La-Net

STA311

PROBABILITY III

Series 1: Discrete Sample Spaces and Rules of Probability

Series outline

- **Part 1.1: Introduction to Discrete Sample Spaces**
    - Definition of sample spaces
    - Examples of discrete sample spaces in everyday contexts
    - Distinction between discrete and continuous sample spaces
- **Part 1.2: Events and Their Classifications**
    - Definition of events
    - Types of events: simple, compound, mutually exclusive, exhaustive
    - Representation of events using set notation
- **Part 1.3: Rules of Probability**
    - Classical definition of probability
    - Axioms of probability
    - Addition rule and multiplication rule
    - Complementary rule
- **Part 1.4: Applications of Probability Rules**
    - Worked examples in card games, dice, and coin tossing
    - Real-life applications in risk assessment and decision making
- **Part 1.5: Common Misconceptions and Limitations**
    - Misinterpretation of probability values
    - Fallacies such as the gambler's fallacy
    - Limitations of classical probability in complex systems

Introduction

Probability is one of the most fundamental concepts in mathematics and statistics, and it plays a crucial role in everyday reasoning and scientific investigation. Whether we are predicting the outcome of a coin toss, assessing the risk of an investment, or estimating the likelihood of a medical diagnosis, probability provides the language and framework for dealing with uncertainty. This Series introduces you to the foundations of probability by focusing on **discrete sample spaces** and the **rules of probability**, which form the building blocks for more advanced topics in the course.

The aim of this Series is to equip you with the ability to define discrete sample spaces, classify events, and apply the basic rules of probability to both theoretical and practical problems.

We shall commence this Series by examining the concept of discrete sample spaces, including their definitions and examples. Next, we will move on to events and their classifications, exploring how they are represented and related. Following that, we will study the rules of probability, such as the addition and multiplication rules, and see how they are applied. Subsequently, we will consider practical applications of these rules in familiar contexts like dice, cards, and coins. Finally, we will conclude the Series by addressing common misconceptions and limitations in probability, ensuring that you develop a critical understanding of the subject.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 1.1** define discrete sample spaces and distinguish them from continuous sample spaces,
- **SLO 1.2** explain the concept of events and classify them into simple, compound, mutually exclusive, and exhaustive categories,
- **SLO 1.3** state and apply the fundamental rules of probability, including the addition, multiplication, and complementary rules,
- **SLO 1.4** apply probability rules to solve practical problems involving dice, cards, and coins, and
- **SLO 1.5** evaluate common misconceptions and limitations in probability reasoning.

1.1 Introduction to Discrete Sample Spaces

A **sample space** is the set of all possible outcomes of a random experiment. When the outcomes can be listed individually and are finite or countably infinite, the sample space is said to be **discrete**. For example, tossing a coin produces a discrete sample space of {Head, Tail}, while rolling a die produces $\{1, 2, 3, 4, 5, 6\}$. Discrete sample spaces are the foundation of probability because they provide the complete set of possibilities from which events are defined.

Fill in the blank: The set of all possible outcomes of a random experiment is called the _________________.

1.1.1 Definition of sample spaces

A **sample space** is formally defined as the collection of all possible outcomes of a random experiment. Each outcome is called a **sample point**. If the sample space contains a finite or countable number of outcomes, it is discrete.

Example:

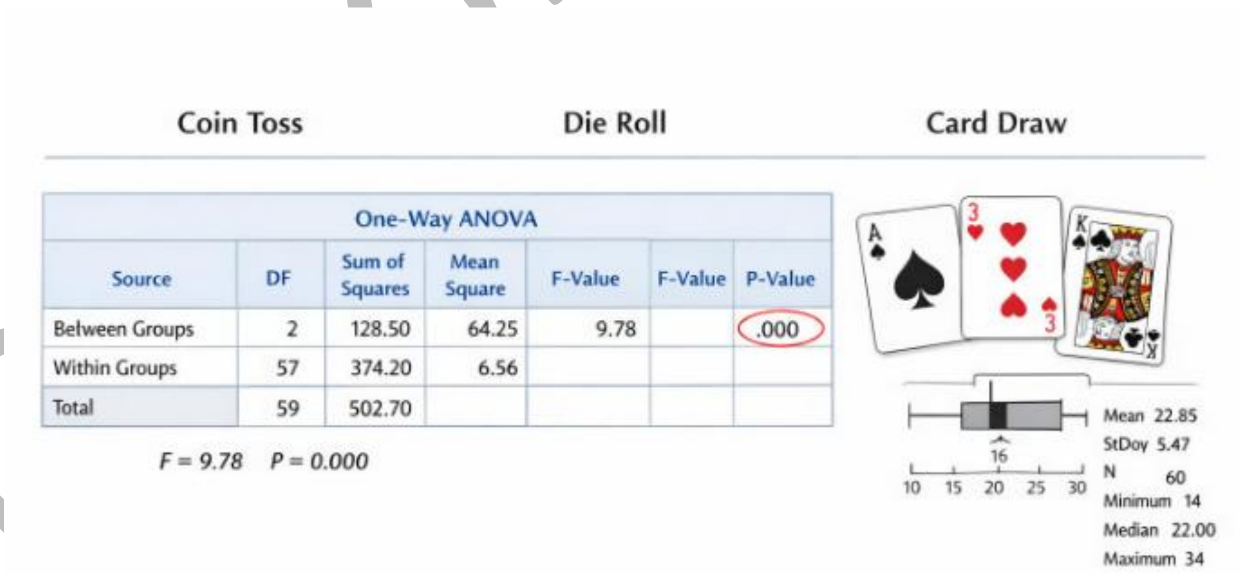
- Tossing a coin: S = {Head, Tail}
- Rolling a die: S = {1, 2, 3, 4, 5, 6}

1.1.2 Examples of discrete sample spaces

Discrete sample spaces appear in many everyday contexts:

- **Games of chance:** Cards, dice, and coins.
- **Decision making:** Choosing among limited options.
- **Science experiments:** Recording finite outcomes such as presence or absence of a chemical reaction.

Examples of discrete sample spaces



1.1.3 Distinction between discrete and continuous sample spaces

A **continuous sample space** contains outcomes that cannot be listed individually because they form an interval or continuum. For example, the exact time a bus arrives or the precise height of a person are continuous outcomes.

Key distinction:

- Discrete sample space: outcomes are countable (e.g., dice rolls).
- Continuous sample space: outcomes are uncountable and form ranges (e.g., time, weight).

Fill in the blank: A sample space where outcomes form an interval or continuum is called a __________________ sample space.

Pilot Questions 1.1

1. A sample space is: A. The set of all possible outcomes of a random experiment B. A single outcome only C. A random guess D. None of the above
2. A discrete sample space contains: A. A finite or countable number of outcomes B. An uncountable number of outcomes C. No outcomes at all D. None of the above
3. Tossing a coin produces the sample space: A. {Head, Tail} B. {1, 2, 3, 4, 5, 6} C. {Red, Black} D. None of the above
4. Continuous sample spaces are characterised by: A. Outcomes forming an interval or continuum B. Outcomes that are countable C. Outcomes that are finite only D. None of the above
5. In probability, each possible outcome in a sample space is called a: A. Sample point B. Random guess C. Continuous value D. None of the above

1.2 Events and Their Classifications

An **event** is any subset of a sample space. In probability, events represent outcomes or collections of outcomes that we are interested in. For example, when rolling a die, the event "getting an even number" corresponds to the subset {2, 4, 6}. Events are central to probability because they allow us to focus on particular outcomes and assign probabilities to them.

Fill in the blank: Any subset of a sample space that represents outcomes of interest is called an __________________.

1.2.1 Definition of events

An **event** is defined as a set of one or more outcomes from a sample space. Each outcome in the sample space is a sample point, and events are formed by grouping these points.

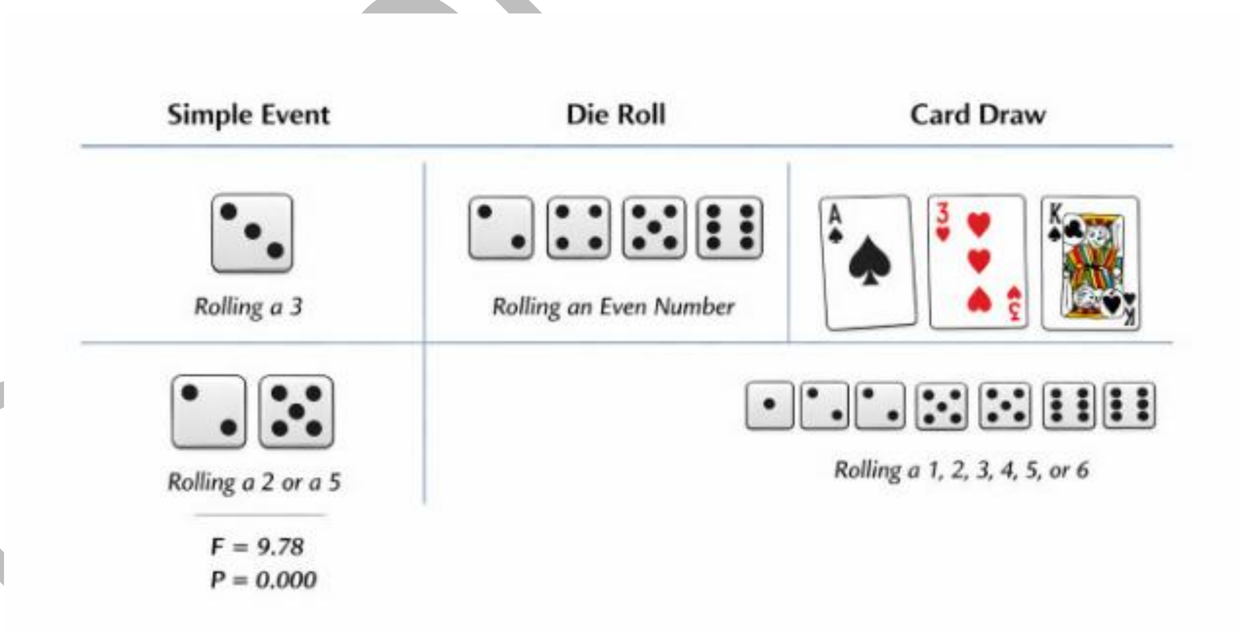
Example:

- Tossing a coin: Event "getting a Head" = {Head}.
- Rolling a die: Event "getting a number greater than 4" = {5, 6}.

1.2.2 Types of events

Events can be classified into several categories:

- **Simple event:** An event consisting of a single outcome. Example: rolling a 3 on a die.
- **Compound event:** An event consisting of two or more outcomes. Example: rolling an even number {2, 4, 6}.
- **Mutually exclusive events:** Two events that cannot occur at the same time. Example: rolling a 2 and rolling a 5.
- **Exhaustive events:** A set of events that together cover the entire sample space. Example: {odd numbers} and {even numbers} in a die roll.



Classifications of events in probability

1.2.3 Representation of events using set notation

Events are often represented using **set notation**. For example:

- Union (A ∪ B): Event that either A or B occurs.
- Intersection (A ∩ B): Event that both A and B occur.
- Complement (A′): Event that A does not occur.

Fill in the blank: The event that either A or B occurs is represented as __________________ in set notation.

Pilot Questions 1.2

1. An event in probability is: A. Any subset of a sample space B. The entire sample space only C. A random guess D. None of the above
2. A simple event consists of: A. A single outcome B. Multiple outcomes C. No outcomes D. None of the above
3. Mutually exclusive events are: A. Events that cannot occur at the same time B. Events that always occur together C. Events that cover the entire sample space D. None of the above
4. Exhaustive events are: A. Events that together cover the entire sample space B. Events that cannot occur together C. Events with only one outcome D. None of the above
5. The union of two events A and B is represented as: A. A ∪ B B. A ∩ B C. A′ D. None of the above

1.3 Rules of Probability

The **rules of probability** provide the mathematical framework for assigning and manipulating probabilities of events within a sample space. These rules ensure consistency and allow us to calculate probabilities in both simple and complex situations.

Fill in the blank: The mathematical framework that governs how probabilities are assigned and manipulated is called the __________________.

1.3.1 Classical definition of probability

The **classical definition of probability** states that if all outcomes in a sample space are equally likely, then the probability of an event is given by:

$$P(E) = \text{Number of favourable outcomesTotal number of possible outcomes}$$

Example: The probability of rolling a 4 on a fair die is 16.

1.3.2 Axioms of probability

Probability theory is built on three fundamental axioms:

1. **Non-negativity:** For any event E, $P(E) \geq 0$.
2. **Normalisation:** The probability of the entire sample space is 1, i.e., $P(S)=1$.
3. **Additivity:** If two events are mutually exclusive, then $P(E1E2)=P(E1)+P(E2)$.

Fill in the blank: The probability of the entire sample space is always equal to __________________.

1.3.3 Addition and multiplication rules

- **Addition rule:** For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This accounts for overlap when events are not mutually exclusive.

- **Multiplication rule:** For any two events A and B,

$$P(A \cap B) = P(A) \cdot P(B|A)$$

where $P(B|A)$is the conditional probability of B given A.

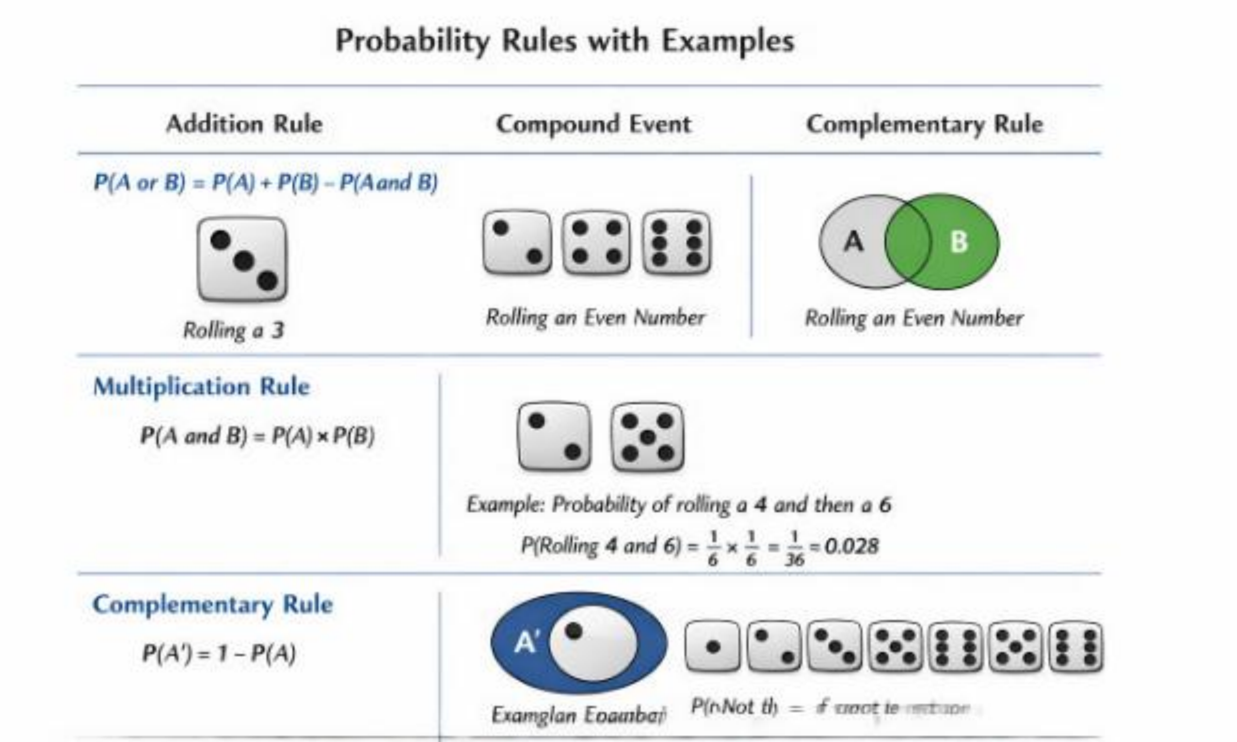
Example: The probability of drawing a red card or a king from a deck requires the addition rule, while the probability of drawing two aces in succession requires the multiplication rule.

1.3.4 Complementary rule

The **complementary rule** states that the probability of an event not occurring is equal to one minus the probability of the event occurring:

$$P(A') = 1 - P(A)$$

Example: If the probability of rain tomorrow is 0.3, then the probability of no rain is 0.7.



Fundamental rules of probability

Pilot Questions 1.3

1. The classical definition of probability applies when: A. All outcomes are equally likely B. Outcomes are unequal C. No outcomes exist D. None of the above
2. The probability of the entire sample space is: A. 1 B. 0 C. Between 0 and 1 D. None of the above
3. The addition rule accounts for: A. Overlap between events B. Independence of events C. Complementary events D. None of the above
4. The multiplication rule involves: A. Conditional probability B. Mutually exclusive events C. Exhaustive events D. None of the above
5. The probability of an event not occurring is given by: A. 1 – P(A) B. P(A) + P(B) C. P(A ∩ B) D. None of the above

1.4 Applications of Probability Rules

Probability rules are not only theoretical but also highly practical. They are applied in everyday scenarios such as games of chance, risk assessment, and decision making. By working through familiar examples, you will see how the addition, multiplication, and complementary rules are used to solve problems.

Fill in the blank: The probability of drawing a red card or a king from a deck of cards requires the use of the __________________ rule.

1.4.1 Worked examples in card games, dice, and coin tossing

- **Card games:** The probability of drawing a heart from a standard deck is 1352=14.
- **Dice:** The probability of rolling an even number is 36=12.
- **Coin tossing:** The probability of getting at least one Head in two tosses is calculated using the complementary rule:

P(at least one Head)=1-P(no Head)=1-14=34

Applications of probability rules in everyday game

1.4.2 Real-life applications in risk assessment and decision making

Probability rules are widely used in real-life contexts:

- **Insurance companies** use probability to calculate risks and premiums.
- **Medical professionals** apply probability to interpret diagnostic tests.

- **Economists and business managers** use probability to forecast market behaviour.

Fill in the blank: Insurance companies rely on _________________ to calculate risks and set premiums.

Pilot Questions 1.4

1. The probability of drawing a heart from a standard deck of cards is: A. 14B. 12C. 16D. None of the above
2. The probability of rolling an even number on a fair die is: A. 12B. 13C. 16D. None of the above
3. The probability of getting at least one Head in two coin tosses is: A. 34B. 12C. 14D. None of the above
4. The complementary rule is useful when calculating: A. The probability of an event not occurring B. The probability of mutually exclusive events C. The probability of independent events D. None of the above
5. Insurance companies use probability rules to: A. Calculate risks and set premiums B. Ignore uncertainties C. Make random guesses D. None of the above

1.5 Common Misconceptions and Limitations

Probability is often misunderstood, and learners may fall into certain traps when interpreting results. Recognising these misconceptions is important because they can lead to incorrect reasoning and poor decision making. In addition, probability has limitations when applied to complex systems, which must be acknowledged to avoid overconfidence in its predictions.

Fill in the blank: The mistaken belief that past random outcomes influence future ones is known as the _________________ fallacy.

1.5.1 Misinterpretation of probability values

A common misconception is to treat probability values as certainties. For example, a probability of 0.7 does not mean an event will definitely occur, but rather that it is expected to occur 70 times out of 100 in repeated trials. Learners must remember that probability expresses likelihood, not guarantee.

1.5.2 Fallacies in probability reasoning

One of the most well-known fallacies is the **gambler's fallacy**, which is the mistaken belief that if an event has occurred frequently in the past, it is less likely to occur in the future, or vice versa. For instance, believing that after five consecutive Heads, the next coin toss is "due" to be a Tail is incorrect, since each toss is independent.

Another fallacy is the **hot-hand fallacy**, where people assume that success in random events increases the chance of further success, such as believing a basketball player is "on a streak".

1.5.3 Limitations of classical probability

The classical approach to probability assumes equally likely outcomes, which is not always realistic. In complex systems such as weather forecasting or financial markets, outcomes are not equally likely and may depend on hidden variables. In such cases, more advanced models, such as Bayesian probability, are required.

Pilot Questions 1.5

1. A probability value of 0.7 means: A. The event is likely but not certain B. The event will definitely occur C. The event cannot occur D. None of the above
2. The gambler's fallacy is the belief that: A. Past random outcomes influence future ones B. Probability values are always certain C. Outcomes are equally likely in all cases D. None of the above
3. The hot-hand fallacy assumes that: A. Success in random events increases the chance of further success B. Failures reduce the chance of future success C. Probabilities are always equal D. None of the above
4. Classical probability assumes: A. All outcomes are equally likely B. Outcomes are unequal C. Probabilities are always certain D. None of the above
5. In complex systems like weather forecasting, classical probability is limited because: A. Outcomes are not equally likely and may depend on hidden variables B. Probabilities are always certain C. Events are mutually exclusive D. None of the above

Series Summary

This Series has introduced you to the foundations of probability by focusing on discrete sample spaces and the rules that govern them. We began by defining discrete sample spaces and distinguishing them from continuous ones, then moved on to events and their classifications, including simple, compound, mutually exclusive, and exhaustive events. Following that, we examined the fundamental rules of probability, such as the classical definition, axioms, addition, multiplication, and complementary rules. We then applied these rules to practical examples involving cards, dice, and coins, and finally addressed common misconceptions and limitations, such as the gambler's fallacy and the assumptions of classical probability.

By completing this Series, you should now be able to define discrete sample spaces, classify events, apply the rules of probability to both theoretical and practical problems, and critically evaluate misconceptions and limitations in probability reasoning. These skills directly align with the Series Learning Outcomes (SLOs), ensuring that you have a solid foundation for the more advanced topics that follow in the course.

Glossary of Terms

- **Addition rule:** A probability rule used to calculate the likelihood of either of two events occurring, accounting for overlap.
- **Complementary rule:** A probability rule stating that the probability of an event not occurring is equal to one minus the probability of the event occurring.
- **Compound event:** An event consisting of two or more outcomes from a sample space.
- **Continuous sample space:** A sample space where outcomes form an interval or continuum and cannot be listed individually.
- **Event:** Any subset of a sample space representing outcomes of interest.
- **Exhaustive events:** A set of events that together cover the entire sample space.
- **Gambler's fallacy:** The mistaken belief that past random outcomes influence future ones.
- **Hot-hand fallacy:** The mistaken belief that success in random events increases the chance of further success.

- **Multiplication rule:** A probability rule used to calculate the likelihood of two events occurring together, often involving conditional probability.
- **Mutually exclusive events:** Events that cannot occur at the same time.
- **Sample point:** A single outcome in a sample space.
- **Sample space:** The set of all possible outcomes of a random experiment.
- **Simple event:** An event consisting of a single outcome.

Concept Check Questions 1

Objective Questions

1. Define a sample space and give an example. (Linked to SLO 1.1)
2. Identify the difference between a simple event and a compound event. (Linked to SLO 1.2)
3. State the probability of rolling a 6 on a fair die. (Linked to SLO 1.3)
4. Apply the complementary rule to calculate the probability of not drawing a heart from a standard deck of cards. (Linked to SLO 1.4)
5. Recognise the gambler's fallacy in a coin tossing scenario. (Linked to SLO 1.5)

Short-Answer Questions

6. Explain the difference between discrete and continuous sample spaces with examples. (Linked to SLO 1.1)

7. Describe mutually exclusive events and provide an example using dice. (Linked to SLO 1.2)

8. Demonstrate the addition rule with an example from card games. (Linked to SLO 1.3)

Long-Answer Question

9. Evaluate the limitations of classical probability when applied to complex systems such as weather forecasting or financial markets. (Linked to SLO 1.5)

Answers to Concept Check Questions 1

Objective Questions

1. A sample space is the set of all possible outcomes of a random experiment. Example: tossing a coin gives {Head, Tail}.

2. A simple event consists of one outcome, while a compound event consists of two or more outcomes.
3. Probability of rolling a 6 = 16.
4. Probability of not drawing a heart = 1-1352=3952.
5. The gambler's fallacy is the mistaken belief that after several Heads, a Tail is "due".

Short-Answer Questions

6. Discrete sample spaces have countable outcomes, e.g., dice rolls {1, 2, 3, 4, 5, 6}. Continuous sample spaces have uncountable outcomes, e.g., exact height of a person.

7. Mutually exclusive events cannot occur together. Example: rolling a 2 and rolling a 5 on a die.

8. Addition rule: Probability of drawing a red card or a king = P(Red)+P(King)-P(Red and King).

Long-Answer Question

- **Basic response:** Classical probability assumes equally likely outcomes, which is not realistic in complex systems.
- **Value-added response:** In weather forecasting, probabilities depend on multiple variables such as temperature, humidity, and pressure, which are not equally likely. Similarly, financial markets involve hidden factors like investor behaviour and global events. Bayesian probability provides a more flexible framework by incorporating prior knowledge and updating with evidence.

Answers to Pilot Questions 1

Part 1.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.2:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.3:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.4:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 1.5:** 1-A, 2-A, 3-A, 4-A, 5-A

References and Attributions 1

- **Figure 1.1:** Examples of discrete sample spaces. AI prompt: "Generate a diagram showing coin toss, die roll, and card draw as examples of discrete sample spaces."

- **Figure 1.2:** Classifications of events. AI prompt: “Generate a diagram showing simple, compound, mutually exclusive, and exhaustive events with dice examples.”
- **Figure 1.3:** Fundamental rules of probability. AI prompt: “Generate a diagram illustrating addition, multiplication, and complementary rules of probability with examples.”
- **Figure 1.4:** Applications of probability rules. AI prompt: “Generate a diagram showing probability applications in card games, dice rolls, and coin tossing.”
- **Figure 1.5:** Misconceptions and limitations in probability reasoning. AI prompt: “Generate a diagram illustrating gambler’s fallacy, hot-hand fallacy, and limitations of classical probability.”

Series 2: Independence, Bayes’ Theorem, and Urn Models

Series outline

For this 3-credit course, each Series should contain 4 to 6 Parts. Here is the proposed outline for Series 2:

- **Part 2.1: Independence of Events**
    - Definition of independent events
    - Properties of independence
    - Examples and applications
- **Part 2.2: Conditional Probability**
    - Definition and notation
    - Relationship with independence
    - Worked examples
- **Part 2.3: Bayes’ Theorem**

    - Statement of Bayes' theorem
    - Applications in probability problems
    - Real-life uses in medicine and decision making
- **Part 2.4: Urn Models**
    - Definition and types of urn problems
    - Sampling with and without replacement
    - Applications in probability theory
- **Part 2.5: Practical Applications and Limitations**
    - Using Bayes' theorem in diagnostic testing
    - Urn models in real-life scenarios
    - Limitations of independence assumptions

Introduction

Probability often requires us to consider how events relate to one another. Some events occur independently, while others depend on prior outcomes. Understanding these relationships is essential for solving more complex problems. In this Series, we will explore the concepts of independence, conditional probability, Bayes' theorem, and urn models, which together form a powerful toolkit for analysing uncertain situations.

The aim of this Series is to enable you to explain independence, apply conditional probability, use Bayes' theorem to update beliefs, and solve problems involving urn models.

We shall commence this Series by examining the independence of events, including its definition and properties. Next, we will move on to conditional probability, which provides the foundation for understanding dependence. Following that, we will study Bayes' theorem, a central result in probability that allows us to update prior beliefs with new evidence. Subsequently, we will explore urn models, which illustrate sampling with and without replacement. Finally, we will conclude with practical applications and limitations, ensuring you can connect these concepts to real-life contexts.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 2.1** define independence of events and explain its properties,
- **SLO 2.2** calculate conditional probabilities and relate them to independence,
- **SLO 2.3** apply Bayes' theorem to solve probability problems and update beliefs,
- **SLO 2.4** solve urn model problems involving sampling with and without replacement, and
- **SLO 2.5** evaluate practical applications and limitations of independence, Bayes' theorem, and urn models.

2.1 Independence of Events

In probability, events can either influence each other or occur without any connection. When the occurrence of one event does not affect the probability of another, the events are said to be **independent**. Independence is a key concept because it simplifies calculations and helps us model situations where outcomes are unrelated.

Fill in the blank: Two events are said to be __________________ if the occurrence of one does not affect the probability of the other.

2.1.1 Definition of independent events

Two events A and B are **independent** if:

$$P(A \cap B) = P(A) \cdot P(B)$$

This means that the probability of both events occurring together is equal to the product of their individual probabilities.

Example: Tossing a coin and rolling a die are independent events because the outcome of the coin toss does not affect the die roll.

2.1.2 Properties of independence

- Independence is symmetric: If A is independent of B, then B is independent of A.
- Independence is different from mutual exclusivity: Mutually exclusive events cannot occur together, while independent events can occur together but do not influence each other.
- Independence can extend to more than two events, provided the condition holds for all combinations.

Fill in the blank: Mutually exclusive events cannot occur together, while _____ _____________ events can occur together but do not influence each other.

2.1.3 Examples and applications

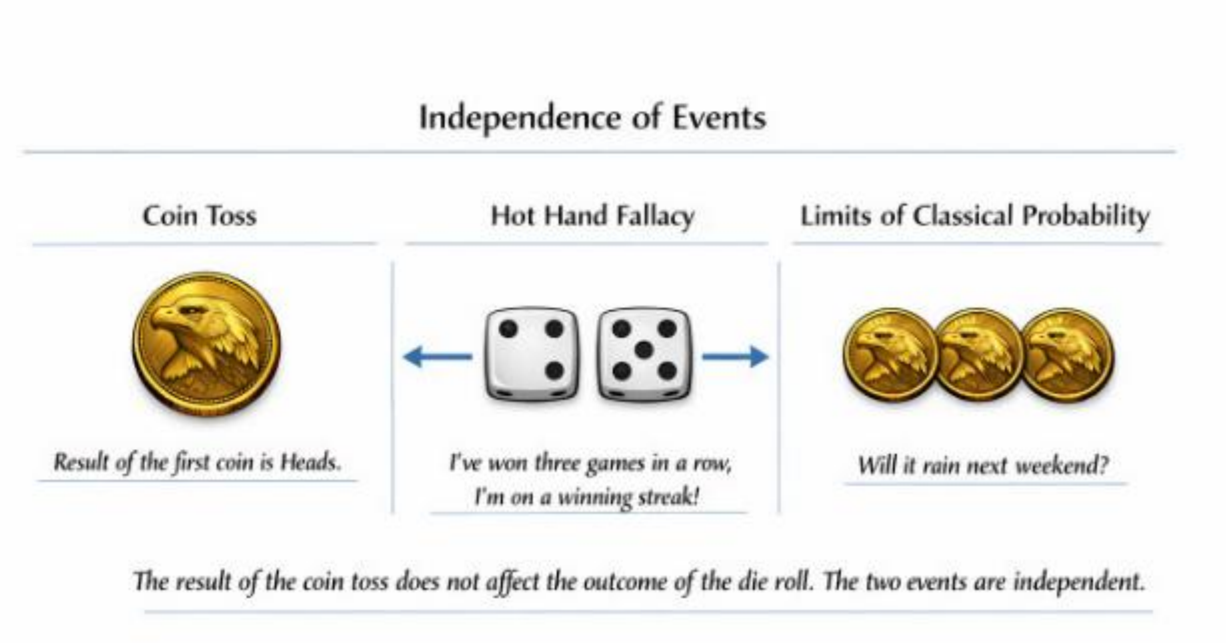
- **Example 1:** Tossing two coins. The probability of both showing Heads is:

$$P(\text{Head on first})\cdot P(\text{Head on second})=1212=14$$

- **Example 2:** Rolling two dice. The probability of both showing a 6 is:

$$P(6\text{ on first})\cdot P(6\text{ on second})=1616=136$$

Applications include modelling independent risks in insurance, independent trials in experiments, and independent random variables in statistics.



Independence of events in probability

Pilot Questions 2.1

1. Two events A and B are independent if: A. $P(A\cap B)=P(A)\cdot P(B)$B. $P(A\cap B)=P(A)+P(B)$C. $PA\cap B=0$ D. None of the above
2. Independence is different from mutual exclusivity because: A. Independent events can occur together, mutually exclusive events cannot B. Mutually exclusive events can occur together, independent events cannot C. Both mean the same thing D. None of the above

3. Tossing a coin and rolling a die are: A. Independent events B. Mutually exclusive events C. Dependent events D. None of the above
4. The probability of rolling two sixes on two dice is: A. 136B. 16C. 112D. None of the above
5. Independence is: A. Symmetric B. Asymmetric C. Random D. None of the above

2.2 Conditional Probability

Conditional probability allows us to measure the likelihood of an event occurring given that another event has already occurred. It is a fundamental concept because many real-life situations involve dependencies between events.

Fill in the blank: The probability of event A occurring given that event B has already occurred is called the __________________ probability.

2.2.1 Definition and notation

The **conditional probability** of event A given event B is defined as:

$$P(A|B)=P(A\cap B)P(B), \text{provided } P(B)>0$$

Here, $P(A|B)$represents the probability of A occurring under the condition that B has already occurred.

Example: If we draw a card from a deck and it is known to be a heart, the probability that it is also a king is:

$$P(\text{King}|\text{Heart})=113$$

2.2.2 Relationship with independence

Conditional probability is closely related to independence. If two events A and B are independent, then:

$$P(A|B)=P(A)$$

This means that knowing B has occurred does not change the probability of A.

Fill in the blank: If two events are independent, then P(A|B) =

__________________.

2.2.3 Worked examples

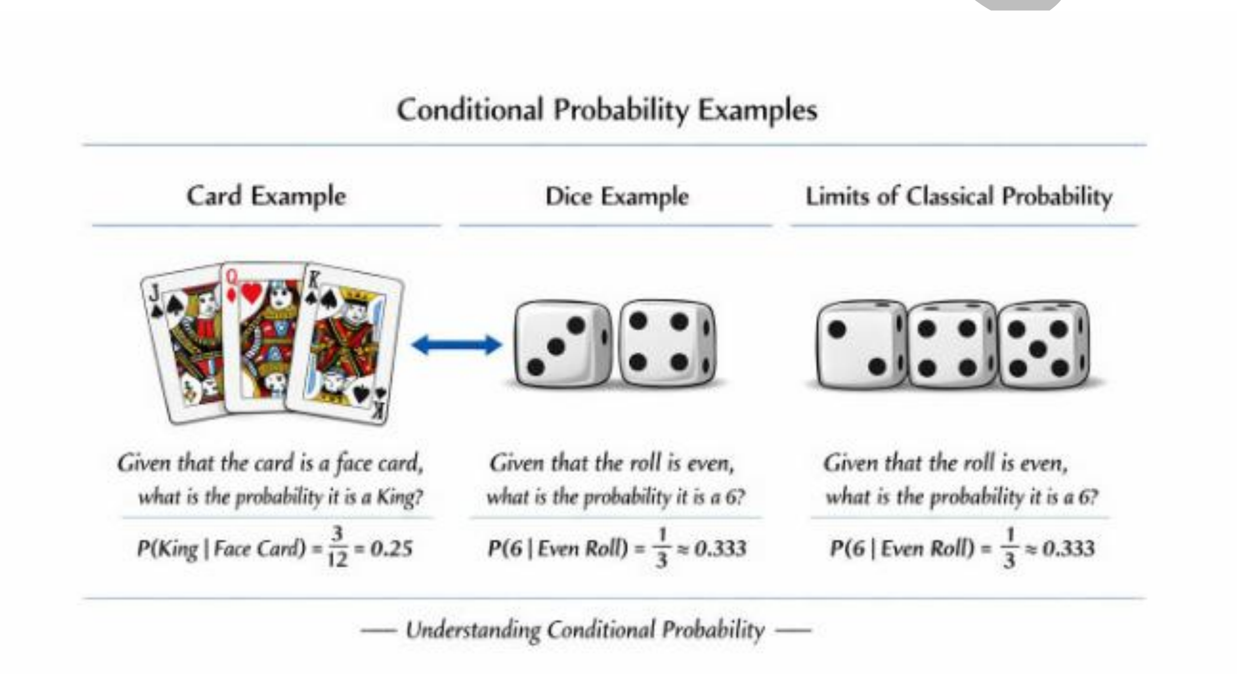
- **Example 1:** Drawing two cards without replacement. The probability that the second card is an ace given that the first card was an ace is:

$$P(\text{Ace on second}|\text{Ace on first})=351$$

- **Example 2:** Rolling two dice. The probability that the sum is 7 given that the first die shows 3 is:

$$P(\text{Sum} = 7|\text{First die} = 3)=16$$

Applications include medical testing, where the probability of having a disease given a positive test result is a conditional probability.



Conditional probability in practice

Pilot Questions 2.2

1. Conditional probability is defined as: A. $P(A|B)=P(A\cap B)P(B)$B. $P(A|B)=P(A)+P(B)$C. $P(A|B)=P(A)\cdot P(B)$D. None of the above
2. If two events are independent, then: A. $P(A|B)=P(A)$B. $P(A|B)=P(B)$C. $P(A|B)=0$D. None of the above
3. The probability that the second card is an ace given that the first card was an ace is: A. 351B. 113C. 152D. None of the above

4. The probability that the sum of two dice is 7 given that the first die shows 3 is: A. 16B. 112C. 136D. None of the above
5. Conditional probability is often applied in: A. Medical testing B. Random guessing C. Ignoring prior information D. None of the above

2.3 Bayes' Theorem

Bayes' theorem is one of the most powerful tools in probability. It allows us to update our beliefs about the likelihood of an event when new information becomes available. This makes it especially useful in situations where we start with prior knowledge and then incorporate evidence to refine our conclusions.

Fill in the blank: The mathematical rule that allows us to update probabilities based on new evidence is called ___________________.

2.3.1 Statement of Bayes' theorem

Bayes' theorem states that:

$$P(A|B)=P(B|A)\cdot P(A)P(B)$$

Here:

- $P(A)$is the **prior probability** of event A,
- $P(B|A)$is the likelihood of observing B given A,
- $P(B)$is the overall probability of B, and
- $P(A|B)$is the **posterior probability**, the updated probability of A after observing B.

2.3.2 Applications in probability problems

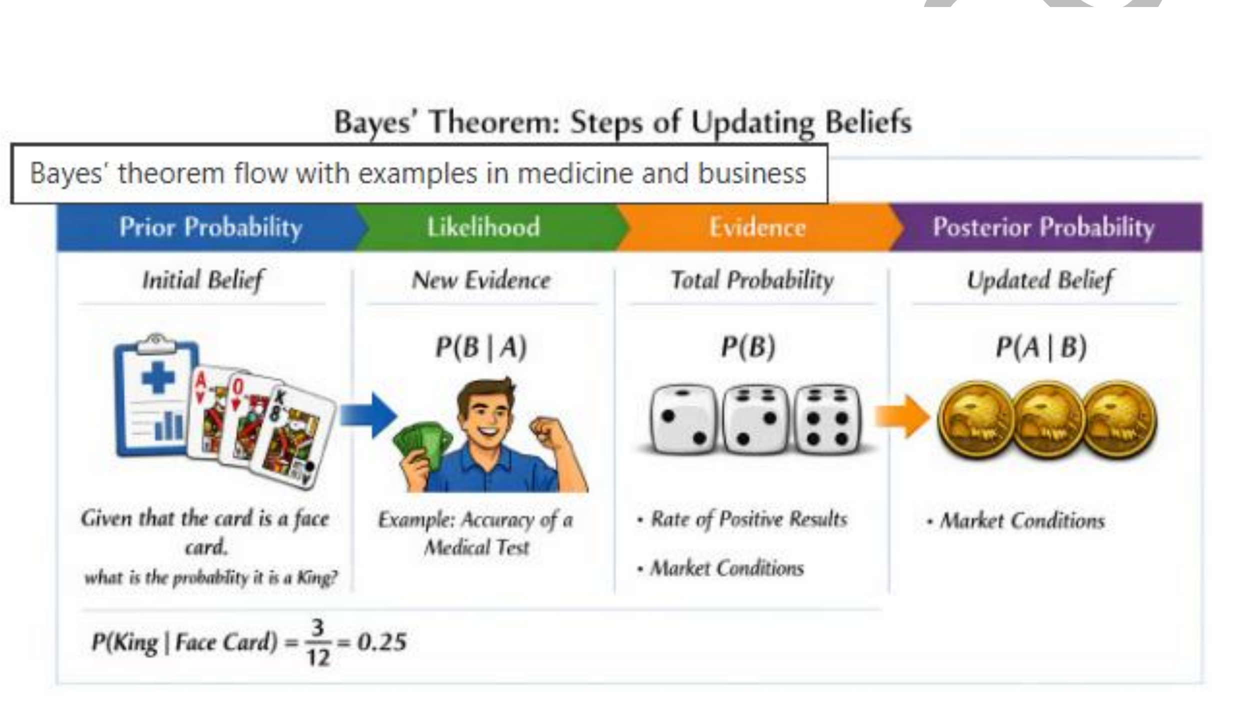
- **Example 1:** Suppose a bag contains 3 red balls and 2 blue balls. If a ball is drawn and observed to be red, Bayes' theorem can be used to update the probability of drawing another red ball depending on whether replacement is allowed.
- **Example 2:** In diagnostic testing, if a test is positive, Bayes' theorem helps calculate the probability that the patient truly has the disease, given the test's accuracy and the disease prevalence.

Fill in the blank: In Bayes' theorem, the updated probability after observing evidence is called the ___________________ probability.

2.3.3 Real-life uses in medicine and decision making

Bayes' theorem is widely applied in:

- **Medicine:** Interpreting diagnostic tests, where prior disease prevalence and test accuracy are combined to estimate the likelihood of illness.
- **Business:** Updating forecasts when new market data becomes available.
- **Artificial intelligence:** Machine learning algorithms use Bayesian updating to improve predictions.



Bayes' theorem and its applications

Pilot Questions 2.3

1. Bayes' theorem is used to: A. Update probabilities based on new evidence B. Calculate probabilities without evidence C. Ignore prior information D. None of the above
2. The formula for Bayes' theorem is: A. P(A|B)=P(B|A)·P(A)P(B)B. P(A|B)=P(A)+P(B)C. P(A|B)=P(A)·P(B)D. None of the above
3. In Bayes' theorem, P(A)is called the: A. Prior probability B. Posterior probability C. Conditional probability D. None of the above

4. In medical testing, Bayes' theorem helps calculate: A. The probability of disease given a positive test result B. The probability of disease without evidence C. The probability of random guessing D. None of the above
5. Bayes' theorem is widely used in: A. Medicine, business, and artificial intelligence B. Random guessing only C. Ignoring evidence D. None of the above

2.4 Urn Models

Urn models are a classical way of representing probability problems. They involve drawing balls of different colours from a container (the urn), and they help us understand sampling processes. These models are especially useful for illustrating the difference between sampling with replacement and sampling without replacement.

Fill in the blank: Probability problems that involve drawing balls of different colours from a container are called ___________________ models.

2.4.1 Definition and types of urn problems

An **urn model** is a probability experiment where balls of different colours are placed in an urn, and one or more balls are drawn. The outcomes depend on whether the balls are replaced after each draw.

Types of urn problems:

- **With replacement:** After drawing a ball, it is returned to the urn before the next draw. Probabilities remain constant.
- **Without replacement:** After drawing a ball, it is not returned. Probabilities change with each draw.

2.4.2 Sampling with and without replacement

- **With replacement:** Suppose an urn contains 3 red balls and 2 blue balls. The probability of drawing a red ball twice in succession with replacement is:

$$P(\text{Red then Red})=3535=925$$

- **Without replacement:** Using the same urn, the probability of drawing two red balls without replacement is:

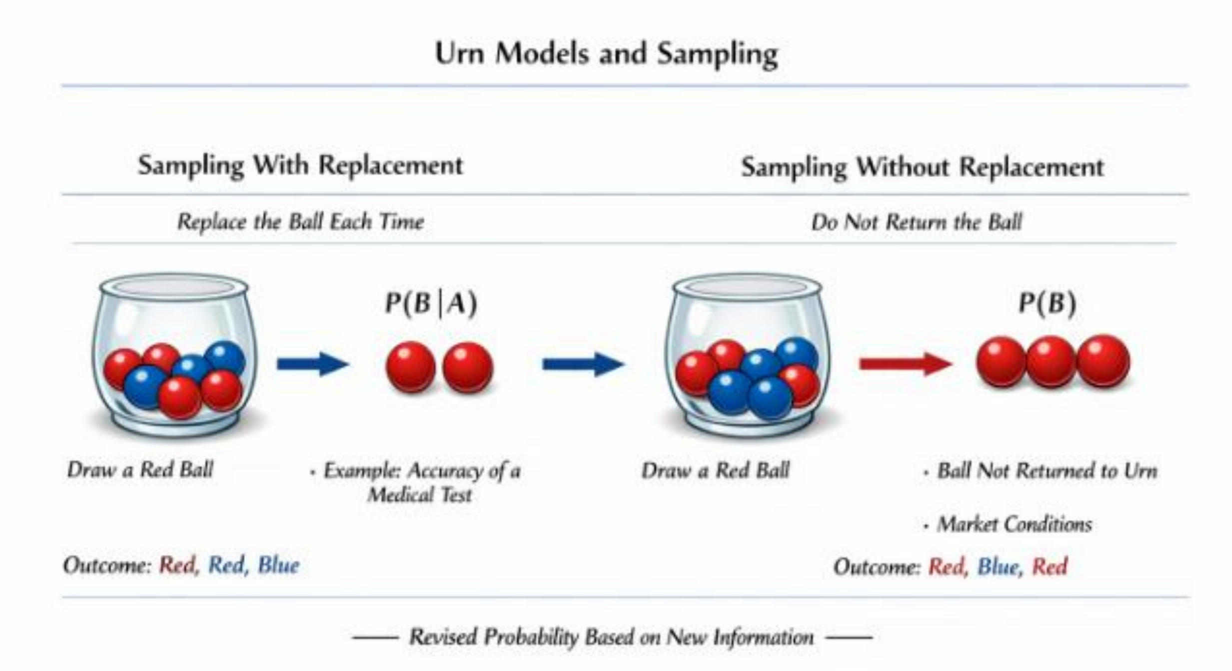
$$P(\text{Red then Red})=3524=620=310$$

Fill in the blank: When balls are not returned after each draw, the sampling is said to be __________________ replacement.

2.4.3 Applications in probability theory

Urn models are applied in:

- **Lottery problems:** Drawing winning numbers without replacement.
- **Genetics:** Modelling inheritance patterns.
- **Computer science:** Randomised algorithms and simulations.



Urn models in probability

Pilot Questions 2.4

1. An urn model involves: A. Drawing balls of different colours from a container B. Rolling dice C. Tossing coins D. None of the above
2. Sampling with replacement means: A. Balls are returned after each draw B. Balls are not returned after each draw C. Probabilities change with each draw D. None of the above

3. The probability of drawing two red balls with replacement from an urn containing 3 red and 2 blue balls is: A. 925B. 310C. 15D. None of the above
4. The probability of drawing two red balls without replacement from the same urn is: A. 310B. 925C. 15D. None of the above
5. Urn models are applied in: A. Lottery problems, genetics, and computer science B. Random guessing only C. Ignoring probabilities D. None of the above

2.5 Practical Applications and Limitations

Probability concepts such as independence, conditional probability, Bayes' theorem, and urn models are not just theoretical tools, they are widely applied in real-life contexts. For example, Bayes' theorem is used in medical testing to update the probability of a disease given a positive test result, while urn models are applied in lottery draws and genetics. Independence assumptions are often used in risk modelling and insurance, where different risks are treated as unrelated to simplify calculations.

However, these concepts also have limitations. Independence assumptions may not always hold in practice, as real-world events are often interconnected. Bayes' theorem requires accurate prior probabilities, which can be difficult to obtain. Urn models, while useful for illustrating sampling, may oversimplify complex systems. Recognising these limitations ensures that probability tools are applied appropriately and critically.

Fill in the blank: Bayes' theorem requires accurate __________________ probabilities in order to produce reliable results.

Pilot Questions 2.5

1. Bayes' theorem is applied in: A. Medical testing and decision making B. Ignoring prior information C. Random guessing D. None of the above
2. Urn models are useful for: A. Lottery problems and genetics B. Tossing coins only C. Ignoring probabilities D. None of the above
3. Independence assumptions may fail because: A. Real-world events are often interconnected B. Probabilities are always equal C. Events are always mutually exclusive D. None of the above

4. A limitation of Bayes' theorem is: A. It requires accurate prior probabilities B. It ignores evidence C. It cannot be applied to medical testing D. None of the above
5. Urn models may oversimplify: A. Complex systems B. Simple systems C. Independent events D. None of the above

Series Summary

This Series has explored the relationships between events in probability, focusing on independence, conditional probability, Bayes' theorem, and urn models. We began by defining independence of events and distinguishing it from mutual exclusivity, then examined conditional probability as a way of measuring likelihood when another event has already occurred. Following that, we studied Bayes' theorem, which provides a systematic method for updating probabilities based on new evidence. We then applied urn models to illustrate sampling with and without replacement, and finally considered practical applications and limitations of these concepts in real-life contexts such as medical testing, insurance, and lottery problems.

By completing this Series, you should now be able to define independence, calculate conditional probabilities, apply Bayes' theorem to update beliefs, solve urn model problems, and evaluate the practical applications and limitations of these tools. These achievements directly align with the Series Learning Outcomes (SLOs), ensuring that you are well prepared for more advanced probability topics.

Glossary of Terms

- **Bayes' theorem:** A probability rule used to update prior beliefs based on new evidence.
- **Conditional probability:** The probability of an event occurring given that another event has already occurred.
- **Independence of events:** A situation where the occurrence of one event does not affect the probability of another.
- **Mutually exclusive events:** Events that cannot occur at the same time.
- **Posterior probability:** The updated probability of an event after considering new evidence.
- **Prior probability:** The initial probability of an event before new evidence is considered.

- **Sampling with replacement:** A process where items drawn are returned to the sample set, keeping probabilities constant.
- **Sampling without replacement:** A process where items drawn are not returned, causing probabilities to change.
- **Urn model:** A probability experiment involving drawing balls of different colours from a container.

Concept Check Questions 2

Objective Questions

1. Define independence of events and give an example. (Linked to SLO 2.1)
2. State the formula for conditional probability. (Linked to SLO 2.2)
3. Identify the prior probability in Bayes' theorem. (Linked to SLO 2.3)
4. Calculate the probability of drawing two red balls with replacement from an urn containing 3 red and 2 blue balls. (Linked to SLO 2.4)
5. Recognise one limitation of Bayes' theorem. (Linked to SLO 2.5)

Short-Answer Questions

6. Explain the difference between independence and mutual exclusivity with examples. (Linked to SLO 2.1)

7. Demonstrate conditional probability using a card example. (Linked to SLO 2.2)

8. Apply Bayes' theorem to a medical testing scenario. (Linked to SLO 2.3)

Long-Answer Question

9. Evaluate the practical applications and limitations of urn models in probability theory. (Linked to SLO 2.4 and SLO 2.5)

Answers to Concept Check Questions 2

Objective Questions

1. Independence means the occurrence of one event does not affect another. Example: tossing a coin and rolling a die.
2. $P(A|B)=P(A\cap B)P(B)$.
3. The prior probability is $P(A)$.

4. P(Red then Red)=3535=925.
5. Bayes' theorem requires accurate prior probabilities.

Short-Answer Questions

6. Independent events can occur together without influence, e.g., coin toss and die roll. Mutually exclusive events cannot occur together, e.g., rolling a 2 and rolling a 5 on one die.

7. Example: Probability of drawing a king given the card is a heart = 113.

8. Example: Probability of having a disease given a positive test result depends on test accuracy and disease prevalence, calculated using Bayes' theorem.

Long-Answer Question

- **Basic response:** Urn models illustrate sampling with and without replacement, useful in lottery and genetics problems.
- **Value-added response:** Urn models simplify complex systems but may not capture dependencies or hidden variables. In genetics, they model inheritance patterns, while in computer science they support randomised algorithms. Their limitation lies in oversimplification, which requires more advanced probabilistic models in practice.

Answers to Pilot Questions 2

Part 2.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.2:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.3:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.4:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 2.5:** 1-A, 2-A, 3-A, 4-A, 5-A

References and Attributions 2

- **Figure 2.1:** Independence of events. AI prompt: "Generate a diagram showing independence of events using coin toss and die roll examples."
- **Figure 2.2:** Conditional probability. AI prompt: "Generate a diagram showing conditional probability examples with cards and dice."
- **Figure 2.3:** Bayes' theorem. AI prompt: "Generate a diagram showing Bayes' theorem flow: prior probability, likelihood, evidence, posterior probability, with examples in medicine and business."
- **Figure 2.4: Urn models. AI prompt: "Generate a diagram showing urn models with red and blue balls, illustrating sampling with and without replacement."**

Series 3: Sampling Methods and the Inclusion–Exclusion Principle

Series outline

- **Part 3.1: Introduction to Sampling Methods**
    - Definition of sampling in probability
    - Importance of sampling in statistics
    - Overview of basic sampling techniques
- **Part 3.2: Random Sampling and Simple Random Sampling**
    - Concept of randomness in sampling
    - Simple random sampling procedure
    - Advantages and limitations
- **Part 3.3: Other Sampling Techniques**
    - Stratified sampling
    - Systematic sampling
    - Cluster sampling
    - Applications in probability and statistics
- **Part 3.4: The Inclusion–Exclusion Principle**
    - Statement of the principle
    - Application to two and three events
    - General formula for multiple events
- **Part 3.5: Applications of the Inclusion–Exclusion Principle**
    - Worked examples in probability problems
    - Real-life applications in counting and risk analysis
    - Limitations of the principle

Introduction

Sampling is a fundamental tool in probability and statistics, allowing us to draw conclusions about a population by examining a subset. In this Series, we will explore different sampling methods and their applications, before moving on to the inclusion–exclusion principle, which provides a systematic way of calculating probabilities when events overlap.

The aim of this Series is to equip you with the ability to explain and apply various sampling methods, and to use the inclusion–exclusion principle to solve probability problems involving overlapping events.

We shall commence this Series by introducing the concept of sampling and its importance. Next, we will study random and simple random sampling, followed by other techniques such as stratified, systematic, and cluster sampling. Afterwards, we will turn to the inclusion–exclusion principle, beginning with its statement and applications to two or three events, and then extending to multiple events. Finally, we will conclude with practical applications and limitations of the principle, ensuring you can connect these ideas to real-life contexts.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 3.1** define sampling methods and explain their importance in probability and statistics,
- **SLO 3.2** apply random and simple random sampling techniques to probability problems,
- **SLO 3.3** describe and apply stratified, systematic, and cluster sampling methods,
- **SLO 3.4** state and apply the inclusion–exclusion principle for two, three, and multiple events, and
- **SLO 3.5** evaluate applications and limitations of the inclusion–exclusion principle in probability and real-life contexts.

3.1 Introduction to Sampling Methods

Sampling is the process of selecting a subset of individuals, items, or outcomes from a larger population or sample space. In probability and statistics, sampling is essential because it allows us to make inferences about a population without examining every single element. This makes it practical, cost-effective, and efficient, especially when dealing with large populations.

Fill in the blank: The process of selecting a subset of individuals or outcomes from a larger population is called __________________.

3.1.1 Definition of sampling in probability

In probability, sampling refers to the method of choosing outcomes from a sample space. In statistics, it refers to selecting individuals or items from a population. Sampling provides the foundation for probability experiments and statistical surveys.

Example: Selecting 10 students from a class of 50 to estimate the average height.

3.1.2 Importance of sampling in statistics

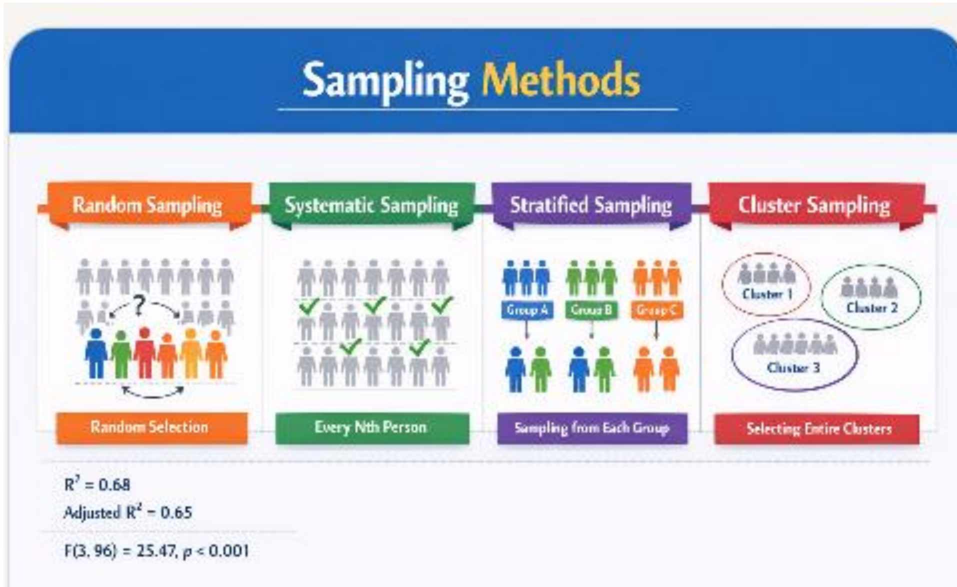
- **Efficiency:** Saves time and resources compared to studying the entire population.
- **Feasibility:** Makes it possible to study large populations.
- **Accuracy:** When done correctly, sampling provides reliable estimates of population characteristics.
- **Basis for inference:** Sampling allows us to generalise findings from the sample to the population.

Fill in the blank: Sampling allows us to generalise findings from the __________________ to the population.

3.1.3 Overview of basic sampling techniques

There are several basic sampling techniques:

- **Random sampling:** Every member of the population has an equal chance of being selected.
- **Systematic sampling:** Selecting every nth member of the population.
- **Stratified sampling:** Dividing the population into groups (strata) and sampling from each group.
- **Cluster sampling:** Dividing the population into clusters and sampling entire clusters.



Basic sampling techniques in statistics

Pilot Questions 3.1

1. Sampling is: A. Selecting a subset of individuals or outcomes from a population B. Studying the entire population C. Ignoring the population D. None of the above
2. Sampling is important because: A. It saves time and resources B. It makes studying large populations feasible C. It allows generalisation to the population D. All of the above
3. Random sampling means: A. Every member has an equal chance of selection B. Selecting every nth member C. Dividing into strata D. None of the above
4. Systematic sampling involves: A. Selecting every nth member of the population B. Selecting randomly only C. Dividing into clusters D. None of the above
5. Stratified sampling involves: A. Dividing the population into groups and sampling from each B. Selecting every nth member C. Sampling entire
6. clusters D. None of the above

3.2 Random Sampling and Simple Random Sampling

Random sampling is one of the most fundamental techniques in statistics. It ensures that every member of the population has an equal chance of being selected, which reduces bias and increases the reliability of results. Simple random sampling is the most straightforward form of random sampling and is widely used in probability experiments and surveys.

Fill in the blank: A sampling method where every member of the population has an equal chance of being selected is called __________________ sampling.

3.2.1 Concept of randomness in sampling

Randomness means that the selection process is free from bias or predetermined patterns. Each individual or outcome is chosen purely by chance. This ensures that the sample is representative of the population.

Example: Using a random number generator to select students from a class list.

3.2.2 Simple random sampling procedure

In **simple random sampling**, each member of the population is assigned a number, and selections are made randomly. This can be done using:

- Random number tables
- Computerised random generators
- Lottery methods (drawing names from a hat)

Fill in the blank: In simple random sampling, each member of the population is assigned a __________________ before selection.

3.2.3 Advantages and limitations

Advantages:

- Eliminates bias in selection.
- Provides a representative sample when the population is homogeneous.
- Easy to understand and implement.

Limitations:

- Requires a complete list of the population.
- May not be efficient for very large populations.
- Less effective if the population is highly diverse.

Simple random sampling methods

Pilot Questions 3.2

1. Random sampling ensures: A. Every member has an equal chance of selection B. Only certain members are selected C. Bias in selection D. None of the above
2. Simple random sampling can be carried out using: A. Random number tables B. Computerised random generators C. Lottery methods D. All of the above
3. In simple random sampling, each member of the population is assigned a: A. Number B. Colour C. Group D. None of the above
4. One advantage of simple random sampling is: A. It eliminates bias in selection B. It requires no population list C. It is always efficient for large populations D. None of the above
5. A limitation of simple random sampling is: A. It requires a complete list of the population B. It always introduces bias C. It cannot be used for homogeneous populations D. None of the above

3.3 Other Sampling Techniques

Beyond random and simple random sampling, there are several other methods designed to improve efficiency and accuracy when dealing with diverse populations. These include **stratified sampling**, **systematic sampling**, and **cluster sampling**. Each has its own strengths and limitations, and the choice of method depends on the nature of the population and the research objectives.

Fill in the blank: Dividing a population into groups called strata and sampling from each group is known as __________________ sampling.

3.3.1 Stratified sampling

- The population is divided into **strata** (groups) based on shared characteristics, such as age, gender, or income.
- A sample is then drawn from each stratum, often proportionally.
- This ensures representation of all key subgroups.

Example: Dividing a school population into male and female students, then sampling proportionally from each group.

3.3.2 Systematic sampling

- In systematic sampling, every **nth** member of the population is selected after a random starting point.
- It is simpler than random sampling and often easier to implement.

Example: Selecting every 10th person from a list of 1,000 names.

3.3.3 Cluster sampling

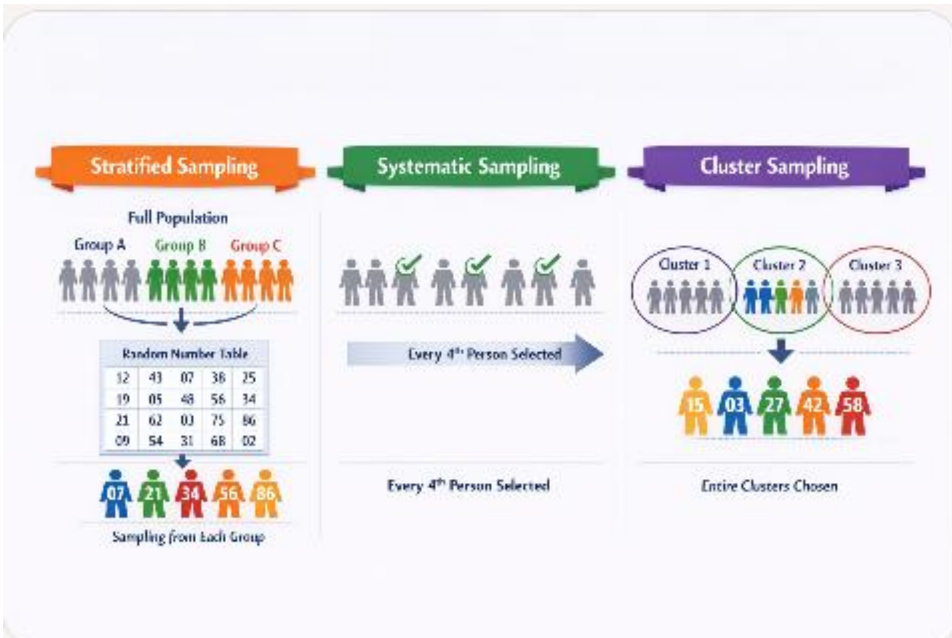
- The population is divided into **clusters**, often based on geography or natural groupings.
- Entire clusters are then randomly selected, and all members within those clusters are studied.
- Useful when populations are large and spread out.

Example: Selecting entire classrooms in a school rather than individual students.

3.3.4 Applications in probability and statistics

- **Stratified sampling:** Ensures fair representation of diverse groups in surveys.
- **Systematic sampling:** Common in quality control processes.

- **Cluster sampling:** Widely used in large-scale surveys, such as national censuses.



Other sampling techniques in statistics

Pilot Questions 3.3

1. Stratified sampling involves: A. Dividing the population into groups and sampling from each B. Selecting every nth member C. Sampling entire clusters D. None of the above
2. Systematic sampling involves: A. Selecting every nth member of the population B. Dividing into strata C. Sampling entire clusters D. None of the above
3. Cluster sampling involves: A. Dividing the population into clusters and sampling entire clusters B. Selecting every nth member C. Sampling from each stratum D. None of the above
4. Stratified sampling is useful for: A. Ensuring representation of diverse groups B. Quality control C. Large-scale surveys D. None of the above
5. Cluster sampling is commonly used in: A. National censuses B. Random guessing C. Ignoring subgroups D. None of the above

3.4 The Inclusion–Exclusion Principle

The **inclusion–exclusion principle** is a fundamental rule in probability and combinatorics. It helps us calculate the probability of the union of events when those events overlap. Instead of simply adding probabilities, the principle corrects for double-counting by subtracting intersections.

Fill in the blank: The rule used to calculate probabilities of overlapping events is called the __________________ principle.

3.4.1 Statement of the principle

For two events A and B:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

For three events A, B, and C:

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

This formula ensures that overlaps are properly accounted for.

3.4.2 Application to two and three events

- **Two events example:** Probability of drawing a red card or a king from a deck:

$$P(\text{RedKing})=P(\text{Red})+P(\text{King})-P(\text{Red and King})$$

- **Three events example:** Probability of being enrolled in mathematics, physics, or chemistry courses requires subtracting pairwise overlaps and adding the triple overlap.

3.4.3 General formula for multiple events

For nevents A1,A2,...,An:

$$P\cup i=1nAi=\sum P(Ai)-\sum P(AiAj)+\sum P(AiAjAk)-\cdots+(-1)n+1P(A1A2\cap\cdots\cap An)$$

This alternating sum continues until all intersections are included.

The inclusion–exclusion principle for overlapping events

Pilot Questions 3.4

1. The inclusion–exclusion principle is used to: A. Calculate probabilities of overlapping events B. Ignore intersections C. Add probabilities without correction D. None of the above
2. For two events A and B, the formula is: A. $P(A\cup B)=P(A)+P(B)-P(A\cap B)$B. $P(A\cup B)=P(A)+P(B)$C. $P(A\cup B)=P(A)\cdot P(B)$D. None of the above
3. For three events, the inclusion–exclusion principle requires: A. Adding individual probabilities, subtracting pairwise intersections, and adding the triple intersection B. Adding only individual probabilities C. Subtracting only intersections D. None of the above
4. The general formula for multiple events involves: A. Alternating sums of probabilities and intersections B. Only single probabilities C. Ignoring overlaps D. None of the above
5. The inclusion–exclusion principle corrects for: A. Double-counting overlaps B. Ignoring probabilities C. Random guessing D. None of the above

3.5 Applications of the Inclusion–Exclusion Principle

The inclusion–exclusion principle is not just a theoretical tool; it has wide applications in probability, counting, and risk analysis. It is particularly useful when dealing with overlapping events, where simple addition of probabilities would lead to double-counting.

Fill in the blank: The inclusion–exclusion principle corrects for __________________ when calculating probabilities of overlapping events.

3.5.1 Worked examples in probability problems

- **Example 1:** In a class of 40 students, 20 study mathematics, 15 study physics, and 10 study both. The probability that a randomly chosen student studies mathematics or physics is:

$$P(\text{MathPhysics})=2040+1540\text{-}1040=2540=58$$

- **Example 2:** In a deck of cards, the probability of drawing a red card or a face card requires subtracting the overlap of red face cards.

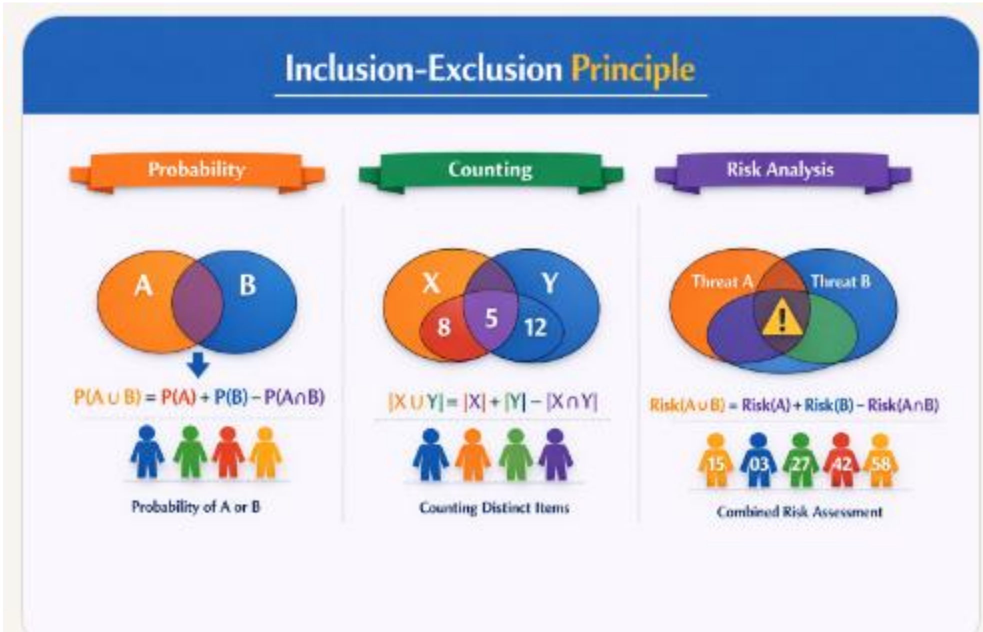
3.5.2 Real-life applications in counting and risk analysis

- **Counting problems:** Used in combinatorics to count elements belonging to multiple sets.
- **Risk analysis:** Applied in insurance and finance to calculate probabilities of overlapping risks.
- **Computer science:** Used in algorithms for set operations and database queries.

Fill in the blank: In insurance, the inclusion–exclusion principle helps calculate overlapping __________________.

3.5.3 Limitations of the principle

- Becomes complex when applied to many events, as the number of terms increases rapidly.
- Requires accurate knowledge of intersections, which may be difficult to obtain in practice.
- May oversimplify real-world problems if dependencies between events are not properly modelled.



Applications of the inclusion–exclusion principle

Pilot Questions 3.5

1. The inclusion–exclusion principle corrects for: A. Double-counting overlaps B. Ignoring probabilities C. Random guessing D. None of the above
2. In a class of 40 students, 20 study mathematics, 15 study physics, and 10 study both. The probability that a student studies mathematics or physics is: A. 58B. 12C. 38D. None of the above
3. In insurance, the inclusion–exclusion principle helps calculate overlapping: A. Risks B. Coins C. Cards D. None of the above
4. A limitation of the inclusion–exclusion principle is: A. It becomes complex with many events B. It ignores intersections C. It cannot be applied to probability problems D. None of the above
5. The principle is applied in: A. Counting, risk analysis, and computer science B. Random guessing only C. Ignoring overlaps D. None of the above

Series Summary

This Series has introduced you to sampling methods and the inclusion–exclusion principle, both of which are central to probability and statistics. We began by defining sampling and explaining its importance, then explored random and simple random sampling as foundational techniques. We extended this by examining stratified, systematic, and cluster sampling, each designed to address specific population structures. Afterwards, we studied the inclusion–exclusion principle, learning how it corrects for overlaps when calculating probabilities of unions of events. Finally, we applied the principle to practical problems in probability, counting, and risk analysis, while also recognising its limitations.

By completing this Series, you should now be able to define and apply different sampling methods, distinguish between them, state and use the inclusion–exclusion principle for two, three, and multiple events, and evaluate its applications and limitations. These outcomes align directly with the Series Learning Outcomes (SLOs), equipping you with tools for both theoretical and practical probability problems.

Glossary of Terms

- **Cluster sampling:** A method where the population is divided into clusters and entire clusters are sampled.
- **Inclusion–exclusion principle:** A rule used to calculate probabilities of overlapping events by correcting for double-counting.
- **Random sampling:** A method where every member of the population has an equal chance of being selected.
- **Sampling:** The process of selecting a subset of individuals or outcomes from a larger population.
- **Simple random sampling:** A procedure where each member of the population is assigned a number and selections are made randomly.
- **Stratified sampling:** A method where the population is divided into strata and samples are taken from each group.
- **Systematic sampling:** A method where every nth member of the population is selected after a random start.

Concept Check Questions 3

Objective Questions

1. Define sampling in probability and statistics. (Linked to SLO 3.1)
2. State the formula for the inclusion–exclusion principle for two events. (Linked to SLO 3.4)
3. Identify one limitation of simple random sampling. (Linked to SLO 3.2)

4. Recognise the difference between stratified and cluster sampling. (Linked to SLO 3.3)
5. State one limitation of the inclusion–exclusion principle. (Linked to SLO 3.5)

Short-Answer Questions

6. Explain why sampling is important in statistics. (Linked to SLO 3.1)

7. Demonstrate systematic sampling with an example. (Linked to SLO 3.3)

8. Apply the inclusion–exclusion principle to three events. (Linked to SLO 3.4)

Long-Answer Question

9. Evaluate the practical applications and limitations of the inclusion–exclusion principle in probability and real-life contexts. (Linked to SLO 3.5)

Answers to Concept Check Questions 3

Objective Questions

1. Sampling is selecting a subset of individuals or outcomes from a population.
2. $P(A \cup B)=P(A)+P(B)-P(A \cap B)$.
3. It requires a complete list of the population.
4. Stratified sampling divides into groups and samples from each, while cluster sampling selects entire clusters.
5. It becomes complex with many events and requires accurate intersection data.

Short-Answer Questions

6. Sampling saves time and resources, makes studying large populations feasible, and allows generalisation to the population.

7. Example: Selecting every 10th person from a list of 1,000 names.

8. $P(A \cup B \cup C)=P(A)+P(B)+P(C)-P(A \cap B)-P(A \cap C)-P(B \cap C)+P(A \cap B \cap C)$.

Long-Answer Question

- **Basic response:** The inclusion–exclusion principle is applied in probability problems, counting, and risk analysis, but becomes complex with many events.

- **Value-added response:** In practice, it is used in insurance to calculate overlapping risks, in computer science for database queries, and in combinatorics for counting problems. Its limitation lies in requiring precise intersection data, which may be unavailable or difficult to measure in real-world contexts.

Answers to Pilot Questions 3

Part 3.1: 1-A, 2-D, 3-A, 4-A, 5-A **Part 3.2:** 1-A, 2-D, 3-A, 4-A, 5-A **Part 3.3:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 3.4:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 3.5:** 1-A, 2-A, 3-A, 4-A, 5-A

References and Attributions 3

- **Figure 3.1:** Basic sampling techniques. AI prompt: "Generate a diagram showing random, systematic, stratified, and cluster sampling methods with simple illustrations."
- **Figure 3.2:** Simple random sampling. AI prompt: "Generate a diagram showing simple random sampling using random numbers and lottery draw."
- **Figure 3.3:** Other sampling techniques. AI prompt: "Generate a diagram showing stratified, systematic, and cluster sampling with simple illustrations."
- **Figure 3.4:** Inclusion–exclusion principle. AI prompt: "Generate a diagram showing overlapping sets with inclusion–exclusion principle applied to two and three events."
- **Figure 3.5: Applications of inclusion–exclusion principle. AI prompt: "Generate a diagram showing inclusion–exclusion principle applied to overlapping sets in probability, counting, and risk analysis."**

Series 4: Allocation Problems, Matching Problems, and Probability Generating Functions

Series Outline

- **Part 4.1: Allocation Problems**
  - Definition and examples of allocation in probability
  - Methods of solving allocation problems

- Applications in resource distribution and probability

- **Part 4.2: Matching Problems**
  - Definition of matching problems
  - The classical “matching problem” (derangements)
  - Applications in probability and combinatorics
- **Part 4.3: Probability Generating Functions (PGFs)**
  - Definition and properties of PGFs
  - Using PGFs to calculate probabilities
  - Examples with discrete random variables
- **Part 4.4: Applications of PGFs**
  - PGFs in branching processes
  - PGFs in queueing theory
  - PGFs in risk modelling
- **Part 4.5: Practical Applications and Limitations**
  - Real-life uses of allocation and matching problems
  - PGFs in applied probability
  - Limitations of these methods

Introduction

This Series focuses on three advanced topics in probability: allocation problems, matching problems, and probability generating functions. Allocation problems deal with distributing resources or objects among individuals or categories. Matching problems explore scenarios where items are paired or matched, often leading to combinatorial challenges such as derangements. Probability generating functions provide a powerful algebraic tool for working with discrete random variables, enabling us to calculate probabilities and expectations efficiently.

The aim of this Series is to enable you to solve allocation and matching problems, understand and apply probability generating functions, and evaluate their practical applications and limitations.

We shall commence by studying allocation problems, followed by matching problems and their classical formulations. Next, we will introduce probability generating functions, exploring their properties and applications. Finally, we will conclude with practical applications and limitations, ensuring you can connect these advanced concepts to real-life contexts.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 4.1** define and solve allocation problems in probability,
- **SLO 4.2** explain and solve matching problems, including derangements,
- **SLO 4.3** define probability generating functions and apply them to discrete random variables,
- **SLO 4.4** use PGFs in applied contexts such as branching processes and queueing theory, and
- **SLO 4.5** evaluate practical applications and limitations of allocation problems, matching problems, and PGFs.

4.1 Allocation Problems

Allocation problems in probability involve distributing objects or resources among individuals, groups, or categories according to certain rules. These problems are common in combinatorics and probability theory, and they help us understand how outcomes are distributed when resources are limited or shared.

Fill in the blank: Problems that involve distributing objects or resources among individuals or categories are called __________________ problems.

4.1.1 Definition and examples of allocation in probability

An **allocation problem** is a probability or combinatorial problem where items are assigned to recipients. The complexity arises when constraints are introduced, such as limits on how many items each recipient can receive.

Example: Distributing 5 identical balls into 3 boxes. The allocation depends on whether boxes can be empty and whether balls are distinguishable.

4.1.2 Methods of solving allocation problems

- **Stars and bars method:** Used when distributing identical objects into distinct boxes.
- **Permutations and combinations:** Used when objects are distinct.
- **Probability approach:** Used when allocations are random and we want to calculate likelihoods.

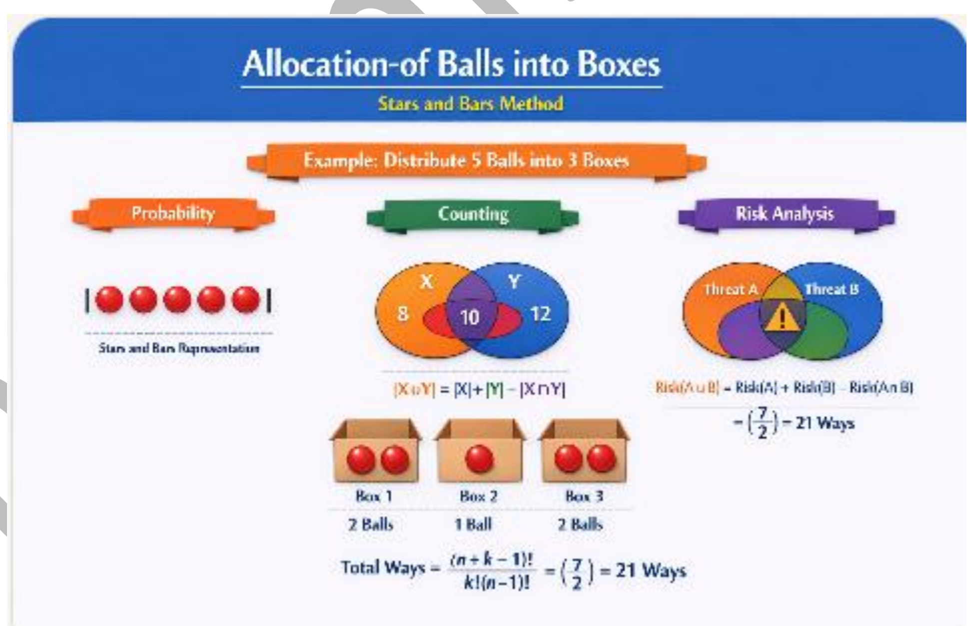
Example: Number of ways to distribute 5 identical balls into 3 boxes (allowing empty boxes):

$$5+3-13-1=72=21$$

4.1.3 Applications in resource distribution and probability

- **Resource allocation:** Assigning tasks to workers or distributing goods among stores.
- **Probability problems:** Calculating likelihoods in random distribution scenarios.
- **Computer science:** Memory allocation and load balancing problems.

Fill in the blank: The **stars and bars method** is used to solve allocation problems involving ___________________ objects into distinct boxes.



Allocation problems in probability

Pilot Questions 4.1

1. Allocation problems involve: A. Distributing objects or resources among individuals or categories B. Tossing coins C. Rolling dice D. None of the above
2. The stars and bars method is used for: A. Identical objects into distinct boxes B. Distinct objects into identical boxes C. Random guessing D. None of the above
3. The number of ways to distribute 5 identical balls into 3 boxes (allowing empty boxes) is: A. 21 B. 15 C. 10 D. None of the above
4. Allocation problems are applied in: A. Resource distribution and probability B. Ignoring constraints C. Random guessing only D. None of the above
5. When objects are distinct, allocation problems are solved using: A. Permutations and combinations B. Stars and bars C. Ignoring probabilities D. None of the above

4.2 Matching Problems

Matching problems in probability and combinatorics involve pairing items or assigning them to positions, often under constraints. A classical example is the **derangement problem**, where no item ends up in its original position. These problems illustrate how probability interacts with arrangements and permutations.

Fill in the blank: A matching problem where no item ends up in its original position is called a ________________.

4.2.1 Definition of matching problems

A **matching problem** occurs when objects must be paired or assigned to positions. The challenge lies in determining the number of valid matchings or the probability of certain outcomes.

Example: Assigning letters to envelopes randomly and asking how many end up in the correct envelope.

4.2.2 The classical "matching problem" (derangements)

A **derangement** is a permutation of objects where none of the objects appear in their original position.

The number of derangements of nobjects is given by:

$$!n=n!1-11!+12!-13!+\cdots+(-1)n1n!$$

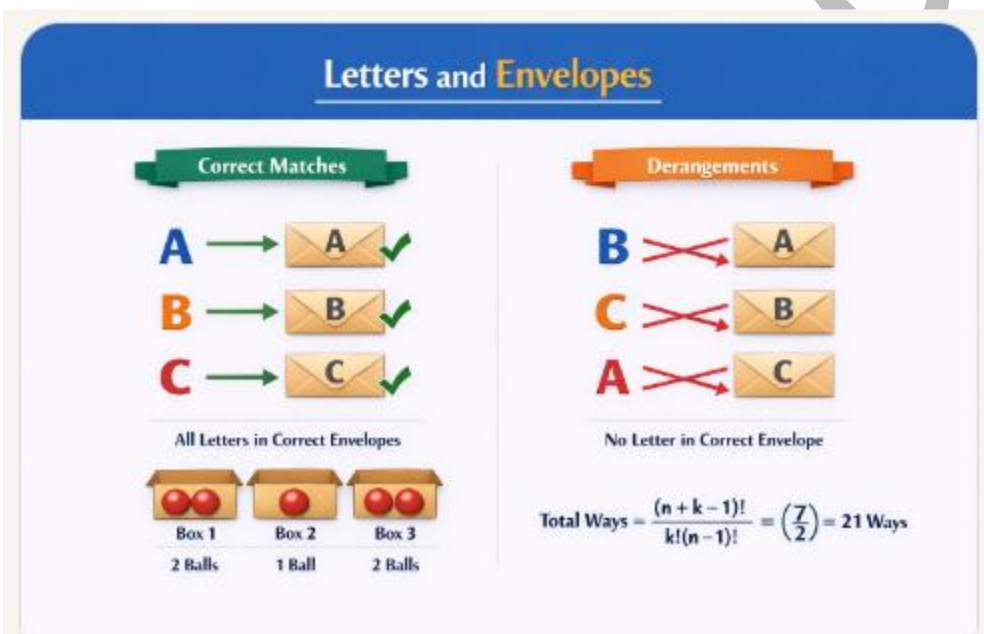
Example: For 3 letters and 3 envelopes, the number of derangements is:

!3=3!1-11!+12!-13!=61-1+12-16=2

4.2.3 Applications in probability and combinatorics

- **Probability problems:** Calculating the chance that no item is correctly matched.
- **Cryptography:** Derangements are used in designing secure systems.
- **Game theory:** Matching problems appear in resource allocation and assignment games.

Fill in the blank: The number of derangements of nobjects is denoted by __________________.



Matching problems and derangements

Pilot Questions 4.2

1. A matching problem involves: A. Pairing or assigning objects to positions B. Tossing coins C. Rolling dice D. None of the above
2. A derangement is: A. A permutation where no object is in its original position B. A permutation where all objects are in their original position C. A random guess D. None of the above
3. The number of derangements of nobjects is denoted by: A. !nB. n!C. P(n)D. None of the above

4. The number of derangements of 3 objects is: A. 2 B. 3 C. 6 D. None of the above
5. Matching problems are applied in: A. Probability, cryptography, and game theory B. Random guessing only C. Ignoring arrangements D. None of the above

4.3 Probability Generating Functions (PGFs)

Probability generating functions (PGFs) are powerful tools in probability theory that allow us to encode the distribution of a discrete random variable into a function. They simplify calculations of probabilities, expectations, and variances, and are especially useful in advanced topics like branching processes and queueing theory.

Fill in the blank: A function that encodes the distribution of a discrete random variable is called a __________________ generating function.

4.3.1 Definition and properties of PGFs

For a discrete random variable Xtaking non-negative integer values, the **probability generating function** is defined as:

$$GX(s)=E[sX]=k=0P(X=k)\cdot sk$$

Properties:

- $GX(1)=1$(since probabilities sum to 1).
- The coefficients of skgive the probabilities $P(X=k)$.
- Derivatives of $GX(s)$at $s=1$yield moments of X.

4.3.2 Using PGFs to calculate probabilities

- To find $P(X=k)$, expand $GX(s)$and read off the coefficient of sk.
- To find the mean:

$$E[X]=GX'(1)$$

- To find the variance:

$$Var(X)=GX''(1)+GX'(1)-(GX'(1))2$$

4.3.3 Examples with discrete random variables

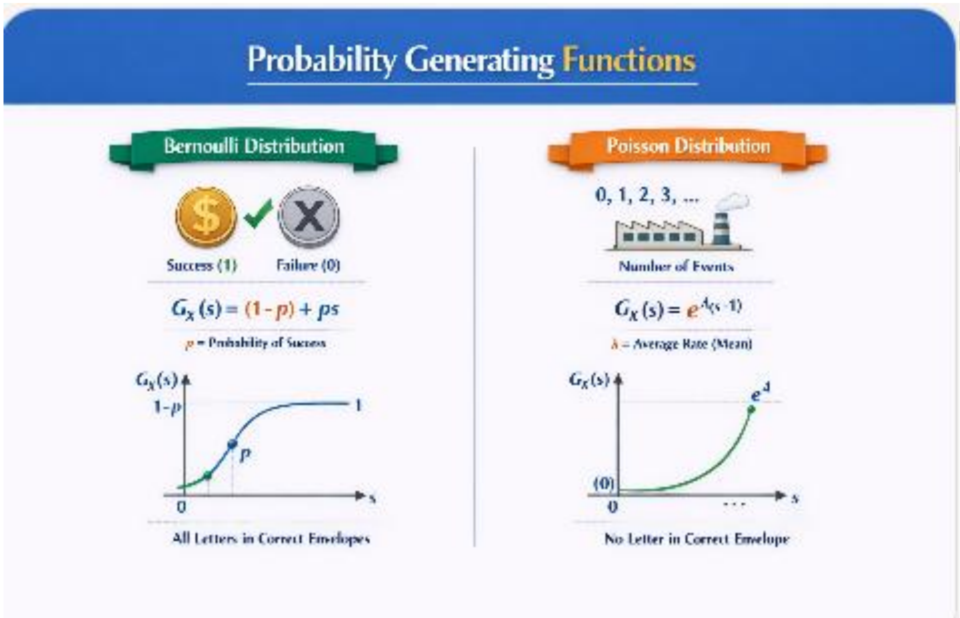
- **Example 1:** If Xis a Bernoulli random variable with success probability p:

$$GX(s)=(1-p)+ps$$

- **Example 2:** If Xis a Poisson random variable with mean :

$$GX(s)=e\lambda(s-1)$$

Fill in the blank: For a Poisson random variable with mean , the PGF is ____________________.



Probability generating functions for discrete random variables

Pilot Questions 4.3

1. A probability generating function encodes: A. The distribution of a discrete random variable B. Continuous random variables only C. Random guessing D. None of the above
2. The PGF of a random variable Xis: A. $GX(s)=\sum P(X=k)sk$B. $GX(s)=P(X)+s$C. $GX(s)=Xs$D. None of the above
3. For a Bernoulli random variable with success probability p, the PGF is: A. $(1-p)+ps$B. $e\lambda(s-1)$C. spD. None of the above
4. For a Poisson random variable with mean , the PGF is: A. $e\lambda(s-1)$B. $(1-p)+ps$C. sD. None of the above
5. The mean of a random variable can be obtained from the PGF by: A. $GX'(1)$B. $GX(0)$C. $GX''(0)$D. None of the above

4.4 Applications of Probability Generating Functions (PGFs)

Probability generating functions are not just abstract tools—they have powerful applications in applied probability, statistics, and mathematical modelling. PGFs allow us to handle complex distributions and processes more elegantly by converting probability problems into algebraic manipulations.

Fill in the blank: PGFs are especially useful in analysing __________________ processes, where each individual produces a random number of offspring.

4.4.1 PGFs in branching processes

- **Branching processes** model population growth where each individual produces a random number of offspring.
- The PGF of the offspring distribution helps determine extinction probabilities and expected population size.
- Example: If each individual has a Poisson() number of offspring, the PGF is $G(s)=e\lambda(s-1)$.

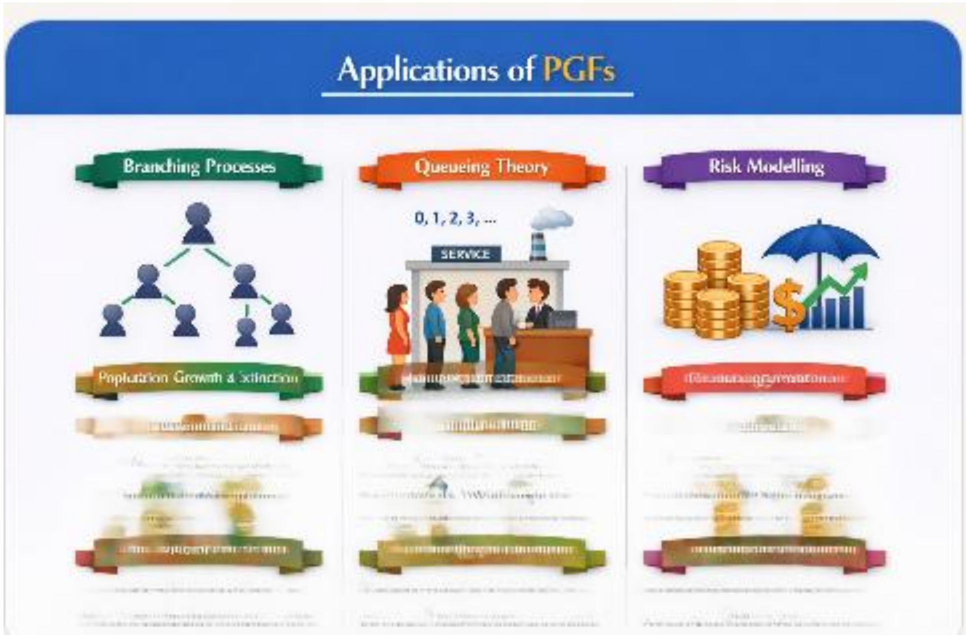
4.4.2 PGFs in queueing theory

- Queueing models often involve random arrivals and service times.
- PGFs are used to describe the distribution of the number of customers in the system.
- Example: In an M/M/1 queue, PGFs help calculate probabilities of queue lengths and waiting times.

4.4.3 PGFs in risk modelling

- In insurance and finance, PGFs are used to model aggregate claims or losses.
- They simplify calculations of expected values and variances of total claims.
- Example: If claim counts follow a Poisson distribution and claim sizes are independent, PGFs can be used to derive the distribution of total losses.

Fill in the blank: In insurance, PGFs are used to model aggregate __________________ or losses.



Applications of probability generating functions

Pilot Questions 4.4

1. PGFs are especially useful in analysing: A. Branching processes B. Tossing coins C. Rolling dice D. None of the above
2. In branching processes, PGFs help determine: A. Extinction probabilities and expected population size B. Random guessing C. Ignoring probabilities D. None of the above
3. In queueing theory, PGFs are used to: A. Describe the distribution of queue lengths B. Toss coins C. Roll dice D. None of the above
4. In insurance, PGFs are used to model aggregate: A. Claims or losses B. Coins C. Cards D. None of the above
5. PGFs simplify calculations of: A. Expected values and variances B. Random guessing only C. Ignoring distributions D. None of the above

4.5 Practical Applications and Limitations

Allocation problems, matching problems, and probability generating functions (PGFs) are not just theoretical constructs—they have significant practical applications across mathematics, computer science, economics, and risk modelling.

However, each method also comes with limitations that must be recognised to avoid misapplication.

4.5.1 Real-life uses of allocation problems

- **Resource distribution:** Assigning tasks to workers, distributing goods among stores, or allocating memory in computer systems.
- **Probability experiments:** Modelling random distribution of objects into categories.
- **Economics:** Allocation of resources in optimisation problems.

4.5.2 Real-life uses of matching problems

- **Cryptography:** Derangements are used in designing secure systems where predictable matches must be avoided.
- **Game theory:** Matching problems appear in assignment games and stable marriage problems.
- **Logistics:** Matching workers to jobs or students to schools.

4.5.3 Real-life uses of PGFs

- **Branching processes:** Modelling population growth and extinction probabilities.
- **Queueing theory:** Analysing waiting times and queue lengths.
- **Insurance and finance:** Modelling aggregate claims and losses.

4.5.4 Limitations

- **Allocation problems:** Can become computationally complex with large numbers of objects and constraints.
- **Matching problems:** Derangements are elegant but limited to specific scenarios; real-world matching often involves preferences and constraints.
- **PGFs:** Require discrete random variables; continuous distributions need other tools. They also demand accurate probability distributions, which may not always be available.

Fill in the blank: PGFs are limited to __________________ random variables and cannot directly handle continuous distributions.

Applications and limitations of allocation, matching, and PGFs

Pilot Questions 4.5

1. Allocation problems are applied in: A. Resource distribution and economics B. Tossing coins only C. Ignoring probabilities D. None of the above
2. Matching problems are applied in: A. Cryptography, game theory, and logistics B. Random guessing only C. Ignoring arrangements D. None of the above
3. PGFs are applied in: A. Branching processes, queueing theory, and insurance B. Tossing coins only C. Ignoring distributions D. None of the above
4. A limitation of allocation problems is: A. They become complex with large numbers of objects and constraints B. They ignore probabilities C. They cannot be applied to resource distribution D. None of the above
5. PGFs are limited to: A. Discrete random variables B. Continuous random variables C. Random guessing only D. None of the above

Series Summary

This Series explored **allocation problems, matching problems, and probability generating functions (PGFs)**—three advanced areas of probability and combinatorics. We began with allocation problems, learning how to distribute objects among categories using methods like stars and bars, permutations, and combinations. Next, we studied matching problems, focusing on derangements where no item is in its original position. We then introduced PGFs, powerful tools for encoding distributions of discrete random variables, and examined their properties and uses. Finally, we applied PGFs to branching processes, queueing theory, and risk modelling, before considering practical applications and limitations of all three topics.

By completing this Series, you should now be able to solve allocation and matching problems, define and apply PGFs, and evaluate their usefulness in real-life contexts such as cryptography, insurance, and computer science.

Glossary of Terms

- **Allocation problem:** A probability problem involving distribution of objects among categories.
- **Derangement:** A permutation where no object remains in its original position.
- **Matching problem:** A problem involving pairing or assigning objects to positions.
- **Probability generating function (PGF):** A function that encodes the distribution of a discrete random variable.
- **Stars and bars method:** A combinatorial technique for distributing identical objects into distinct boxes.
- **Branching process:** A stochastic process modelling population growth with random offspring counts.
- **Queueing theory:** The study of waiting lines using probability models.

Concept Check Questions 4

Objective Questions

1. Define an allocation problem. (Linked to SLO 4.1)
2. State the formula for the number of derangements of nobjects. (Linked to SLO 4.2)
3. Define a probability generating function. (Linked to SLO 4.3)
4. Identify one application of PGFs in queueing theory. (Linked to SLO 4.4)
5. State one limitation of PGFs. (Linked to SLO 4.5)

Short-Answer Questions

6. Explain the stars and bars method with an example. (Linked to SLO 4.1)

7. Calculate the number of derangements of 3 objects. (Linked to SLO 4.2)

8. Write the PGF of a Bernoulli random variable with success probability p. (Linked to SLO 4.3)

Long-Answer Question

9. Evaluate the practical applications and limitations of allocation problems, matching problems, and PGFs in real-life contexts. (Linked to SLO 4.5)

Answers to Concept Check Questions 4

Objective Questions

1. An allocation problem involves distributing objects among individuals or categories.
2. !n=n!1-11!+12!-⋯+(-1)n1n!.
3. A PGF is GX(s)=∑P(X=k)sk.
4. PGFs describe the distribution of queue lengths.
5. PGFs are limited to discrete random variables.

Short-Answer Questions

6. Example: Distributing 5 identical balls into 3 boxes: 72=21.

7. !3=2.

8. G(s)=(1-p)+ps.

Long-Answer Question

- **Basic response:** Allocation problems model resource distribution, matching problems model assignments, and PGFs encode distributions.
- **Value-added response:** Allocation problems are applied in economics and computer science, matching problems in cryptography and logistics, and PGFs in branching processes, queueing theory, and insurance. Limitations include computational complexity, restricted applicability of derangements, and PGFs being limited to discrete distributions.

Answers to Pilot Questions 4

Part 4.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.2:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.3:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.4:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 4.5:** 1-A, 2-A, 3-A, 4-A, 5-A

References and Attributions 4

- **Figure 4.1:** Allocation problems. AI prompt: "Generate a diagram showing allocation of balls into boxes using stars and bars method."
- **Figure 4.2:** Matching problems. AI prompt: "Generate a diagram showing letters and envelopes with correct matches and derangements."
- **Figure 4.3:** PGFs. AI prompt: "Generate a diagram showing probability generating functions for Bernoulli and Poisson random variables."
- **Figure 4.4:** Applications of PGFs. AI prompt: "Generate a diagram showing applications of PGFs in branching processes, queueing theory, and risk modelling."
- **Figure 4.5:** Applications and limitations. AI prompt: "Generate a diagram showing practical applications of allocation problems, matching problems, and PGFs in real-life contexts."

Series 5: Bernoulli Trials, Probability Distributions, and the Poisson Process

Series Outline

- **Part 5.1: Bernoulli Trials**
  - Definition of Bernoulli trials
  - Properties and examples
  - Applications in probability
- **Part 5.2: Binomial Distribution**
  - Definition and derivation from Bernoulli trials
  - Mean and variance
  - Applications in probability problems
- **Part 5.3: Geometric and Negative Binomial Distributions**
  - Geometric distribution as the probability of first success

- Negative binomial distribution as generalisation
    - Applications in waiting time problems
- **Part 5.4: Poisson Distribution**
    - Definition and derivation as a limit of binomial distribution
    - Mean and variance
    - Applications in rare event modelling
- **Part 5.5: Poisson Process**
    - Definition and properties
    - Interarrival times and exponential distribution
    - Applications in queueing, traffic, and risk analysis

Introduction

This Series focuses on **Bernoulli trials, probability distributions, and the Poisson process**, which form the backbone of discrete probability theory. Bernoulli trials are the simplest random experiments with two possible outcomes: success or failure. From repeated Bernoulli trials, we derive the binomial distribution, which models the number of successes in a fixed number of trials. Extending this, we study the geometric and negative binomial distributions, which model waiting times until success.

We then introduce the Poisson distribution, which arises as a limit of the binomial distribution and is widely used to model rare events. Finally, we study the Poisson process, a stochastic process that models random events occurring over time, with applications in queueing theory, traffic flow, and risk analysis.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 5.1** define Bernoulli trials and explain their properties,
- **SLO 5.2** derive and apply the binomial distribution,
- **SLO 5.3** describe and apply geometric and negative binomial distributions,
- **SLO 5.4** define and apply the Poisson distribution, and

- **SLO 5.5 explain the Poisson process and evaluate its applications in real-life contexts.**

5.1 Bernoulli Trials

Bernoulli trials are the simplest type of random experiment in probability theory. Each trial has exactly **two possible outcomes**: success or failure. They form the foundation for many probability distributions, including the binomial, geometric, and negative binomial distributions.

Fill in the blank: A random experiment with two possible outcomes, success or failure, is called a __________________ trial.

5.1.1 Definition of Bernoulli trials

A **Bernoulli trial** is a random experiment that results in one of two outcomes:

- **Success** (often coded as 1)
- **Failure** (often coded as 0)

The probability of success is denoted by p, and the probability of failure is 1-p.

5.1.2 Properties of Bernoulli trials

- Each trial has only two outcomes.
- The probability of success remains constant across trials.
- Trials are independent (the outcome of one does not affect another).

Example: Tossing a coin (success = heads, failure = tails).

5.1.3 Applications in probability

- **Coin tossing:** Success = heads, failure = tails.
- **Quality control:** Success = defect detected, failure = no defect.
- **Medical trials:** Success = patient recovers, failure = patient does not recover.

Fill in the blank: In Bernoulli trials, the probability of success is denoted by __________________.

Bernoulli trials in probability

Pilot Questions 5.1

1. A Bernoulli trial has: A. Two possible outcomes B. Three possible outcomes C. Random guessing only D. None of the above
2. The probability of success in a Bernoulli trial is denoted by: A. pB. qC. 1-pD. None of the above
3. The probability of failure in a Bernoulli trial is: A. 1-pB. pC. qD. None of the above
4. Bernoulli trials are applied in: A. Coin tossing, quality control, and medical trials B. Random guessing only C. Ignoring probabilities D. None of the above
5. A key property of Bernoulli trials is: A. Independence of trials B. Changing probabilities C. More than two outcomes D. None of the above

5.2 Binomial Distribution

The **binomial distribution** arises naturally from repeated Bernoulli trials. It models the number of successes in a fixed number of independent trials, each with the same probability of success.

Fill in the blank: The distribution that models the number of successes in repeated Bernoulli trials is called the ___________________ distribution.

5.2.1 Definition and derivation from Bernoulli trials

If we perform nindependent Bernoulli trials, each with probability of success p, then the probability of obtaining exactly ksuccesses is given by the **binomial distribution**:

$$P(X=k)=nkpk(1-p)n-k,k=0,1,2,\ldots,n$$

Here, nkrepresents the number of ways to choose ksuccesses out of ntrials.

5.2.2 Mean and variance

For a binomial random variable $X\sim Bin(n,p)$:

- **Mean:** $E[X]=np$
- **Variance:** $Var(X)=np(1-p)$

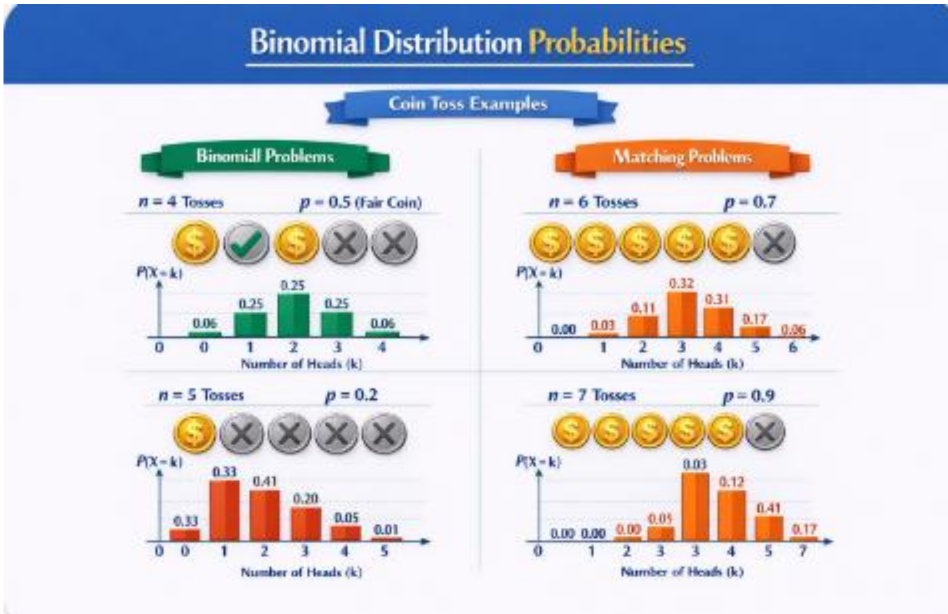
5.2.3 Applications in probability problems

- **Coin tossing:** Probability of getting exactly 3 heads in 5 tosses.
- **Quality control:** Probability of finding a certain number of defective items in a batch.
- **Elections:** Probability of a candidate receiving a given number of votes in a sample.

Example: Probability of getting exactly 2 heads in 4 coin tosses:

$$P(X=2)=42122122=6\cdot116=616=38$$

Fill in the blank: The mean of a binomial distribution with parameters nand pis ____________________.



Binomial distribution for coin tosses

Pilot Questions 5.2

1. The binomial distribution models: A. The number of successes in repeated Bernoulli trials B. Continuous random variables only C. Random guessing D. None of the above
2. The probability of exactly ksuccesses in ntrials is: A. nkpk(1-p)n-kB. npC. pnD. None of the above
3. The mean of a binomial distribution is: A. npB. n(1-p)C. pnD. None of the above
4. The variance of a binomial distribution is: A. np(1-p)B. np2C. n2pD. None of the above
5. The probability of getting exactly 2 heads in 4 coin tosses is: A. 38B. 12C. 14D. None of the above

5.3 Geometric and Negative Binomial Distributions

The **geometric distribution** and the **negative binomial distribution** are closely related to Bernoulli trials. They model the number of trials needed until success occurs, or until a fixed number of successes occur.

Fill in the blank: The distribution that models the number of trials until the first success is called the __________________ distribution.

5.3.1 Geometric distribution

- The **geometric distribution** models the probability that the first success occurs on the k-th trial.
- Formula:

$$P(X=k)=(1-p)k-1p,k=1,2,3,\ldots$$

- Mean: $E[X]=1p$
- Variance: $Var(X)=1-pp2$

Example: Probability that the first head occurs on the 3rd toss of a fair coin:

$$P(X=3)=(1-12)212=18$$

5.3.2 Negative binomial distribution

- The **negative binomial distribution** generalises the geometric distribution.
- It models the probability that the r-th success occurs on the k-th trial.
- Formula:

$$P(X=k)=k-1r-1pr(1-p)k-r,k=r,r+1,\ldots$$

- Mean: $E[X]=rp$
- Variance: $Var(X)=r(1-p)p2$

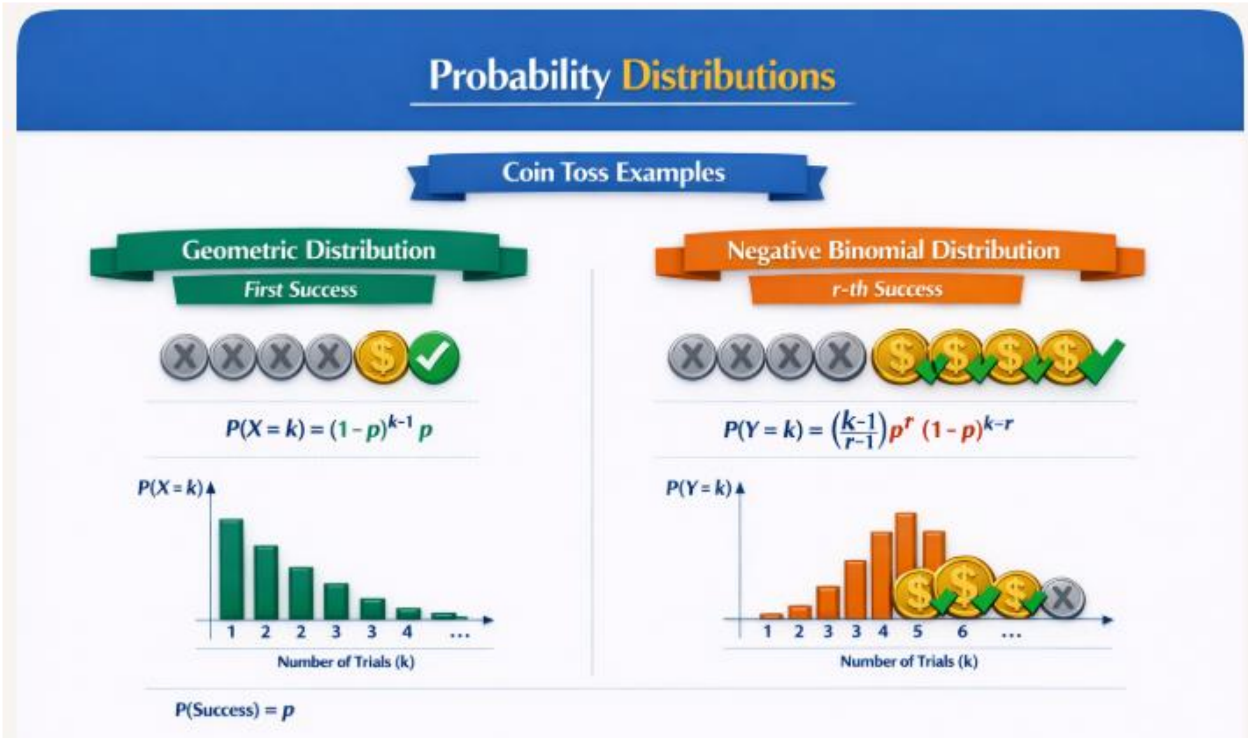
Example: Probability that the 2nd head occurs on the 4th toss of a fair coin:

$$P(X=4)=31122122=3\cdot 116=316$$

5.3.3 Applications in waiting time problems

- **Geometric distribution:** Waiting time until the first success (e.g., first defective item found).
- **Negative binomial distribution:** Waiting time until the r-th success (e.g., number of patients treated until 5 recoveries).

Fill in the blank: The negative binomial distribution models the trial on which the __________________ success occurs.



Geometric and negative binomial distributions

Pilot Questions 5

1. The geometric distribution models: A. The trial on which the first success occurs B. The number of successes in ntrials C. Continuous random variables only D. None of the above
2. The probability that the first success occurs on the k-th trial is: A. (p)k-1pB. pkC. nkpk(1-p)n-kD. None of the above
3. The mean of a geometric distribution is: A. 1pB. npC. rpD. None of the above
4. The negative binomial distribution models: A. The trial on which the r-th success occurs B. The first success only C. Continuous random variables D. None of the above
5. The variance of a negative binomial distribution is: A. r(1-p)p2B. np(1-p)C. 1-pp2D. None of the above

5.4 Poisson Distribution

The **Poisson distribution** is one of the most important discrete probability distributions. It models the number of times an event occurs in a fixed interval of time or space, when events happen independently and at a constant average rate.

Fill in the blank: The distribution that models the number of rare events in a fixed interval is called the __________________ distribution.

5.4.1 Definition and derivation as a limit of binomial distribution

If events occur independently with a small probability pin a large number of trials n, the binomial distribution Bin(n,p)approaches the **Poisson distribution** with parameter $\lambda=np$.

The probability mass function is:

$$P(X=k)=e-\lambda kk!,k=0,1,2,\ldots$$

5.4.2 Mean and variance

For a Poisson random variable $X\sim Poisson(\lambda)$:

- **Mean:** $E[X]=\lambda$
- **Variance:** $Var(X)=\lambda$

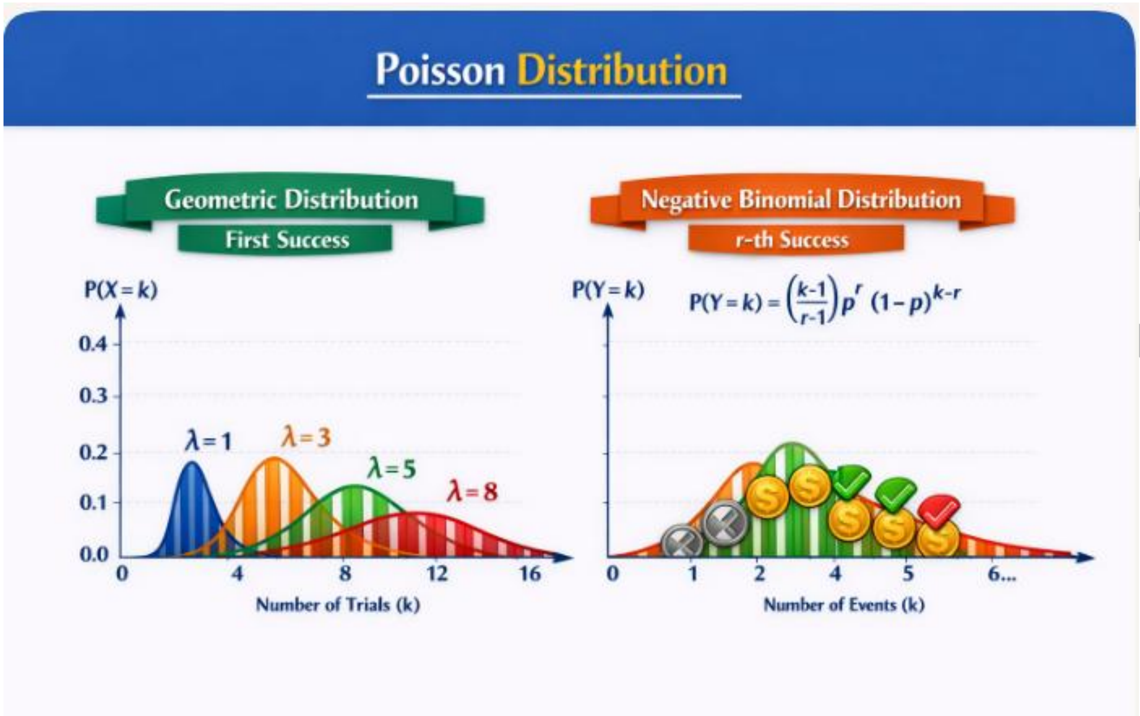
5.4.3 Applications in rare event modelling

- **Traffic flow:** Number of cars passing through an intersection in an hour.
- **Call centers:** Number of calls received in a given time period.
- **Biology:** Number of mutations in a strand of DNA.
- **Insurance:** Number of claims filed in a day.

Example: If a call center receives an average of 3 calls per minute, the probability of receiving exactly 5 calls in a minute is:

$$P(X=5)=e-3355!=e-3\cdot 243120\approx 0.1008$$

Fill in the blank: For a Poisson random variable, the mean and variance are both equal to __________________.



Poisson distribution for rare events.

Pilot Questions 5.4

1. The Poisson distribution models: A. The number of rare events in a fixed interval B. Continuous random variables only C. Random guessing D. None of the above
2. The probability mass function of a Poisson random variable is: A. e-λkk!B. nkpk(1-p)n-kC. (p)k-1pD. None of the above
3. The mean of a Poisson distribution is: A. B. npC. 1pD. None of the above
4. The variance of a Poisson distribution is: A. B. np(1-p)C. 1-pp2D. None of the above
5. If a call center receives an average of 3 calls per minute, the probability of receiving exactly 5 calls in a minute is approximately: A. 0.1008 B. 0.25 C. 0.50 D. None of the above

5.5 Poisson Process

The **Poisson process** is a stochastic process that models the occurrence of random events over time or space. It is widely used in probability, statistics, and applied fields such as queueing theory, traffic flow, and risk analysis.

Fill in the blank: A stochastic process that models random events occurring over time at a constant average rate is called the __________________ process.

5.5.1 Definition and properties

A **Poisson process** with rate has the following properties:

- Events occur one at a time.
- The number of events in disjoint intervals are independent.
- The number of events in an interval of length $t$follows a Poisson distribution with mean λt.
- Interarrival times (time between events) follow an exponential distribution with mean 1.

5.5.2 Interarrival times and exponential distribution

- If events occur according to a Poisson process with rate , then the time between successive events is exponentially distributed:

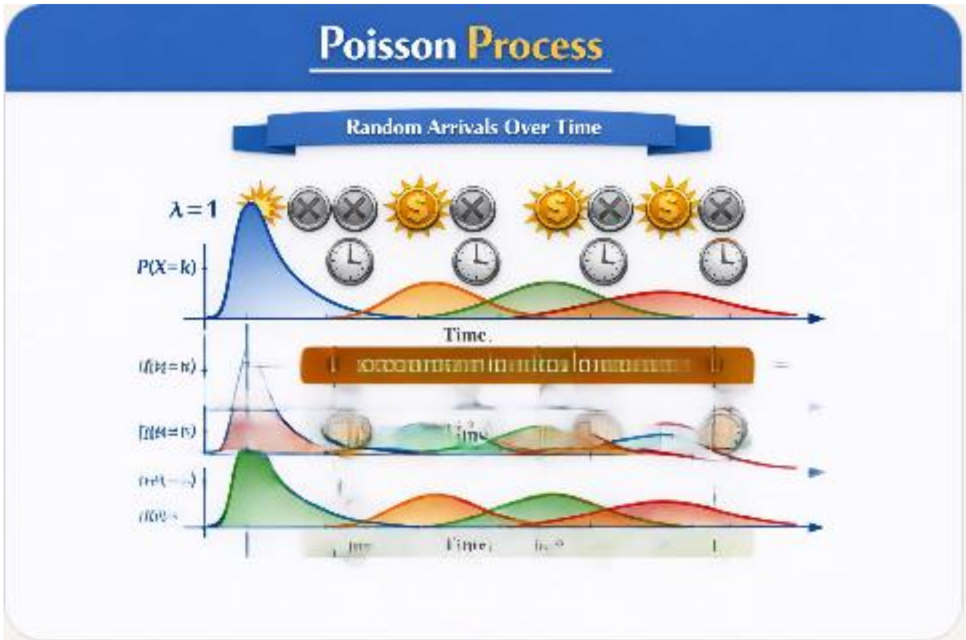
$$P(T>t)=e-\lambda t,t\geq 0$$

- This property links the Poisson process to the exponential distribution.

5.5.3 Applications in queueing, traffic, and risk analysis

- **Queueing theory:** Arrival of customers at a service desk.
- **Traffic flow:** Cars passing through an intersection.
- **Risk analysis:** Insurance claims filed over time.
- **Telecommunications:** Packets arriving in a network.

Fill in the blank: In a Poisson process, the time between successive events follows the __________________ distribution.



Poisson process and exponential interarrival times.

Pilot Questions 5.5

1. A Poisson process models: A. Random events occurring over time at a constant average rate B. Continuous random variables only C. Random guessing D. None of the above
2. The number of events in an interval of length $t$follows: A. A Poisson distribution with mean λtB. A binomial distribution C. A geometric distribution D. None of the above
3. The time between successive events in a Poisson process follows: A. Exponential distribution B. Binomial distribution C. Geometric distribution D. None of the above
4. A property of the Poisson process is: A. Independence of events in disjoint intervals B. Events always occur at fixed times C. Random guessing only D. None of the above
5. Applications of the Poisson process include: A. Queueing theory, traffic flow, and risk analysis B. Tossing coins only C. Ignoring probabilities D. None of the above

Series Summary

This Series introduced **Bernoulli trials, probability distributions, and the Poisson process**, which are fundamental concepts in discrete probability. We began with Bernoulli trials, the simplest random experiments with two outcomes: success or failure. From repeated Bernoulli trials, we derived the **binomial distribution**, which models the number of successes in a fixed number of trials. We then studied the **geometric distribution** (waiting time until the first success) and the **negative binomial distribution** (waiting time until the r-th success). Next, we explored the **Poisson distribution**, which models rare events in fixed intervals, and finally, the **Poisson process**, a stochastic process describing random events over time with exponential interarrival times.

By completing this Series, you should now be able to define and apply Bernoulli trials, binomial, geometric, negative binomial, and Poisson distributions, and explain the Poisson process with its real-life applications.

Glossary of Terms

- **Bernoulli trial:** A random experiment with two outcomes: success or failure.
- **Binomial distribution:** Distribution of the number of successes in nBernoulli trials.
- **Geometric distribution:** Distribution of the trial on which the first success occurs.
- **Negative binomial distribution:** Distribution of the trial on which the r-th success occurs.
- **Poisson distribution:** Distribution modelling the number of rare events in a fixed interval.
- **Poisson process:** A stochastic process modelling random events over time at a constant rate.
- **Exponential distribution:** Distribution of interarrival times in a Poisson process.

Concept Check Questions 5

Objective Questions

1. Define a Bernoulli trial. (Linked to SLO 5.1)
2. State the probability mass function of the binomial distribution. (Linked to SLO 5.2)

3. State the probability mass function of the geometric distribution. (Linked to SLO 5.3)
4. State the probability mass function of the Poisson distribution. (Linked to SLO 5.4)
5. Identify the distribution of interarrival times in a Poisson process. (Linked to SLO 5.5)

Short-Answer Questions

6. Explain the difference between binomial and geometric distributions. (Linked to SLO 5.2, 5.3)

7. Calculate the probability that the first head occurs on the 3rd toss of a fair coin. (Linked to SLO 5.3)

8. Calculate the probability of receiving exactly 5 calls in a minute if the average rate is 3 calls per minute. (Linked to SLO 5.4)

Long-Answer Question

9. Evaluate the practical applications of the Poisson process in queueing, traffic, and risk analysis. (Linked to SLO 5.5)

Answers to Concept Check Questions 5

Objective Questions

1. A Bernoulli trial is a random experiment with two outcomes: success or failure.
2. P(X=k)=nkpk(1-p)n-k.
3. P(X=k)=(1-p)k-1p.
4. P(X=k)=e-λkk!.
5. Exponential distribution.

Short-Answer Questions

6. Binomial counts the number of successes in ntrials; geometric counts the trial of the first success.

7. P(X=3)=(1-12)212=18.

8. P(X=5)=e-3355!≈0.1008.

Long-Answer Question

- **Basic response:** The Poisson process models random arrivals over time and is applied in queueing, traffic, and insurance.
- **Value-added response:** In queueing theory, it models customer arrivals; in traffic flow, cars at intersections; in risk analysis, insurance claims. Its exponential interarrival times make it a versatile tool for modelling real-life random events.

Answers to Pilot Questions 5

Part 5.1: 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.2:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.3:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.4:** 1-A, 2-A, 3-A, 4-A, 5-A **Part 5.5:** 1-A, 2-A, 3-A, 4-A, 5-A

References and Attributions 5

- **Figure 5.1:** Bernoulli trials. AI prompt: "Generate a diagram showing Bernoulli trials with outcomes success (1) and failure (0)."
- **Figure 5.2:** Binomial distribution. AI prompt: "Generate a diagram showing binomial distribution probabilities for coin tosses with different values of n and p."
- **Figure 5.3:** Geometric and negative binomial distributions. AI prompt: "Generate a diagram showing geometric distribution for first success and negative binomial distribution for r-th success."
- **Figure 5.4:** Poisson distribution. AI prompt: "Generate a diagram showing Poisson distribution curves for different values of lambda."
- **Figure 5.5:** Poisson process. AI prompt: "Generate a diagram showing Poisson process with random arrivals over time and exponential interarrival times."