this line touches the highest attainable indifference curve.
Fill in the blank: The consumer's equilibrium occurs where the budget line is __________ to an indifference curve.

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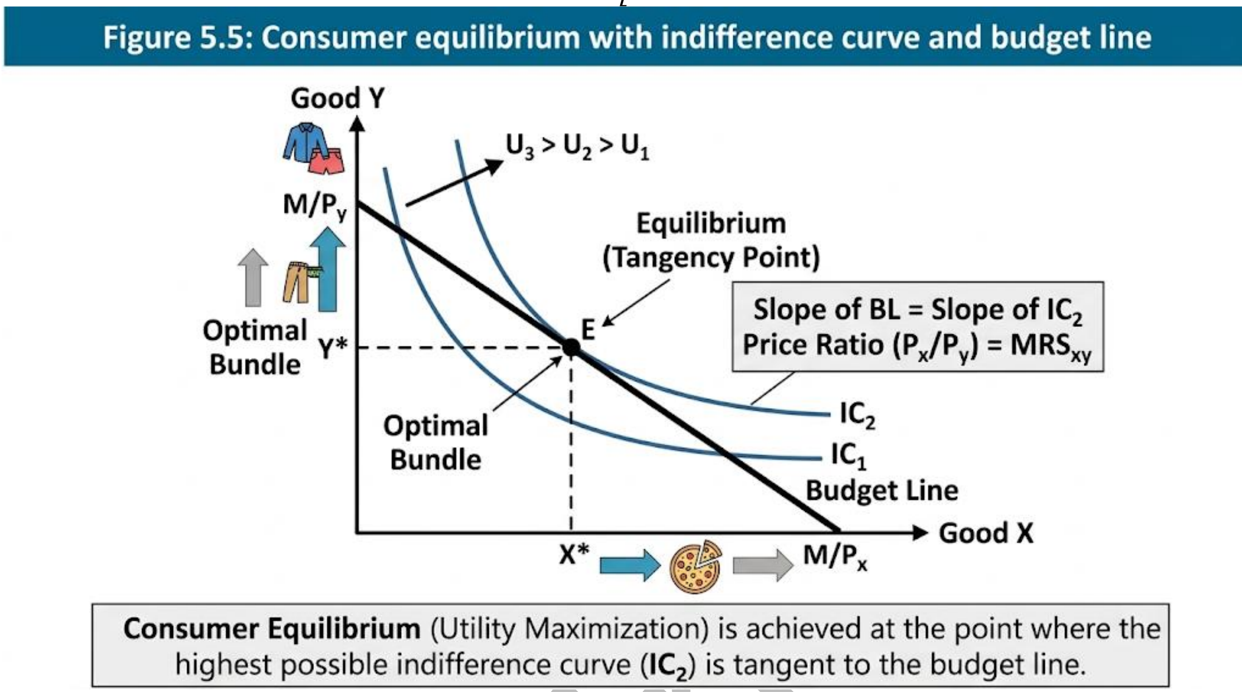

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Diagram showing indifference curves and a budget line tangent at equilibrium.

Figure 5.5: Consumer equilibrium with indifference curve and budget line

5.3.3 Isoquants and isocost lines in production theory

Isoquants and isocost lines are the production-side equivalents of indifference curves and budget lines.

- **Isoquants**: Show combinations of inputs (labour and capital) that produce the same level of output. They slope downward and are convex, reflecting diminishing marginal rate of technical substitution (MRTS).
- **Isocost lines**: Show all combinations of inputs that can be purchased for a given cost. The slope is the ratio of input prices, (-w/r), where (w) is the wage rate and (r) is the rental rate of capital.

Producer equilibrium occurs where an isoquant is tangent to an isocost line. This represents the least-cost combination of inputs to produce a given output.

Worked Example: If total cost is 100, wage rate is 10, and rental rate of capital is 20, the isocost line is (10L + 20K = 100). Tangency with an isoquant gives the optimal input mix.

Fill in the blank: Producer equilibrium occurs where an isoquant is __________ to an isocost line.

[

Figure 5.6: Producer equilibrium with isoquant and isocost line

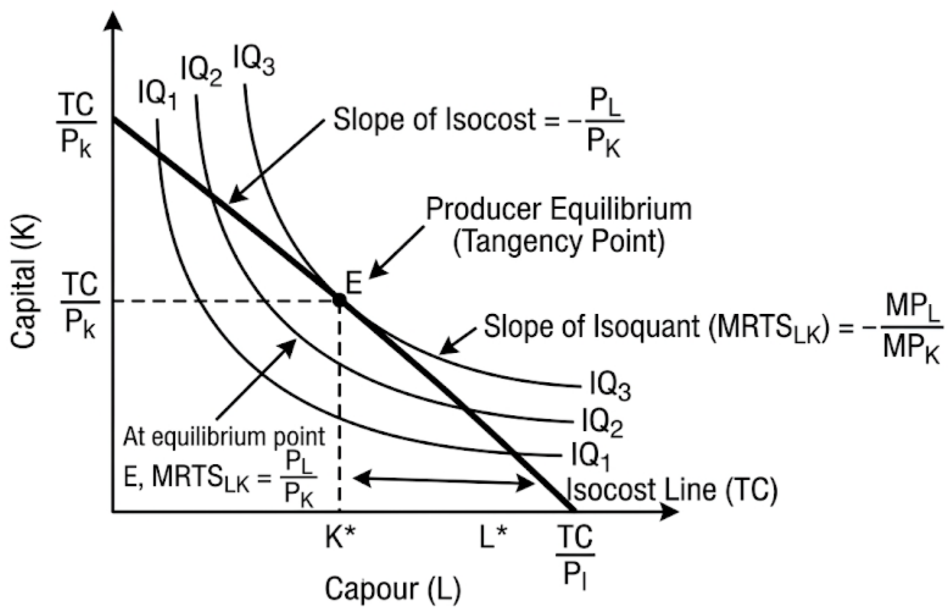

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Diagram showing isoquants and an isocost line tangent at equilibrium.

Figure 5.6: Producer equilibrium with isoquant and isocost line

5.3.4 Applications of advanced graphical tools

Advanced graphical analysis provides deeper insights into economic behaviour:

- **Consumer choice**: Indifference curves and budget lines explain how consumers allocate income between goods.
- **Production decisions**: Isoquants and isocost lines show efficient input combinations for firms.
- **Policy analysis**: Elasticity graphs illustrate the impact of taxes, subsidies, and price controls.
- **Comparative statics**: Graphs show how equilibrium shifts when external conditions change.

Worked Example: A subsidy on good X rotates the budget line outward, allowing the consumer to reach a higher indifference curve and achieve greater utility.
Fill in the blank: A subsidy on a good rotates the budget line __________.

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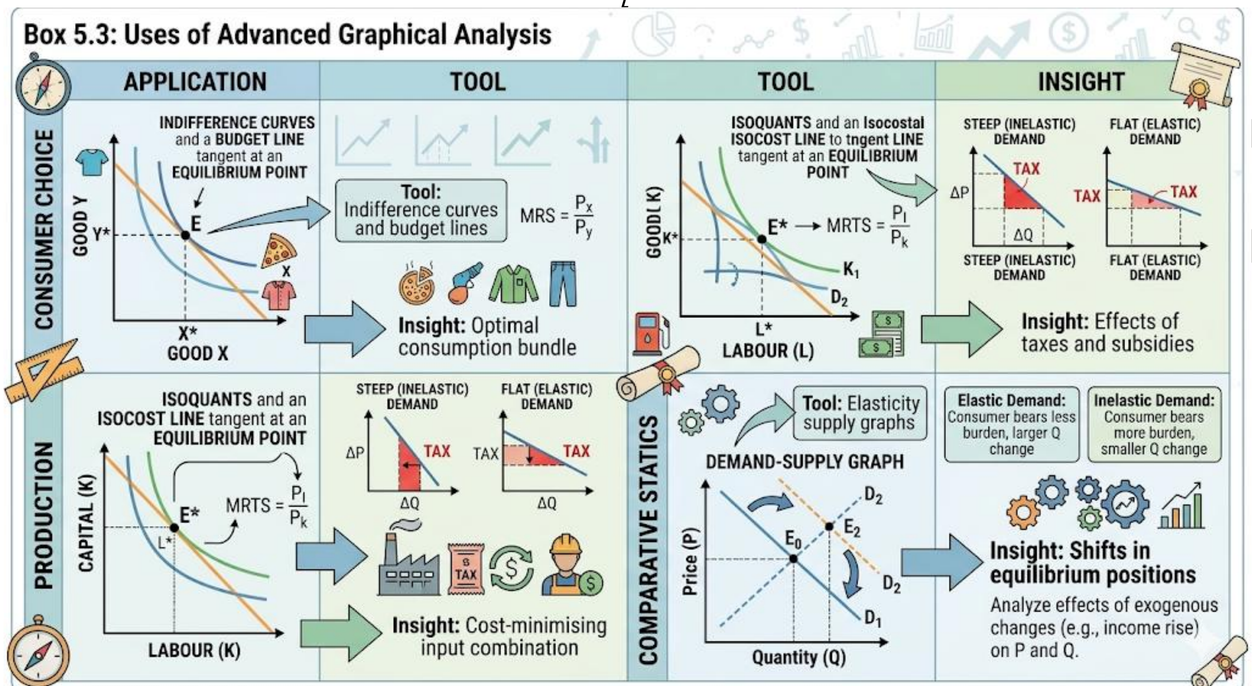

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Box 5.3: Uses of advanced graphical analysis

Application | Tool | Insight

Consumer choice | Indifference curves and budget lines | Optimal consumption bundle

Production | Isoquants and isocost lines | Cost-minimising input combination

Policy | Elasticity graphs | Effects of taxes and subsidies

Comparative statics | Demand-supply graphs | Shifts in equilibrium positions

Pilot questions 5.3

1. What does elasticity measure in economics, and how is it calculated?
2. Explain the difference between slope and elasticity using a demand function example.
3. Define an indifference curve and explain its shape.
4. What condition represents consumer equilibrium in graphical analysis?
5. Distinguish between isoquants and isocost lines in production theory.
6. Explain producer equilibrium using the tangency condition.
7. How does a subsidy affect the budget line in consumer analysis?
8. Give one example of how elasticity graphs can be used in policy evaluation.

5.4 Applications of Mathematical and Graphical Analysis in Economics

5.4.1 Market equilibrium analysis using graphs

Market equilibrium occurs where demand and supply curves intersect, representing the price and quantity at which the market clears. Graphical analysis makes this concept intuitive:

- **Equilibrium price**: The price at which quantity demanded equals quantity supplied.
- **Equilibrium quantity**: The quantity exchanged at equilibrium price.

- **Disequilibrium**: If price is above equilibrium, there is excess supply; if below, excess demand.

Worked Example: Demand ($Q_d = 100 - 2P$), Supply ($Q_s = 20 + 3P$). Setting ($Q_d = Q_s$):
[100 - 2P = 20 + 3P \quad \Rightarrow \quad P = 16, \quad Q = 68]
Thus, equilibrium occurs at price 16 and quantity 68.
Fill in the blank: When price is above equilibrium, there is excess __________.

[

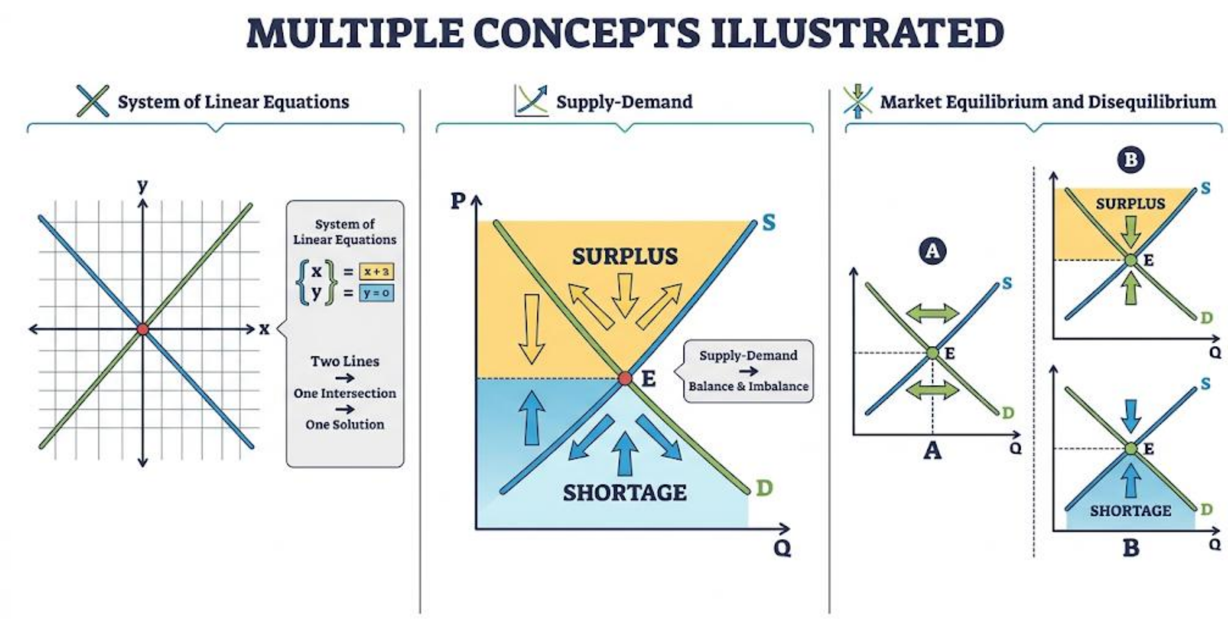

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Diagram showing demand and supply curves intersecting at equilibrium, with surplus and shortage areas highlighted.

Figure 5.7: Market equilibrium and disequilibrium

5.4.2 Comparative statics and policy evaluation

Comparative statics examines how equilibrium changes when external conditions shift. Graphs are used to show the before-and-after positions of demand and supply curves.

- **Tax policy**: A tax shifts the supply curve upward, leading to higher prices and lower quantities.
- **Subsidy policy**: A subsidy shifts the supply curve downward, lowering prices and increasing quantities.
- **Income changes**: Higher income shifts the demand curve outward, raising equilibrium price and quantity.

Worked Example: A tax of 4 units shifts the supply curve from ($Q_s = 20 + 3P$) to ($Q_s = 20 + 3(P - 4)$). Graphically, equilibrium moves to a higher price and lower quantity.
Fill in the blank: A subsidy shifts the supply curve __________.

[

Policy	Graphical effect	Equilibrium outcome
Tax	Supply curve shifts upward	Higher price, lower quantity
Subsidy	Supply curve shifts downward	Lower price, higher quantity
Income increase	Demand curve shifts outward	Higher price, higher quantity

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Table 5.3: Comparative statics in policy analysis

Policy | Graphical effect | Equilibrium outcome
Tax | Supply curve shifts upward | Higher price, lower quantity
Subsidy | Supply curve shifts downward | Lower price, higher quantity
Income increase | Demand curve shifts outward | Higher price, higher quantity

5.4.3 Dynamic graphical illustrations

Graphs can also be used to represent dynamic economic concepts over time:

- **Growth trends**: Upward-sloping time series graphs show increases in GDP, income, or output.
- **Business cycles**: Graphs illustrate expansions, peaks, recessions, and recoveries.
- **Stability analysis**: Graphical tools show whether markets return to equilibrium after shocks.

Worked Example: A time series graph of GDP shows steady growth interrupted by cyclical downturns. Economists use these graphs to identify long-term trends and short-term fluctuations.
Fill in the blank: A graph showing expansions and recessions over time represents the __________ cycle.
[Insert Figure here]
Diagram showing GDP growth trend with cyclical fluctuations.
Figure 5.8: Growth trends and business cycles

5.4.4 Integrating mathematical and graphical tools

Mathematical equations and graphical analysis complement each other:

- Equations provide precise numerical solutions.
- Graphs provide intuitive visual insights.
- Together, they allow economists to analyse equilibrium, policy impacts, and dynamic behaviour.

Worked Example: Solving ($Q_d = 100 - 2P$) and ($Q_s = 20 + 3P$) gives equilibrium numerically, while plotting both functions shows equilibrium visually.
Fill in the blank: Mathematical equations give precise solutions, while graphs provide __________ insights.

[Insert Box here]
Box 5.4: Integration of mathematical and graphical tools
Tool | Role | Example
Equation | Precise calculation | Solving equilibrium algebraically
Graph | Visual interpretation | Showing equilibrium point on curves
Combined | Comprehensive analysis | Policy evaluation with numbers and diagrams

Pilot questions 5.4

1. Define market equilibrium and explain how it is shown graphically.
2. What happens when price is above equilibrium?
3. How does a tax affect the supply curve and equilibrium?
4. Explain the role of comparative statics in policy evaluation.
5. What does a business cycle graph represent?
6. Why is it useful to combine mathematical and graphical tools in economic analysis?

Series Summary

This Series has focused on mathematical relationships and graphical analysis as essential tools for understanding economic concepts. We began by exploring different types of mathematical relationships, including linear, non-linear, direct, and inverse, and examined how they apply to demand, consumption, and production functions. We then moved to graphical representation, learning how to plot demand and supply curves on the Cartesian plane, and distinguishing between movements along curves and shifts of curves. Building on this, we studied advanced graphical tools such as elasticity, indifference curves, budget lines, isoquants, and isocost lines, which provide deeper insights into consumer choice and production decisions. Finally, we applied these mathematical and graphical methods to analyse market equilibrium, evaluate policy changes through comparative statics, and represent dynamic concepts such as growth trends and business cycles.
By completing this Series, you should now be able to define and interpret mathematical relationships, represent economic concepts graphically, and apply advanced tools to analyse consumer and producer behaviour. In line with the Series Learning Outcomes, you are equipped to evaluate equilibrium conditions, assess policy impacts, and illustrate dynamic economic processes using both equations and graphs. This strengthens your ability to combine mathematical precision with visual clarity in economic reasoning.

Glossary of Terms

Budget line: A straight line showing all combinations of goods a consumer can afford given income and prices.
Business cycle: Graphical representation of expansions, peaks, recessions, and recoveries in economic activity over time.
Comparative statics: Analysis of how equilibrium changes when external conditions shift, often shown graphically.
Consumer equilibrium: The point where a budget line is tangent to an indifference curve, maximising utility.
Direct relationship: A relationship where two variables move in the same direction.
Elasticity: A measure of responsiveness of one variable to changes in another, often linked to demand or supply curves.

Indifference curve: A curve showing combinations of goods that give equal satisfaction to a consumer.
Inverse relationship: A relationship where two variables move in opposite directions.
Isoquant: A curve showing combinations of inputs that produce the same level of output.
Isocost line: A straight line showing all combinations of inputs that can be purchased for a given cost.
Linear function: A function represented by a straight line, showing proportional change between variables.
Market equilibrium: The point where demand and supply curves intersect, determining equilibrium price and quantity.
Non-linear function: A function represented by a curve, showing non-proportional change between variables.
Slope: The rate of change of one variable relative to another, represented by the steepness of a curve.

Concept Check Questions [Series 5]

Objective Questions

1. Which of the following best describes a direct relationship?
   a) Variables move in opposite directions
   b) Variables move in the same direction
   c) Variables remain constant
   d) Variables are unrelated
   (Linked to SLO 5.1)
2. The intersection of demand and supply curves represents:
   a) Disequilibrium
   b) Market equilibrium
   c) Comparative statics
   d) Consumer equilibrium
   (Linked to SLO 5.7)
3. An indifference curve is:
   a) Upward sloping
   b) Downward sloping and convex to the origin
   c) A straight line
   d) Parallel to the budget line
   (Linked to SLO 5.5)
4. Isoquants are used to analyse:
   a) Consumer choice
   b) Producer input combinations
   c) Market equilibrium
   d) Comparative statics
   (Linked to SLO 5.6)
5. A subsidy shifts the supply curve:
   a) Upward
   b) Downward
   c) Leftward
   d) Rightward
   (Linked to SLO 5.8)

Short-Answer Questions

1. Define a mathematical relationship in economics. (Linked to SLO 5.1)
2. State the difference between a movement along a curve and a shift of a curve. (Linked to SLO 5.3)
3. What does elasticity measure? (Linked to SLO 5.4)
4. Explain the condition for consumer equilibrium. (Linked to SLO 5.5)
5. Write the equation of a budget line given income of 100, price of good X = 10, and price of good Y = 20. (Linked to SLO 5.5)

Long-Answer Questions

1. Analyse the importance of graphical representation in economics, using demand and supply curves as examples. (Linked to SLO 5.2, SLO 5.7)
   - **Basic response**: Explain how demand and supply curves are plotted and how equilibrium is shown graphically.
   - **Value-added response**: Extend by discussing how shifts in curves illustrate policy impacts and comparative statics.
2. Evaluate the role of indifference curves and budget lines in consumer choice. (Linked to SLO 5.5)
   - **Basic response**: Define indifference curves and budget lines, and explain consumer equilibrium.
   - **Value-added response**: Apply the analysis to show how changes in income or subsidies affect consumer choices.
3. Apply isoquants and isocost lines to production theory. (Linked to SLO 5.6)
   - **Basic response**: Define isoquants and isocost lines, and explain producer equilibrium.
   - **Value-added response**: Illustrate how firms adjust input combinations when costs or technology change.
4. Demonstrate how mathematical and graphical tools can be integrated to analyse market equilibrium and policy changes. (Linked to SLO 5.7, SLO 5.8, SLO 5.9)
   - **Basic response**: Show how equations and graphs both represent equilibrium.
   - **Value-added response**: Discuss dynamic illustrations such as growth trends and business cycles, linking them to policy evaluation.

Answers to Concept Check Questions [Series 5]

Objective Questions

1. b) Variables move in the same direction
2. b) Market equilibrium
3. b) Downward sloping and convex to the origin
4. b) Producer input combinations
5. b) Downward

Short-Answer Questions

1. A mathematical relationship shows how one economic variable changes in response to another.
2. A movement along a curve is caused by a change in the variable on the axis (e.g., price), while a shift occurs when other factors change (e.g., income, technology).
3. Elasticity measures the responsiveness of one variable to changes in another, often expressed as a percentage change.
4. Consumer equilibrium occurs where the budget line is tangent to an indifference curve.
5. The budget line is (10X + 20Y = 100).

Long-Answer Questions

1. **Basic response**: Demand and supply curves show equilibrium graphically.
   Value-added response: Shifts in curves illustrate policy impacts and comparative statics.
2. **Basic response**: Indifference curves and budget lines explain consumer equilibrium.
   Value-added response: Income changes or subsidies shift the budget line, altering consumer choices.
3. **Basic response**: Isoquants and isocost lines show producer equilibrium.
   Value-added response: Firms adjust input combinations when costs or technology change.
4. **Basic response**: Equations and graphs both represent equilibrium.
   Value-added response: Dynamic graphs illustrate growth trends and cycles, supporting policy evaluation.

Answers to Pilot Questions [Series 5]

Part 5.1: Mathematical Relationships in Economics

1. A mathematical relationship shows how one economic variable changes in response to another.
2. Linear functions are straight-line relationships with proportional changes, while non-linear functions are curved and show non-proportional changes.
3. Direct relationship: income and consumption. Inverse relationship: price and demand.
4. A consumption function expresses spending as a function of income, e.g., $C=a+bY$.
5. A simple production function: $Q=f(L,K)$, where L is labour and K is capital.
6. Mathematical relationships are important because they allow economists to predict outcomes, analyse policies, and model dynamic behaviour.

Part 5.2: Graphical Representation of Economic Concepts

1. The Cartesian plane is used to plot economic functions such as demand and supply.
2. Demand curves slope downward (inverse relationship), while supply curves slope upward (direct relationship).
3. The intersection of demand and supply curves represents market equilibrium.
4. A movement along a curve is caused by a change in the variable on the axis (e.g., price), while a shift occurs when other factors change (e.g., income, technology).
5. A tax on producers shifts the supply curve upward.

Part 5.3: Advanced Graphical Analysis

1. Elasticity measures the responsiveness of one variable to changes in another.
2. Slope is the absolute rate of change, while elasticity adjusts slope by considering proportional changes.
3. An indifference curve shows combinations of goods that yield equal satisfaction, and it is downward sloping and convex to the origin.

4. Consumer equilibrium occurs where the budget line is tangent to an indifference curve.
5. Isoquants show combinations of inputs that produce the same output, while isocost lines show combinations of inputs that can be purchased for a given cost.
6. Producer equilibrium occurs where an isoquant is tangent to an isocost line.
7. A subsidy rotates the budget line outward, allowing higher utility.
8. Elasticity graphs can be used to evaluate the impact of taxes or subsidies on demand responsiveness.

Part 5.4: Applications of Mathematical and Graphical Analysis in Economics

1. Market equilibrium is the point where demand and supply curves intersect, showing equilibrium price and quantity.
2. When price is above equilibrium, there is excess supply.
3. A tax shifts the supply curve upward, leading to higher prices and lower quantities.
4. Comparative statics analyse how equilibrium changes when external conditions shift, such as policy changes.
5. A business cycle graph represents expansions, peaks, recessions, and recoveries in economic activity.
6. Combining mathematical and graphical tools is useful because equations provide precise solutions, while graphs give intuitive visual insights.

References and Attributions

Figures

- *Figure 5.1: Linear and non-linear relationships*
  AI-generated suggestion: "Generate a diagram showing linear and non-linear curves on a Cartesian plane."
  Suggested OER search string: "linear vs nonlinear relationships OER economics."
- *Figure 5.2: Linear demand and non-linear utility functions*
  AI-generated suggestion: "Generate a diagram showing a straight-line demand function and a curved utility function."
  Suggested OER search string: "linear demand curve non-linear utility function OER economics."
- *Figure 5.3: Market equilibrium on a demand-supply graph*
  AI-generated suggestion: "Generate a diagram showing demand and supply curves intersecting at equilibrium point."
  Suggested OER search string: "demand supply equilibrium graph OER economics."
- *Figure 5.4: Elastic vs inelastic demand curves*
  AI-generated suggestion: "Generate a diagram showing steep and flat demand curves labelled inelastic and elastic."
  Suggested OER search string: "elastic vs inelastic demand curve OER economics."
- *Figure 5.5: Consumer equilibrium with indifference curve and budget line*
  AI-generated suggestion: "Generate a diagram showing indifference curves and a budget line tangent at equilibrium point."
  Suggested OER search string: "indifference curve budget line consumer equilibrium OER economics."

- *Figure 5.6: Producer equilibrium with isoquant and isocost line*
  AI-generated suggestion: "Generate a diagram showing isoquants and an isocost line tangent at equilibrium point."
  Suggested OER search string: "isoquant isocost line producer equilibrium OER economics."
- *Figure 5.7: Market equilibrium and disequilibrium*
  AI-generated suggestion: "Generate a diagram showing demand and supply curves intersecting at equilibrium, with surplus and shortage areas highlighted."
  Suggested OER search string: "market equilibrium disequilibrium graph OER economics."
- *Figure 5.8: Growth trends and business cycles*
  AI-generated suggestion: "Generate a diagram showing GDP growth trend with cyclical fluctuations."
  Suggested OER search string: "business cycle GDP growth trend OER economics."

Tables

- *Table 5.1: Examples of direct and inverse relationships in economics*
  AI-generated suggestion: "Create a table showing examples of direct and inverse relationships in economics."
  Suggested OER search string: "direct and inverse relationships economics OER."
- *Table 5.2: Distinction between movements and shifts*
  AI-generated suggestion: "Create a table showing difference between movement along a curve and shift of a curve with examples."
  Suggested OER search string: "movement vs shift demand supply curve OER economics."
- *Table 5.3: Comparative statics in policy analysis*
  AI-generated suggestion: "Create a table summarising comparative statics in policy analysis."
  Suggested OER search string: "comparative statics policy analysis OER economics."

Boxes

- *Box 5.1: Key economic relationships*
  AI-generated suggestion: "Create a text box summarising key economic relationships with examples."
  Suggested OER search string: "economic functions demand consumption production cost OER."
- *Box 5.2: Uses of graphical analysis in economics*
  AI-generated suggestion: "Create a text box summarising uses of graphical analysis in economics."
  Suggested OER search string: "graphical analysis economics equilibrium policy OER."
- *Box 5.3: Uses of advanced graphical analysis*
  AI-generated suggestion: "Create a text box summarising uses of advanced graphical analysis in economics."

Suggested OER search string: “applications of indifference curves isoquants elasticity OER economics.”

- *Box 5.4: Integration of mathematical and graphical tools*
  AI-generated suggestion: “Create a text box summarising integration of mathematical and graphical tools in economics.”
  Suggested OER search string: “integration of mathematical and graphical tools OER economics.”

General References

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