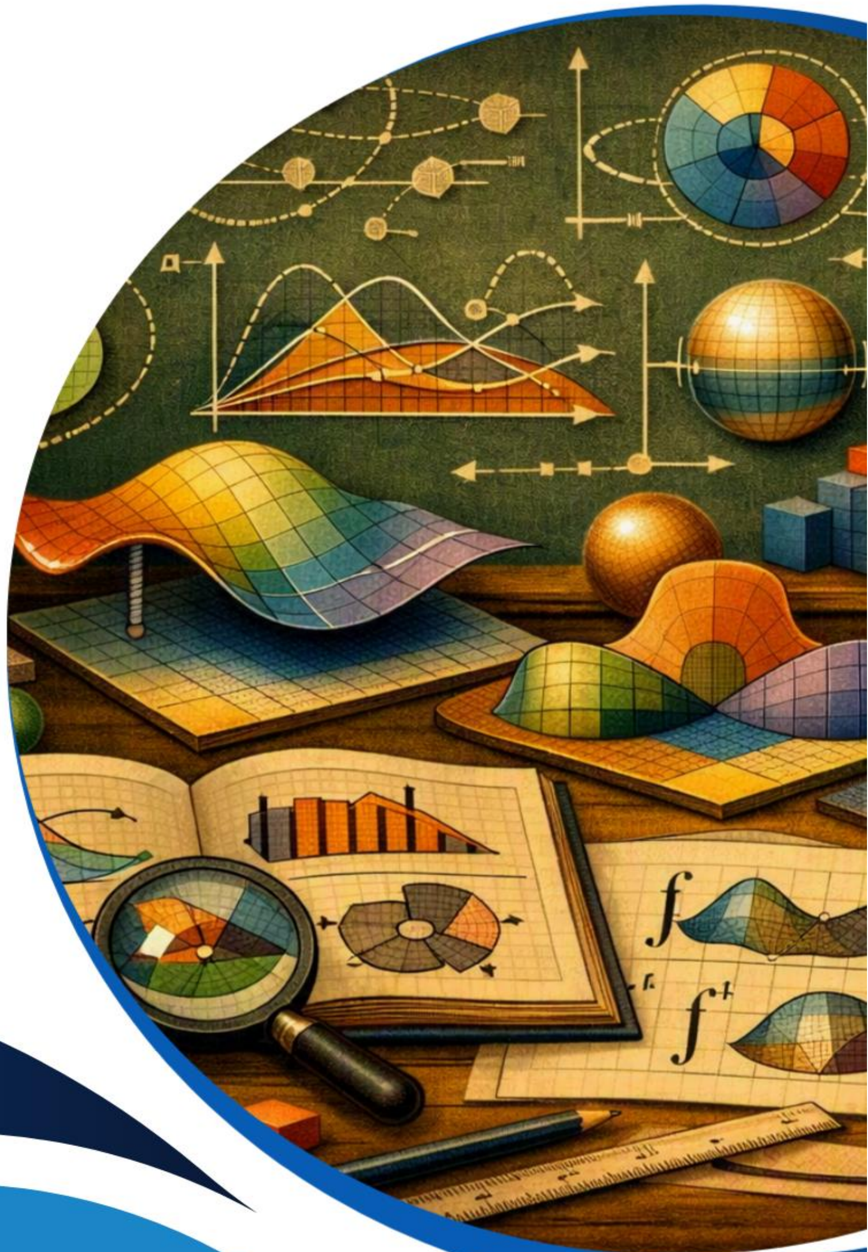
MTH307

REAL ANALYSIS II



Series 1: Riemann Integral of Functions in Rn

Series Outline

Introduction

The study of integration is central to real analysis, and the Riemann integral provides the foundation for understanding how functions can be accumulated over intervals. Extending this concept to functions defined on Rnallows us to handle multidimensional problems, which are essential in mathematics, physics, and engineering. This Series will guide you through the definition, properties, and applications of the Riemann integral in higher dimensions, preparing you for more advanced topics such as bounded variation and Stieltjes integration.

The aim of this Series is to introduce you to the Riemann integral of functions in Rn, to explain its properties, and to demonstrate how it applies to continuous and monopositive functions.

We shall commence this Series by examining the definition and properties of the Riemann integral. Next, we will extend the concept to functions in Rn, supported with illustrative examples. Following that, we will consider continuous and monopositive functions, highlighting their integrability. Finally, we will conclude with applications of the Riemann integral, where you will see how definite integrals in higher dimensions can be evaluated and applied to practical problems.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 1.1 define the Riemann integral and explain the fundamental properties of integrable functions,

SLO 1.2 demonstrate how the Riemann integral extends to functions in Rnwith illustrative examples,

SLO 1.3 describe continuous functions and define monopositive functions in Rn, analysing their integrability,

SLO 1.4 apply the Riemann integral to evaluate definite integrals in higher dimensions and solve practical problems.

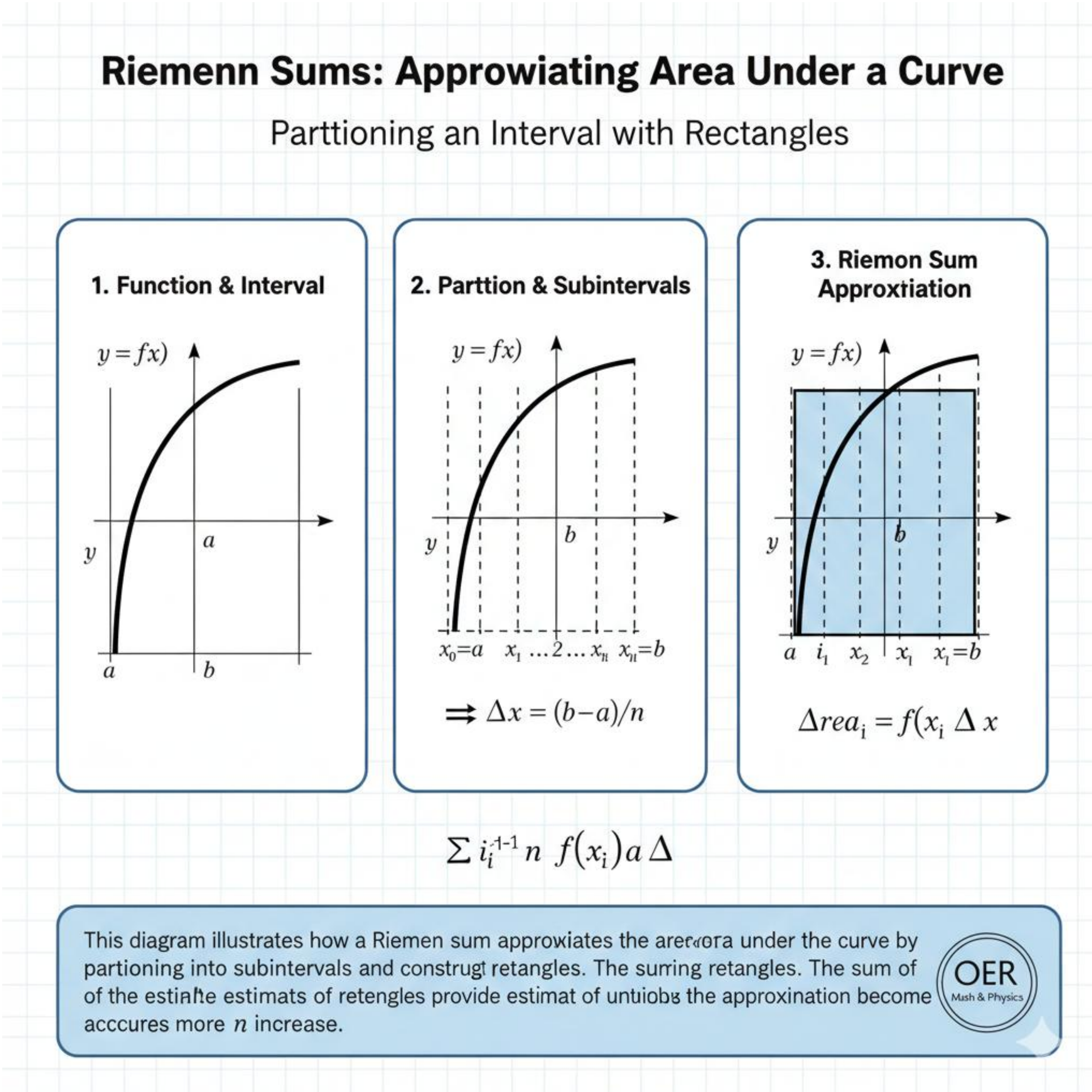
1.1 Definition and Properties of the Riemann Integral

The Riemann integral is a fundamental concept in real analysis. It provides a way of measuring the total accumulation of a function's values over an interval. The basic idea is to partition the interval into smaller subintervals, evaluate the function at chosen points within each subinterval, multiply these values by the lengths of the subintervals, and then sum the results. This sum is called a Riemann sum, and the Riemann integral is defined as the limit of these sums as the partitions become finer.

Fill in the blank: The Riemann integral is defined as the limit of __________ sums as the partitions become finer.

1.1.1 Basic definition of the Riemann integral

A function fdefined on a closed interval bis said to be Riemann integrable if the limit of its Riemann sums exists and is finite. Formally, if we partition binto subintervals and denote the maximum length of these subintervals as the mesh, then as the mesh approaches zero, the Riemann sums converge to a single value. This value is the Riemann integral of fover b.



Partitioning an interval for Riemann sums

Fill in the blank: A function is Riemann integrable if the limit of its __________ sums exists and is finite.

1.1.2 Properties of integrable functions

The Riemann integral has several important properties:

Linearity: If fand gare integrable, then af+bgis also integrable, and

ab(af(x)+bg(x))dx=aabf(x)dx+babg(x)dx

Additivity: If fis integrable on b, then for any cin b,

abf(x)dx=acf(x)dx+cbf(x)dx

Monotonicity: If f(x)≤g(x)for all xin b, then

abf(x)dx≤abg(x)dx

Boundedness: A function must be bounded on bto be Riemann integrable.

Properties of the Riemann integral Content:

- Linearity: Integrals respect addition and scalar multiplication.
- Additivity: Integrals can be split across subintervals.
- Monotonicity: Larger functions yield larger integrals.
- Boundedness: Functions must be bounded to be integrable

Property	Definition / Formula	Short Explanation
Linearity	$\int(cf+g) = c\int f + \int g$	The integral of a sum is the sum of the integrals, and constants can be pulled out.
Additivity	$\int_a^b f = \int_a^c f + \int_c^b f$	An integral over an interval $[a,b]$ can be split at any point c within that interval.
Monotonicity	If $f(x) \leq g(x)$, then $\int f \leq \int g$	If one function is consistently larger than another, its area under the curve is also larger.
Comparison	$m(b-a) \leq \int_a^b f \leq M(b-a)$	If f is bounded by m and M, the integral is bounded by the areas of the minimum and maximum rectangles.
Absolute Value	$\left	\int_{a}^{b} f(x) , dx \right

Fill in the blank: For a function to be Riemann integrable, it must be __________ on the interval.

Pilot Questions 1.1

The Riemann integral is defined as the limit of: A. Derivatives B. Riemann sums C. Infinite series D. None of the above (Linked to SLO 1.1)

A function is Riemann integrable if the limit of its sums: A. Does not exist B. Exists and is finite C. Is infinite D. None of the above (Linked to SLO 1.1)

Which property states that integrals can be split across subintervals? A. Linearity B. Additivity C. Monotonicity D. Boundedness (Linked to SLO 1.1)

If $f(x) \leq g(x)$for all xin b, then: A. abf(x)dx≥abg(x)dxB. abf(x)dx≤abg(x)dxC. abf(x)dx=abg(x)dxD. None of the above (Linked to SLO 1.1)

For a function to be Riemann integrable, it must be: A. Continuous everywhere B. Bounded on the interval C. Differentiable everywhere D. None of the above (Linked to SLO 1.1)

1.2 Riemann Integrability in Rn

The concept of the Riemann integral can be extended beyond single-variable functions to functions defined on higher-dimensional spaces such as R2or R3. In these cases, instead of partitioning an interval, we partition regions such as rectangles or cubes into smaller subregions. The function values are then multiplied by the volumes of these subregions, and the sums are taken. The limit of these sums, as the partitions become finer, defines the Riemann integral in higher dimensions.

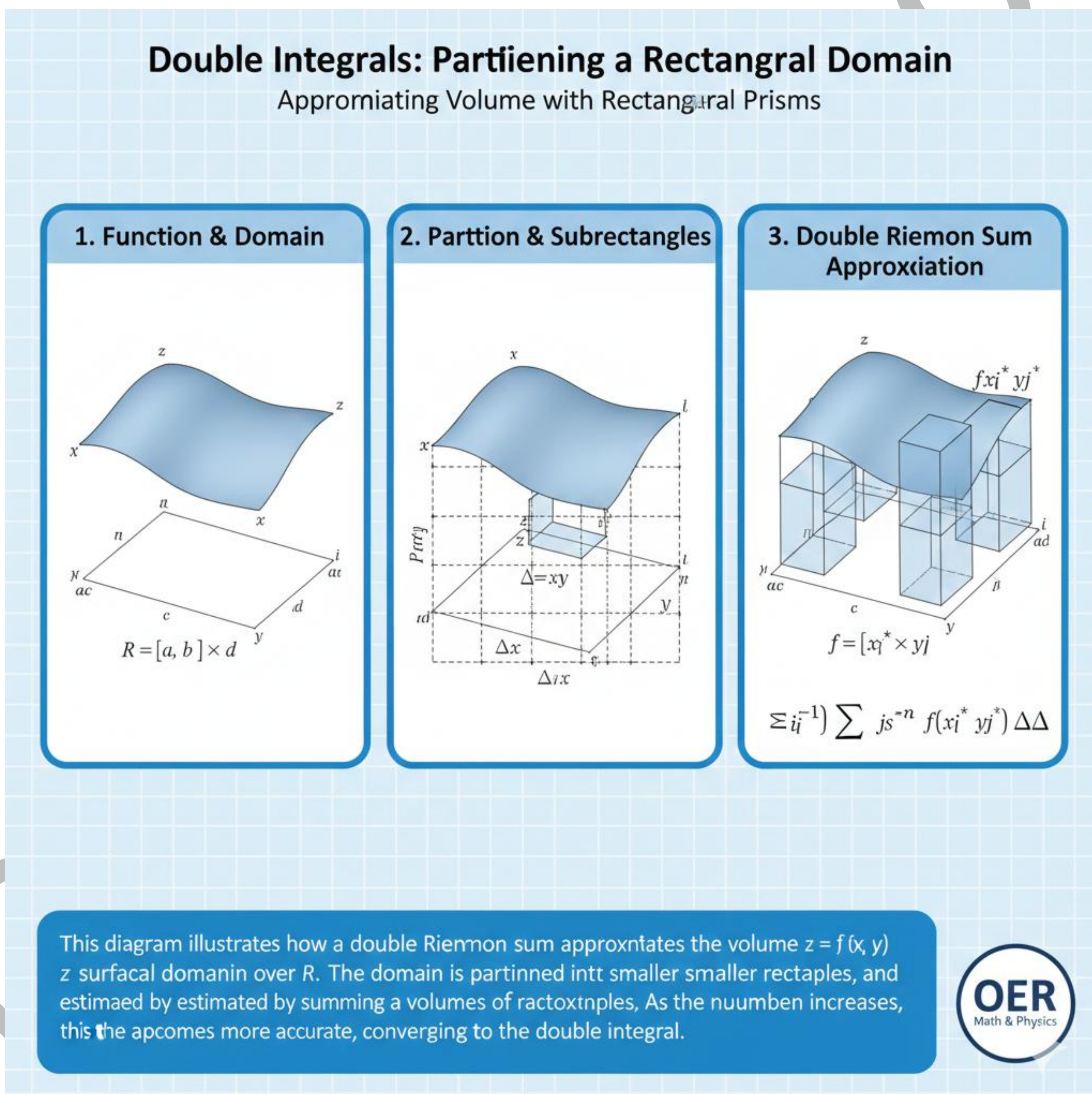
Fill in the blank: In higher dimensions, the Riemann integral is defined by partitioning regions such as __________ or cubes into smaller subregions.

1.2.1 Extension of the Riemann integral to higher dimensions

For a function $f(x,y)$defined on a rectangle $a,b]\times[c,d$, the Riemann integral is obtained by dividing the rectangle into smaller rectangles. Each rectangle contributes an area multiplied by the function value at a chosen point. As the mesh of the partition approaches zero, the sum converges to the double integral:

$$abcdf(x,y)\ dy\ dx$$

This idea generalises to Rn, where the integral is defined over n-dimensional regions.



Partitioning a rectangle in R2for Riemann sums

Fill in the blank: The double integral of a function f(x,y)over a rectangle is written as \int_a^b \int_c^d __________ \, dy \, dx.

1.2.2 Examples of integrable functions in Rn

Consider the function f(x,y)=x+ydefined on the unit square 0,1]×[0,1. The Riemann integral is:

$$0101(x+y)\ dy\ dx$$

Evaluating this gives:

$$01xy+y22|01dx=01x+12dx=12+12=1$$

Thus, the integral of f(x,y)=x+yover the unit square is 1.

[Insert Plate here] Short description: Visual representation of the function f(x,y)=x+yover the unit square. Caption: Plate 1.1 Graph of f(x,y)=x+yover the unit square AI prompt: "Generate a 3D surface plot of the function f(x,y) = x + y over the unit square [0,1] × [0,1]." Search string: "surface plot x+y unit square OER"

Fill in the blank: The integral of f(x,y)=x+yover the unit square 0,1]×[0,1is __________.

Pilot Questions 1.2

In higher dimensions, the Riemann integral is defined by partitioning: A. Intervals B. Rectangles or cubes C. Circles D. None of the above (Linked to SLO 1.2)

The double integral of a function f(x,y)over a rectangle is written as: A. abf(x)dxB. abcdf(x,y)dydxC. cdf(y)dyD. None of the above (Linked to SLO 1.2)

The function f(x,y)=x+yover the unit square integrates to: A. 0 B. 1 C. 2 D. None of the above (Linked to SLO 1.2)

Partitioning a rectangle into smaller rectangles for integration is an example of: A. Riemann sums in higher dimensions B. Taylor series expansion C. Fourier analysis D. None of the above (Linked to SLO 1.2)

The Riemann integral in Rngeneralises the concept of: A. Differentiation B. Integration C. Limits D. None of the above (Linked to SLO 1.2)

1.3 Continuous and Monopositive Functions

The study of continuous functions and monopositive functions is essential in understanding Riemann integrability. These classes of functions often provide straightforward examples of integrable behaviour and help establish general principles.

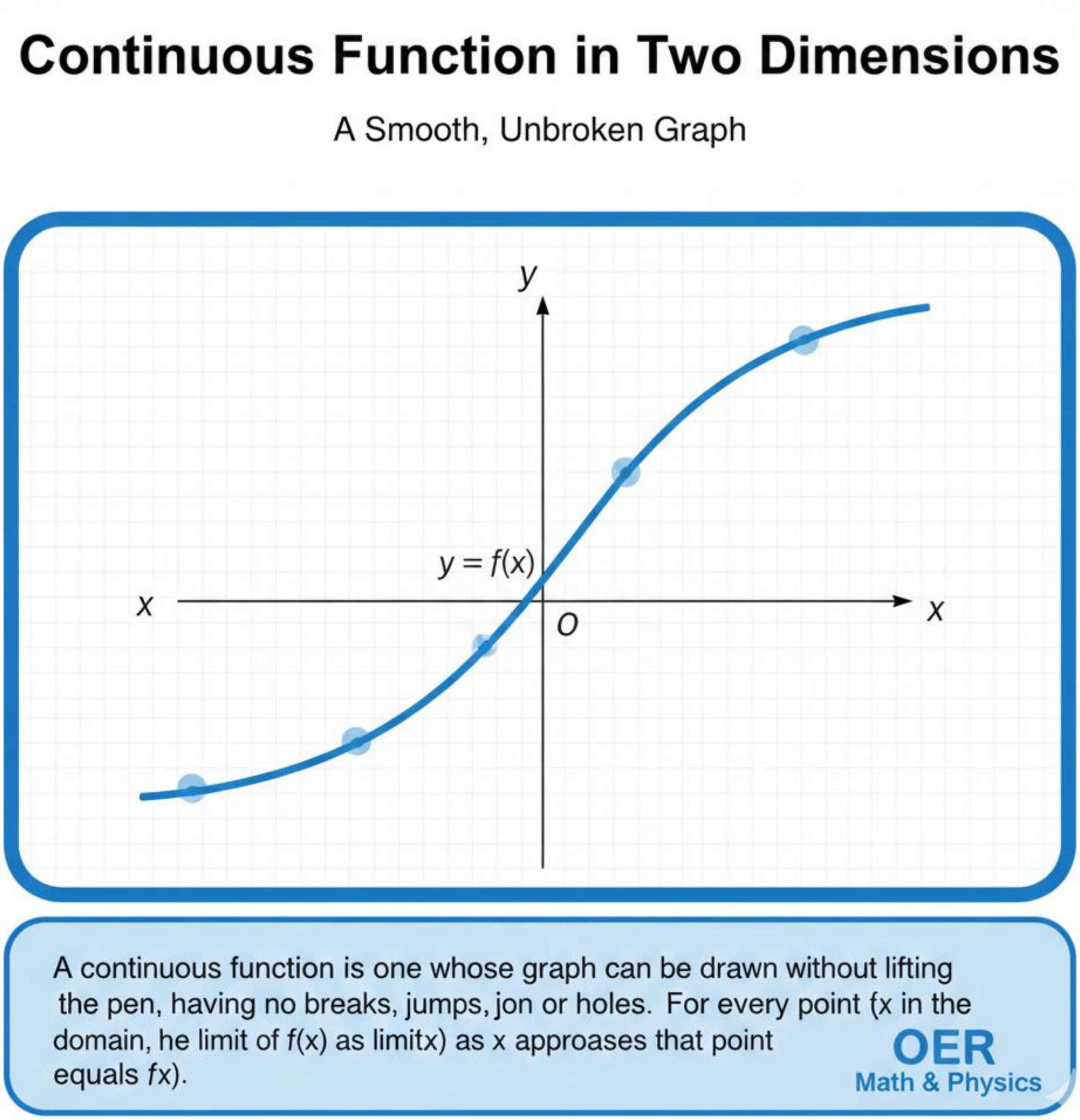
Fill in the blank: Functions that change smoothly without sudden jumps or breaks are called __________ functions.

1.3.1 Definition of continuous functions in Rn

A continuous function is one where small changes in the input lead to small changes in the output. Formally, a function $f:RnR$is continuous at a point $x0$if, for every $\epsilon>0$, there exists a $\delta>0$such that whenever $x-x0<\delta$, we have $|f(x)-f(x0)|<\epsilon$.

Continuity ensures that the function does not have abrupt jumps or breaks, which makes it easier to integrate.

Example of a continuous function in R2



at a point x0if, for every ϵ>0, there exists a __________ such that the condition holds.

1.3.2 Monopositive functions and their integrability

A monopositive function is a function that is non-negative throughout its domain. For example, f(x)=x2is monopositive on Rbecause f(x)≥0for all x. Such functions are particularly useful in integration because their integrals represent areas or volumes that cannot be negative.

Monopositive functions are always bounded below by zero, which makes them easier to handle in Riemann integration. If they are also bounded above and continuous, they are guaranteed to be Riemann integrable.

Characteristics of monopositive functions Content:

• Always non-negative over their domain.

• Represent areas or volumes in integration.

• Easier to integrate due to boundedness below.

Core Characteristics

Feature	Description
Domain Mapping	Maps the Right Half-Plane (RHP) or Upper Half-Plane (UHP) into the RHP.
Analyticity	The function must be analytic (holomorphic) within the defined domain.
Positivity	The real part of the function is non-negative: $\mathrm{Re}\{f(z)\} \geq 0$ for all z in the domain.
Real Values	For many engineering contexts, $f(z)$ is real when z is real (often called "Positive Real").
No Poles in RHP	To maintain analyticity and positivity, there are no poles or zeros in the open right half-plane.

Fill in the blank: A monopositive function is always __________ over its domain.

Pilot Questions 1.3

A function is continuous if: A. It has jumps B. Small changes in input produce small changes in output C. It is always non-negative D. None of the above (Linked to SLO 1.3)

A function is continuous at a point x0if, for every ϵ>0, there exists: A. A derivative B. A limit C. A δ>0D. None of the above (Linked to SLO 1.3)

A monopositive function is always: A. Non-negative B. Continuous C. Differentiable D. None of the above (Linked to SLO 1.3)

The function f(x)=x2is an example of: A. Continuous function B. Monopositive function C. Both continuous and monopositive function D. None of the above (Linked to SLO 1.3)

Monopositive functions are easier to integrate because they are bounded: A. Above by zero B. Below by zero C. By infinity D. None of the above (Linked to SLO 1.3)

1.4 Applications of the Riemann Integral

The Riemann integral is not only a theoretical construct but also a powerful tool for solving practical problems. It allows us to calculate areas, volumes, and other accumulated quantities in mathematics and applied sciences. By extending the concept to functions in Rn, we can evaluate integrals over multidimensional regions, which is essential in physics, engineering, and probability theory.

Fill in the blank: The Riemann integral is widely used to calculate __________, volumes, and accumulated quantities.

1.4.1 Evaluation of definite integrals in Rn

A definite integral represents the total accumulation of a function's values over a specified interval or region. In one dimension, this corresponds to the area under a curve. In higher dimensions, definite integrals represent volumes or hyper-volumes.

For example, consider the function f(x,y)=x2+y2over the unit square 0,1]×[0,1:

0101(x2+y2) dy dx

Evaluating gives:

01x2y+y33|01dx=01x2+13dx=13+13=23

Thus, the integral of f(x,y)=x2+y2over the unit square is 23.

[Insert Figure here] Short description: Diagram showing the surface of f(x,y)=x2+y2over the unit square. Caption: Figure 1.4 Graph of f(x,y)=x2+y2over the unit square AI prompt: “Generate a 3D surface plot of f(x,y) = x^2 + y^2 over the unit square [0,1] × [0,1].” Search string: “surface plot $x^{2+y}2$ unit square OER”

Fill in the blank: The integral of f(x,y)=x2+y2over the unit square 0,1]×[0,1is __________.

1.4.2 Practical examples and problem sets

The Riemann integral is applied in many contexts:

Physics: Computing work done by a variable force, or finding the centre of mass of a body.

Engineering: Calculating stress distributions or heat transfer over a region.

Probability theory: Determining probabilities by integrating density functions.

For instance, in probability, the integral of a probability density function (PDF) over its domain must equal 1. This ensures that the total probability is conserved.

Applications of the Riemann integral Content:

• Physics: Work, energy, and centre of mass

. • Engineering: Stress, heat, and flow analysis.

• Probability: Integration of density functions.

Field	Application	Mathematical Representation
Physics	**Work Done:** Calculating energy transfer when a force varies over a distance.	$W = \int_a^b F(x)\,dx$
Engineering	**Centroids & Moments:** Finding the center of mass or the moment of inertia for non-uniform objects.	$\bar{x} = \frac{1}{M} \int x \cdot \rho(x)\,dx$
Probability	**Expected Value:** Determining the "long-run average" of a continuous random variable.	$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$
Fluid Mechanics	**Hydrostatic Force:** Calculating the total pressure exerted by a fluid on a submerged surface.	$F = \int \rho g h(y) w(y)\,dy$
Statistics	**Cumulative Distribution:** Finding the probability that a variable falls within a specific range.	$P(a \leq X \leq b) = \int_a^b f(x)\,dx$

Fill in the blank: In probability theory, the integral of a probability density function over its domain must equal __________.

Pilot Questions 1.4

A definite integral represents: A. The derivative of a function B. The total accumulation of a function’s values over a region C. The slope of a curve D. None of the above (Linked to SLO 1.4)

The integral of f(x,y)=x2+y2over the unit square is: A. 1 B. 23C. 0 D. None of the above (Linked to SLO 1.4)

In physics, the Riemann integral can be used to compute: A. Work done by a variable force B. The derivative of velocity C. The slope of a tangent D. None of the above (Linked to SLO 1.4)

In probability theory, the integral of a probability density function must equal: A. 0 B. 1 C. Infinity D. None of the above (Linked to SLO 1.4)

The Riemann integral is applied in: A. Physics B. Engineering C. Probability theory D. All of the above (Linked to SLO 1.4)

Series Summary

This Series has introduced you to the Riemann integral of functions in Rn, a cornerstone of real analysis. We began by defining the Riemann integral and exploring its properties such as linearity, additivity, monotonicity, and boundedness. We then extended the concept to higher dimensions, showing how integrals can be evaluated over regions in Rn. Following that, we examined continuous and monopositive functions, highlighting their role in ensuring integrability. Finally, we applied the Riemann integral to practical problems, including evaluating definite integrals and linking the theory to applications in physics, engineering, and probability.

By completing this Series, you should now be able to define the Riemann integral and its properties (SLO 1.1), demonstrate its extension to functions in higher dimensions (SLO 1.2), describe and analyse continuous and monopositive functions (SLO 1.3), and apply the Riemann integral to evaluate definite integrals in multidimensional contexts (SLO 1.4). These outcomes provide a strong foundation for the next Series, where you will study functions of bounded variation and the Riemann–Stieltjes integral.

Glossary of Terms

Boundedness: A property of a function where its values remain within fixed upper and lower limits.

Continuous function: A function where small changes in input produce small changes in output, without jumps or breaks.

Definite integral: The numerical value obtained by integrating a function over a specified interval or region.

Integrable function: A function for which the Riemann integral exists and is finite.

Monopositive function: A function that is non-negative throughout its domain.

Riemann integral: A method of integration defined as the limit of Riemann sums over partitions of an interval or region.

Riemann sum: A sum formed by multiplying function values at chosen points by the lengths or volumes of subintervals or subregions.

Rn: The n-dimensional real space, representing functions defined over multiple variables.

Concept Check Questions 1

Objective questions

The Riemann integral is defined as the limit of: A. Derivatives B. Riemann sums C. Infinite series D. None of the above (Linked to SLO 1.1)

A monopositive function is always: A. Non-negative B. Continuous C. Differentiable D. None of the above (Linked to SLO 1.3)

The integral of f(x,y)=x+yover the unit square is: A. 0 B. 1 C. 2 D. None of the above (Linked to SLO 1.2)

Short-answer questions

Define the Riemann integral and explain its basic idea. (Linked to SLO 1.1)

Give one example of a function in R2that is Riemann integrable. (Linked to SLO 1.2)

Explain why monopositive functions are easier to integrate. (Linked to SLO 1.3)

Long-answer questions

Discuss the extension of the Riemann integral to functions in Rn. (Linked to SLO 1.2)

Basic response: State that the Riemann integral can be extended to higher dimensions by partitioning the domain into subregions.

Value-added response: Provide examples of integrable functions in R2and explain how the partitioning works in practice.

Analyse the integrability of continuous and monopositive functions. (Linked to SLO 1.3)

Basic response: State that continuous and monopositive functions are integrable.

Value-added response: Provide reasoning with examples, showing how boundedness and positivity ensure integrability.

Apply the Riemann integral to evaluate a definite integral in Rn. (Linked to SLO 1.4)

Basic response: State that definite integrals can be evaluated using Riemann sums.

Value-added response: Demonstrate with a worked example, such as integrating a simple function over a square region in R2.

Answers to Concept Check Questions 1

1-B, 2-A, 3-B

The Riemann integral is defined by partitioning the domain into intervals, multiplying function values by interval lengths, and summing the results.

Example: f(x,y)=x+yover the unit square 0,1]×[0,1.

Monopositive functions are easier to integrate because they are bounded below by zero, ensuring non-negative integrals.

Basic response: The Riemann integral extends to Rnby partitioning into subregions. Value-added response: Example: Integrating f(x,y)=x+yover a square region using double sums.

Basic response: Continuous and monopositive functions are integrable. Value-added response: Example: f(x)=x2is continuous and bounded on 1, hence integrable.

Basic response: Definite integrals are evaluated using Riemann sums. Value-added response: Example: 0101(x+y)dxdy=1.

Answers to Pilot Questions 1

Part 1.1: 1-B, 2-B, 3-B, 4-B, 5-B Part 1.2: 1-B, 2-B, 3-B, 4-A, 5-B Part 1.3: 1-B, 2-C, 3-A, 4-C, 5-B Part 1.4: 1-B, 2-B, 3-A, 4-B, 5-D

References and Attributions 1

Apostol, T. M. (1974). *Mathematical Analysis*. Addison-Wesley.

Bartle, R. G., & Sherbert, D. R. (2011). *Introduction to Real Analysis*. Wiley.

Open Educational Resources (OER) diagrams on Riemann sums and integrals.

Illustration prompts and sources:

Figure 1.1 Partitioning an interval for Riemann sums

AI prompt: "Generate a labelled diagram showing partitioning of an interval into subintervals for Riemann sums."

Search string: “Riemann sum partition diagram OER”

Figure 1.2 Partitioning a rectangle in R2for Riemann sums

AI prompt: “Generate a labelled diagram showing a rectangle in R2partitioned into smaller rectangles for double integration.”

Search string: “double integral partition diagram OER”

Figure 1.3 Example of a continuous function in R2

AI prompt: “Generate a simple graph of a continuous function in two dimensions without breaks.”

Search string: “continuous function graph OER”

Figure 1.4 Graph of f(x,y)=x2+y2over the unit square

AI prompt: “Generate a 3D surface plot of f(x,y) = x^2 + y^2 over the unit square [0,1] × [0,1].”

Search string: “surface plot $x^{2+y}2$ unit square OER”

Box 1.1 Properties of the Riemann integral

Box 1.2 Characteristics of monopositive functions

Box 1.3 Applications of the Riemann integral

Series 2: Functions of Bounded Variation and Continuous Monopositive Functions

Series Outline

Part 2.1: Functions of Bounded Variation

Sub-Part 2.1.1: Definition and examples of bounded variation

Sub-Part 2.1.2: Properties and significance in integration

Part 2.2: Continuous Monopositive Functions

Sub-Part 2.2.1: Definition of continuous monopositive functions

Sub-Part 2.2.2: Integrability and applications

Part 2.3: Relationship Between Bounded Variation and Monopositive Functions

Introduction

In the previous Series, we explored the Riemann integral of functions in Rn, focusing on its definition, properties, and applications. Building on that foundation, this Series introduces two important classes of functions: functions of bounded variation and continuous monopositive functions. These functions play a crucial role in determining integrability and in extending the scope of Riemann integration.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 define functions of bounded variation and provide examples,

SLO 2.2 explain the properties of bounded variation and their significance in integration,

SLO 2.3 describe continuous monopositive functions and analyse their integrability,

SLO 2.4 compare bounded variation and monopositive functions in relation to Riemann integration, and

SLO 2.5 apply these concepts to solve practical problems involving integration.

2.1 Functions of Bounded Variation

A function of bounded variation is one whose total variation over a given interval is finite. The concept of variation measures how much a function oscillates or changes in value. If the sum of all these changes remains bounded, the function is said to be of bounded variation. This property is important because such functions are guaranteed to be Riemann–Stieltjes integrable, making them central to advanced integration theory.

Fill in the blank: A function is of bounded variation if its total __________ over an interval is finite.

2.1.1 Definition and examples of bounded variation

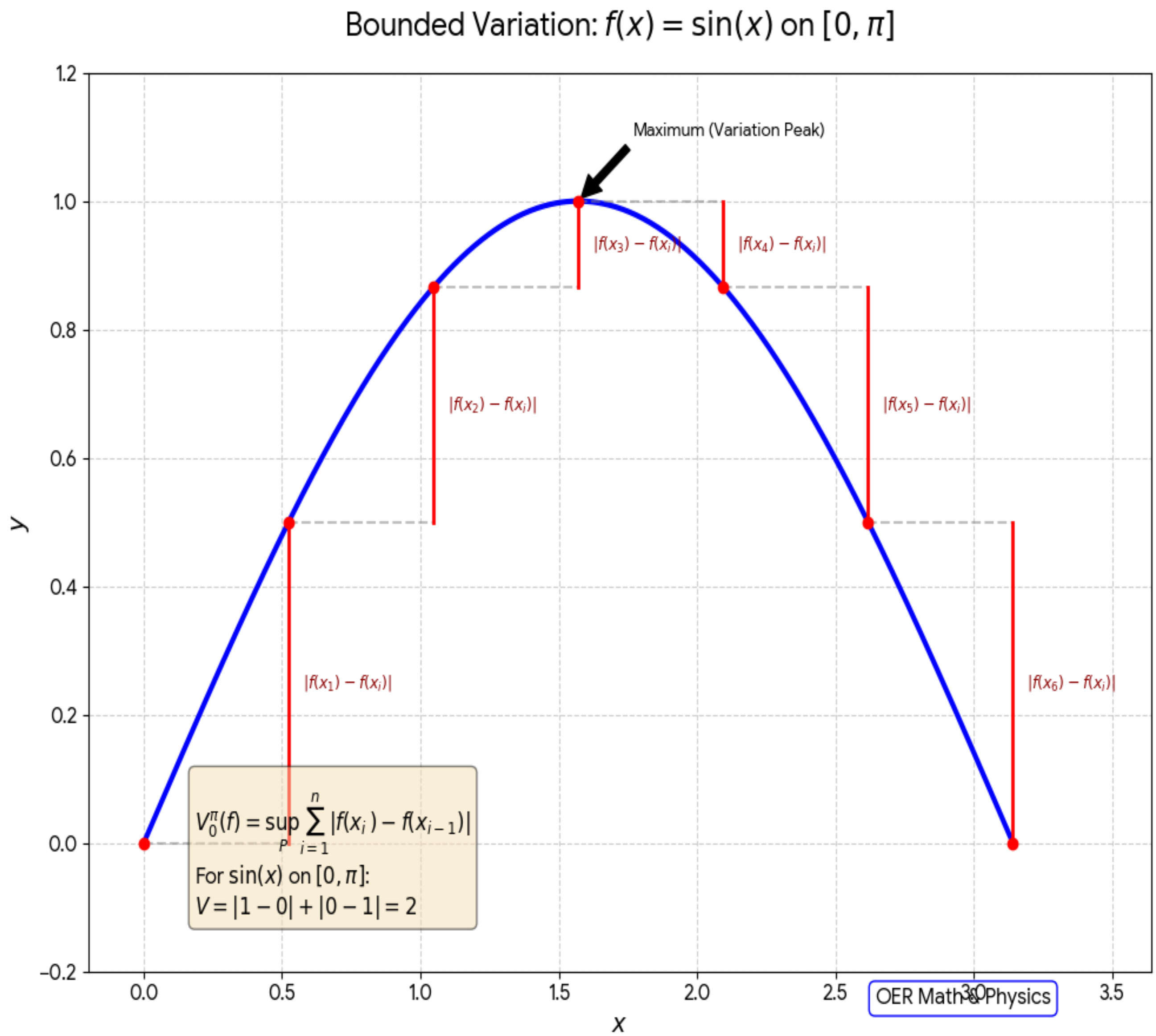
Formally, let fbe a real-valued function defined on b. The total variation of fon bis defined as:

$$Vab(f)=\sup i=1n|f(xi)-f(xi-1)|:a=x0<x1<\cdots<xn=b$$

If $Vab(f)<$, then fis said to be of bounded variation on b.

Example:

- The function $f(x)=x$on 1has bounded variation because the total variation equals 1.
- The function $f(x)=\sin(x)$on also has bounded variation, since the oscillations are finite.



Example of a function of bounded variation

Fill in the blank: The function f(x)=xon 1has bounded variation because its total variation equals __________.

2.1.2 Properties and significance in integration

Functions of bounded variation have several important properties:

Decomposition: Every function of bounded variation can be expressed as the difference of two increasing functions.

Integrability: Functions of bounded variation are Riemann–Stieltjes integrable with respect to continuous functions.

Stability: The sum and product of functions of bounded variation are also of bounded variation.

Applications: They are widely used in Fourier analysis, probability, and measure theory.

Properties of functions of bounded variation Content:

- Can be decomposed into two increasing functions.
- Are Riemann–Stieltjes integrable with respect to continuous functions.
- Closed under addition and multiplication.
- Useful in Fourier analysis and measure theory

Property	Explanation
Jordan Decomposition	Every BV function can be written as the difference of two non-decreasing functions: $f(x) = g(x) - h(x)$.
Differentiability	A function of bounded variation is differentiable **almost everywhere**. Its derivative $f'(x)$ exists and is Lebesgue integrable.
Boundedness	If f is of bounded variation on $[a, b]$, then f is necessarily bounded on that interval.
Discontinuities	A BV function can only have **jump discontinuities**, and the set of these discontinuities is at most countable.
Algebraic Closure	The sum, difference, and product of two BV functions are also of bounded variation.
Composition	If f is BV and g is Lipschitz continuous, then the composition $g \circ f$ is also of bounded variation.

Fill in the blank: Every function of bounded variation can be expressed as the difference of two __________ functions.

Pilot Questions 2.1

A function is of bounded variation if: A. Its total variation is infinite B. Its total variation is finite C. It is continuous everywhere D. None of the above (Linked to SLO 2.1)

The total variation of a function is defined as: A. The supremum of sums of absolute differences B. The derivative of the function C. The integral of the function D. None of the above (Linked to SLO 2.1)

The function f(x)=xon 1has total variation equal to: A. 0 B. 1 C. Infinity D. None of the above (Linked to SLO 2.1)

Every function of bounded variation can be expressed as: A. The sum of two decreasing functions B. The difference of two increasing functions C. The product of two bounded functions D. None of the above (Linked to SLO 2.2)

Functions of bounded variation are integrable with respect to: A. Continuous functions B. Discontinuous functions C. Differentiable functions only D. None of the above (Linked to SLO 2.2)

2.2 Continuous Monopositive Functions

A continuous monopositive function is a function that is both continuous and non-negative throughout its domain. These functions are particularly important in integration because they guarantee that the integral represents a non-negative quantity, such as an area or volume. Their continuity ensures smooth behaviour without abrupt jumps, while their non-negativity ensures that integrals are easier to interpret and evaluate.

Fill in the blank: A continuous monopositive function is always __________ and non-negative throughout its domain.

2.2.1 Definition of continuous monopositive functions

Formally, a function f:RnRis said to be continuous monopositive if:

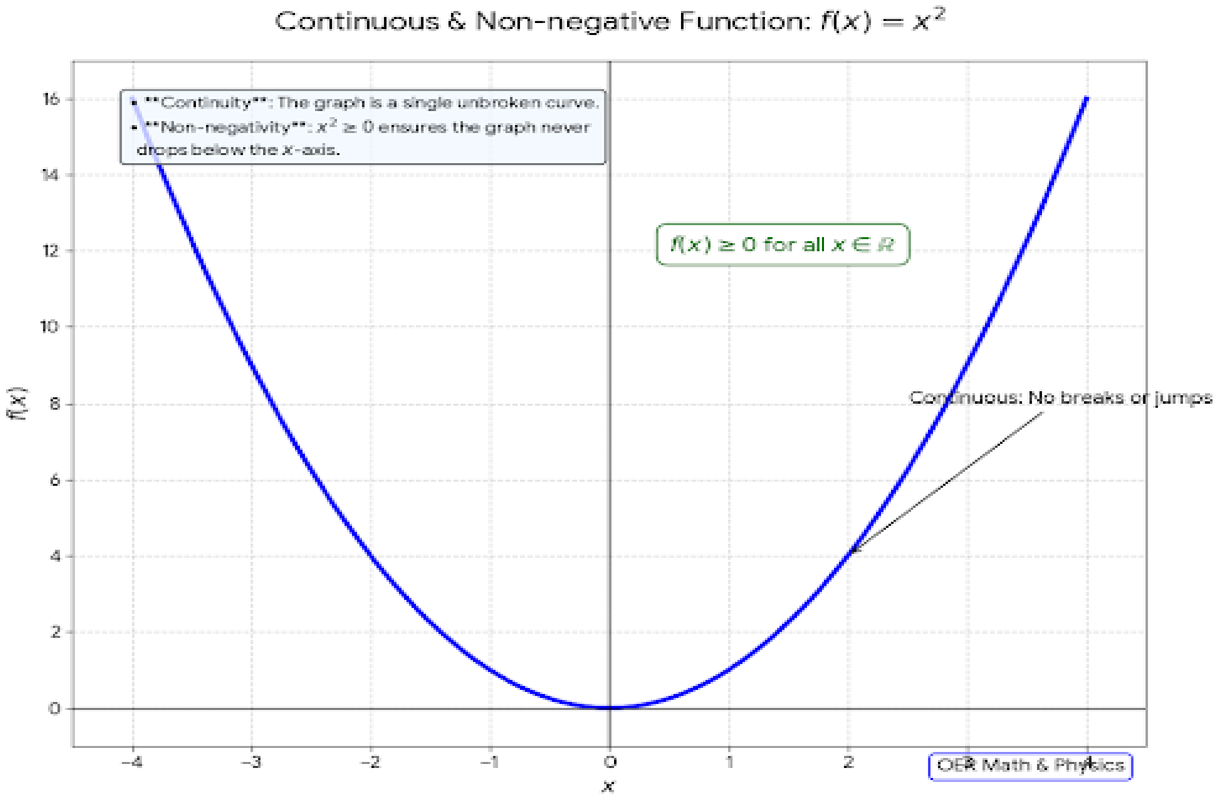
It is continuous at every point in its domain.

It satisfies f(x)≥0for all xin the domain.

Example:

The function f(x)=x2is continuous and monopositive on R.

The function f(x,y)=x2+y2is continuous and monopositive on R2.



Example of a continuous monopositive function

Fill in the blank: The function $f(x)=x^2$is continuous and monopositive because f(x) \geq __________ for all x.

2.2.2 Integrability and applications

Continuous monopositive functions are always Riemann integrable on closed and bounded intervals. Their integrals represent quantities such as:

- Areas under curves in one dimension.
- Volumes under surfaces in two or three dimensions.
- Probabilities when used as density functions in statistics.

For example, the integral of $f(x)=x^2$over 1is:

$$\int_0^1 x^2\,dx=\frac{1}{3}$$

This represents the area under the curve $y=x^2$from 0 to 1.

[Insert Box here] Caption: Box 2.2 Applications of continuous monopositive functions Content:

- Represent areas and volumes in geometry.
- Used in probability as density functions.
- Guarantee non-negative integrals.

Key Applications and Examples

Field	Application	Example Function
Probability	**Probability Density Functions (PDFs):** Used to describe the likelihood of a continuous random variable falling within a range.	The Gaussian (Normal) Distribution curve: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
Physics	**Kinetic Energy:** Representing energy as a function of velocity. Energy cannot be negative.	$K(v) = \frac{1}{2}mv^2$
Statistics	**Mean Squared Error (MSE):** A measure of the quality of an estimator; it is always non-negative, where zero indicates perfect fit.	$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$
Engineering	**Power Dissipation:** Calculating the rate at which energy is lost as heat in a resistor.	$P(I) = I^2R$
Economics	**Cost Functions:** Representing the total cost of production, which typically increases continuously with quantity.	$C(q) = aq^2 + bq + c$ (where $a, b, c > 0$)

Fill in the blank: The integral of f(x)=x2over 1is __________.

Pilot Questions 2.2

A continuous monopositive function is: A. Continuous and non-negative B. Continuous and non-positive C. Discontinuous and non-negative D. None of the above (Linked to SLO 2.3)

The function f(x)=x2is an example of: A. Continuous monopositive function B. Discontinuous function C. Non-integrable function D. None of the above (Linked to SLO 2.3)

Continuous monopositive functions are always: A. Riemann integrable on closed and bounded intervals B. Non-integrable C. Differentiable everywhere D. None of the above (Linked to SLO 2.3)

The integral of f(x)=x2over 1is: A. 1 B. 13C. 0 D. None of the above (Linked to SLO 2.3)

Continuous monopositive functions are useful in: A. Geometry B. Probability C. Both geometry and probability D. None of the above (Linked to SLO 2.3)

2.3 Relationship Between Bounded Variation and Monopositive Functions

The concepts of bounded variation and continuous monopositive functions are closely related in the study of integration. Both provide conditions under which functions are guaranteed to be integrable, but they do so in different ways. Functions of bounded variation are controlled by the finiteness of their oscillations, while continuous monopositive functions are controlled by their smoothness and non-negativity.

Fill in the blank: Functions of bounded variation are controlled by the finiteness of their __________.

2.3.1 Comparative analysis

Bounded variation focuses on the total amount of change a function undergoes. If the variation is finite, the function is well-behaved enough to be integrable.

Continuous monopositive functions focus on smoothness and non-negativity. Their continuity ensures no abrupt jumps, and their non-negativity ensures integrals represent meaningful quantities such as areas or probabilities.

Together, these conditions provide complementary perspectives. While bounded variation allows for functions that may not be strictly monotonic but still integrable, continuous monopositive functions guarantee non-negative integrals that are easier to interpret.

[Insert Table here] Caption: Table 2.1 Comparison between bounded variation and continuous monopositive functions

Feature	Bounded Variation	Continuous Monopositive

Definition	Finite total variation	Continuous and non-negative
Guarantee	Riemann–Stieltjes integrable	Riemann integrable
Behaviour	May oscillate but controlled	Smooth and non-negative
Applications	Fourier analysis, measure theory	Geometry, probability

Fill in the blank: Continuous monopositive functions guarantee integrals that are always __________.

2.3.2 Implications for Riemann integration

The relationship between these two classes of functions is significant for Riemann integration:

Functions of bounded variation extend the scope of integrability beyond strictly continuous functions.

Continuous monopositive functions provide a simpler class where integrability is guaranteed and interpretation is straightforward.

Both classes ensure that integrals are well-defined and finite, which is crucial for applications in analysis and applied mathematics.

Implications for Riemann integration Content:

• Bounded variation extends integrability to oscillating functions.

• Continuous monopositive functions guarantee non-negative integrals.

• Both ensure well-defined and finite integrals.

Core Implications

Property	Implication for Integration	Mathematical Context
Integrability	Every BV function is Riemann integrable on $[a, b]$.	Since BV functions have only countable jump discontinuities, they satisfy Lebesgue's criterion for Riemann integrability.
Existence of Antiderivatives	Continuity ensures the existence of a "primitive."	If f is continuous and non-negative, the area function $F(x) = \int_a^x f(t)dt$ is differentiable and monotonic.
Non-negative Area	Monopositive functions guarantee a non-negative integral.	If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$. This is vital for defining mass, probability, and energy.
Riemann-Stieltjes Link	BV functions serve as "integrators."	We can integrate a continuous function g with respect to a BV function f using $\int g\,df$.

Fill in the blank: Both bounded variation and continuous monopositive functions ensure that integrals are ___________ and finite.

Pilot Questions 2.3

Functions of bounded variation are controlled by: A. Continuity B. Finiteness of oscillations C. Differentiability D. None of the above (Linked to SLO 2.4)

Continuous monopositive functions guarantee integrals that are: A. Negative B. Non-negative C. Infinite D. None of the above (Linked to SLO 2.4)

Which class of functions extends integrability to oscillating functions? A. Continuous monopositive functions B. Functions of bounded variation C. Both A and B D. None of the above (Linked to SLO 2.4)

Continuous monopositive functions are particularly useful in: A. Geometry and probability B. Fourier analysis C. Measure theory D. None of the above (Linked to SLO 2.4)

Both bounded variation and continuous monopositive functions ensure integrals are: A. Undefined B. Well-defined and finite C. Infinite D. None of the above (Linked to SLO 2.4)

2.4 Applications and Problem Sets

Functions of bounded variation and continuous monopositive functions are not only theoretical constructs but also practical tools in analysis. They provide conditions under which integrals are guaranteed to exist and be meaningful. Applications range from pure mathematics, such as Fourier analysis and measure theory, to applied fields like probability and geometry. By working through examples and problem sets, you will see how these functions are used to evaluate integrals and interpret results.

Fill in the blank: Functions of bounded variation and continuous monopositive functions guarantee that integrals are __________ and meaningful.

2.4.1 Worked examples

Example 1: Function of bounded variation Consider f(x)=|x|on -1,1. The total variation is:

$$V\text{-}11(f)=|f(0)-f(-1)|+|f(1)-f(0)|=|0-1|+|1-0|=2$$

Thus, f(x)=|x|is of bounded variation.

Example 2: Continuous monopositive function Consider f(x)=x2on 1. Since f(x)≥0and is continuous, it is monopositive. Its integral is:

$$01x2dx=13$$

Examples of bounded variation and continuous monopositive functions

Section 1: Bounded Variation (BV)

1. **Direct Calculation:** Calculate the total variation $V_a^b(f)$ for the function $f(x) = 3x^2 - 2x + 5$ on the interval $[-1, 2]$.
   - *Hint:* Find the critical points to identify where the function increases and decreases.
2. **The "Wiggle" Test:** Determine if the function $f(x) = x\cos(\frac{\pi}{x})$ for $x \in (0, 1]$ and $f(0) = 0$ is of bounded variation on $[0, 1]$.
   - *Hint:* Compare this to $x^2\cos(\frac{\pi}{x})$.
3. **Jordan Decomposition:** Express $f(x) = |x - 1|$ on the interval $[0, 2]$ as the difference of two non-decreasing functions, $g(x) - h(x)$.
   - *Hint:* Use the positive and negative variation functions.

Section 2: Continuous Monopositive (Non-negative) Functions

4. **Verification:** Prove that the function $f(x) = \frac{1}{1+x^2}$ is both continuous and monopositive for all $x \in \mathbb{R}$.
5. **Probability Application:** Given $f(x) = kx^2$ on the interval $[0, 3]$ and $f(x) = 0$ elsewhere, find the value of k such that $f(x)$ is a valid continuous probability density function.
   - *Hint:* The integral over the entire domain must equal 1.
6. **Optimization:** Show that for any continuous non-negative function $f(x)$, the area function $A(x) = \int_0^x f(t)dt$ is a non-decreasing function.

Section 3: Riemann Integration Implications

7. **Integrability Proof:** Explain why a step function with a finite number of jumps is of bounded variation, and use this to justify its Riemann integrability.
8. **Mean Value Theorem:** For a continuous monopositive function f on $[a, b]$, prove there exists a $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Quick Self-Check

- Can you identify the difference between a function being **bounded** and a function having **bounded variation**? (Example: $\sin(1/x)$ is bounded but not BV near 0).
- Do you see why **monopositivity** is a stricter requirement for physical models than simple continuity?

Fill in the blank: The total variation of f(x)=|x|on -1,1is __________.

2.4.2 Practice exercises

Try solving the following exercises to consolidate your learning:

Show that f(x)=sin(x)on is of bounded variation.

Evaluate 01(x2+y2)dxdyand explain why the function is continuous and monopositive.

Compare the integrability of f(x)=|x|and f(x)=x2on their respective domains.

Discuss how bounded variation and monopositive functions ensure integrability in applied contexts such as probability.

Fill in the blank: The integral of f(x)=x2over 1is __________.

Pilot Questions 2.4

The function f(x)=|x|on -1,1has total variation equal to: A. 1 B. 2 C. 0 D. None of the above (Linked to SLO 2.5)

The function f(x)=x2on 1is: A. Continuous monopositive B. Discontinuous C. Non-integrable D. None of the above (Linked to SLO 2.5)

The integral of f(x)=x2over 1is: A. 1 B. 13C. 0 D. None of the above (Linked to SLO 2.5)

Functions of bounded variation are guaranteed to be: A. Riemann–Stieltjes integrable B. Non-integrable C. Differentiable everywhere D. None of the above (Linked to SLO 2.5)

Continuous monopositive functions are useful in: A. Geometry B. Probability C. Both A and B D. None of the above (Linked to SLO 2.5)

Series Summary

This Series has focused on functions of bounded variation and continuous monopositive functions, both of which are central to understanding integrability. We began by defining functions of bounded variation and examining their properties, such as decomposition into increasing functions and their role in Riemann–Stieltjes integration. We then explored continuous monopositive functions, highlighting their continuity, non-negativity, and guaranteed integrability. Following that, we compared the two classes of functions, showing how they complement each other in ensuring well-defined integrals. Finally, we applied these concepts through worked examples and practice exercises, linking theory to applications in geometry, probability, and analysis.

By completing this Series, you should now be able to define functions of bounded variation (SLO 2.1), explain their properties and significance (SLO 2.2), describe and analyse continuous monopositive functions (SLO 2.3), compare bounded variation and monopositive functions in relation to Riemann integration (SLO 2.4), and apply these concepts to solve practical problems (SLO 2.5).

Glossary of Terms

Bounded variation: A property of a function where the total variation over an interval is finite.

Continuous monopositive function: A function that is continuous everywhere in its domain and non-negative throughout.

Decomposition property: The ability to express a function of bounded variation as the difference of two increasing functions.

Total variation: The supremum of the sums of absolute differences of function values over partitions of an interval.

Riemann–Stieltjes integrable: A function that can be integrated with respect to another function, often requiring bounded variation.

Concept Check Questions 2

Objective questions

A function is of bounded variation if its total variation is: A. Infinite B. Finite C. Undefined D. None of the above (Linked to SLO 2.1)

Continuous monopositive functions are always: A. Non-negative and continuous B. Non-positive and continuous C. Discontinuous D. None of the above (Linked to SLO 2.3)

Functions of bounded variation can be expressed as: A. The sum of two decreasing functions B. The difference of two increasing functions C. The product of two bounded functions D. None of the above (Linked to SLO 2.2)

Short-answer questions

Define total variation of a function. (Linked to SLO 2.1)

Give one example of a continuous monopositive function. (Linked to SLO 2.3)

Explain why bounded variation ensures integrability. (Linked to SLO 2.2)

Long-answer questions

Compare functions of bounded variation and continuous monopositive functions. (Linked to SLO 2.4)

Basic response: State the definitions and highlight differences.

Value-added response: Provide examples and explain their implications for Riemann integration.

Apply the concept of bounded variation to show that f(x)=|x|on -1,1is integrable. (Linked to SLO 2.5)

Basic response: Show that the total variation is finite.

Value-added response: Demonstrate the calculation and link it to integrability.

Analyse the role of continuous monopositive functions in probability theory. (Linked to SLO 2.5)

Basic response: State that they can serve as density functions.

Value-added response: Provide an example of a probability density function and explain how integration ensures total probability equals 1.

Answers to Concept Check Questions 2

1-B, 2-A, 3-B

Total variation is the supremum of sums of absolute differences of function values over partitions of an interval.

Example: f(x)=x2on 1.

Bounded variation ensures integrability because finite oscillations guarantee convergence of Riemann–Stieltjes sums.

Basic response: Bounded variation focuses on finite oscillations, while continuous monopositive functions focus on smoothness and non-negativity. Value-added response: Example: f(x)=|x|is of bounded variation, while f(x)=x2is continuous monopositive. Both are integrable.

Basic response: The total variation of f(x)=|x|on -1,1is 2, hence finite. Value-added response: Since the variation is finite, the function is integrable, and its integral can be evaluated.

Basic response: Continuous monopositive functions can serve as probability density functions. Value-added response: Example: f(x)=2xon 1integrates to 1, making it a valid density function.

Answers to Pilot Questions 2

Part 2.1: 1-B, 2-A, 3-B, 4-B, 5-A Part 2.2: 1-A, 2-A, 3-A, 4-B, 5-C Part 2.3: 1-B, 2-B, 3-B, 4-A, 5-B Part 2.4: 1-B, 2-A, 3-B, 4-A, 5-C

Series 3: The Riemann–Stieltjes Integral

Series Outline

Part 3.1: Definition of the Riemann–Stieltjes Integral

Sub-Part 3.1.1: Formal definition

Sub-Part 3.1.2: Comparison with the Riemann integral

Part 3.2: Properties of the Riemann–Stieltjes Integral

Introduction

In the previous Series, we studied functions of bounded variation and continuous monopositive functions, both of which are essential in ensuring integrability. Building on that foundation, this Series introduces the Riemann–Stieltjes integral, a generalisation of the Riemann integral. Instead of integrating with respect to the variable x, the Riemann–Stieltjes integral allows integration with respect to another function g(x).

This generalisation is powerful because it extends the scope of integration to contexts such as probability theory, where distribution functions serve as integrators, and Fourier analysis, where oscillatory functions are involved.

The aim of this Series is to define the Riemann–Stieltjes integral, explore its properties, and demonstrate its applications.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 3.1 define the Riemann–Stieltjes integral and compare it with the Riemann integral,

SLO 3.2 explain the properties of the Riemann–Stieltjes integral, including linearity and bounded variation,

SLO 3.3 describe the conditions under which functions are Riemann–Stieltjes integrable, SLO 3.4 apply the Riemann–Stieltjes integral in probability theory and Fourier analysis, and

SLO 3.5 evaluate simple Riemann–Stieltjes integrals and solve practice problems.

3.1 Definition of the Riemann–Stieltjes Integral

The Riemann–Stieltjes integral is a generalisation of the Riemann integral. Instead of integrating a function f(x)with respect to the variable x, we integrate with respect to another function g(x). This allows us to capture more complex behaviours, especially when g(x)represents cumulative quantities such as distribution functions in probability.

Formally, the Riemann–Stieltjes integral of fwith respect to gover bis defined as:

$$abf(x)\ dg(x)$$

Here, dg(x)represents the increments of the function g(x)over the interval.

Fill in the blank: The Riemann–Stieltjes integral of fwith respect to gis written as \int_a^b f(x) \, __________.

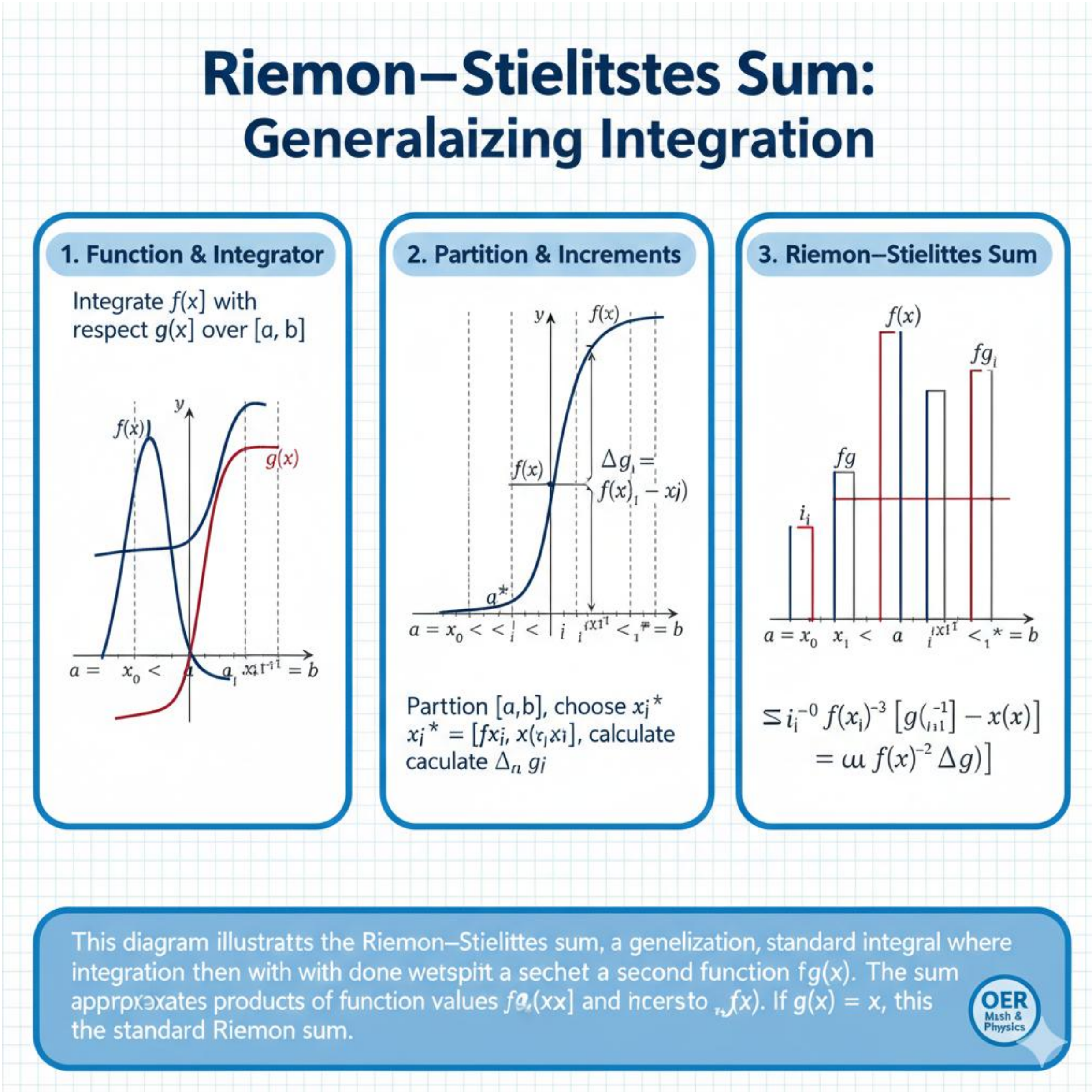
3.1.1 Formal definition

Let $P=\{a=x0<x1<\cdots<xn=b\}$be a partition of b. For each subinterval xi, choose a point i. The Riemann–Stieltjes sum is defined as:

$$S(P,f,g)=i=1nf(i)(g(xi)-g(xi-1))$$

If the limit of these sums exists as the mesh of the partition approaches zero, then the function fis said to be Riemann–Stieltjes integrable with respect to g, and the limit is the integral:

$$abf(x)\ dg(x)$$



Partitioning an interval for Riemann–Stieltjes sums

Fill in the blank: The Riemann–Stieltjes sum is defined as $\sum f(i)(g(xi)-g(xi-1))$, where iis a ___________ in each subinterval.

3.1.2 Comparison with the Riemann integral

The Riemann–Stieltjes integral reduces to the Riemann integral when $g(x)=x$. In that case:

$$abf(x)\ dx=abf(x)\ dg(x)$$

Thus, the Riemann–Stieltjes integral is a broader concept. It allows integration with respect to functions that may not be linear, enabling applications in probability (where g(x)can be a cumulative distribution function) and other areas of analysis.

Comparison between Riemann and Riemann–Stieltjes integrals

Feature	Riemann Integral	Riemann–Stieltjes Integral
Integrator	Variable x	Function g(x)
Definition	abf(x)dx	abf(x)dg(x)
Scope	Limited to linear increments	Generalised to arbitrary increments
Applications	Geometry, calculus	Probability, Fourier analysis

Fill in the blank: The Riemann–Stieltjes integral reduces to the Riemann integral when g(x) = __________.

Pilot Questions 3.1

The Riemann–Stieltjes integral generalises the Riemann integral by integrating with respect to: A. Derivatives B. Another function g(x)C. Infinite series D. None of the above (Linked to SLO 3.1)

The Riemann–Stieltjes sum is defined as: A. ∑f(i)(xi-xi-1)B. ∑f(i)(g(xi)-g(xi-1))C. ∑f(xi)D. None of the above (Linked to SLO 3.1)

The Riemann–Stieltjes integral reduces to the Riemann integral when: A. g(x)=xB. g(x)=f(x)C. g(x)=0D. None of the above (Linked to SLO 3.1)

The Riemann–Stieltjes integral is particularly useful in: A. Geometry only B. Probability and Fourier analysis C. Differentiation D. None of the above (Linked to SLO 3.1)

In the Riemann–Stieltjes sum, irepresents: A. The endpoint of the interval B. A chosen point within the subinterval C. The derivative of the function D. None of the above (Linked to SLO 3.1)

3.2 Properties of the Riemann–Stieltjes Integral

The Riemann–Stieltjes integral inherits many properties from the Riemann integral but also introduces new ones due to the presence of the integrator function g(x). These properties ensure that the integral is well-defined and useful in a wide range of applications.

Fill in the blank: The Riemann–Stieltjes integral inherits many properties from the __________ integral.

3.2.1 Linearity and bounded variation

Linearity: If fand hare integrable with respect to g, then for constants aand b:

ab(af(x)+bh(x)) dg(x)=aabf(x) dg(x)+babh(x) dg(x)

Bounded variation: If the integrator g(x)is of bounded variation on b, then the Riemann–Stieltjes integral exists for every continuous function f(x). This condition ensures that the increments of g(x)are controlled and finite.

Linearity and bounded variation in Riemann–Stieltjes integrals Content:

- Linearity: Integrals respect addition and scalar multiplication.
- Bounded variation: Ensures integrability when g(x)is not smooth.

1. Linearity Properties

The Riemann–Stieltjes integral is linear with respect to both the **integrand** (f) and the **integrator** (g).

Property	Formula	Explanation
Linearity in f	$\int (c_1 f_1 + c_2 f_2)\, dg = c_1 \int f_1\, dg + c_2 \int f_2\, dg$	You can split the sum of two functions being integrated and pull out constants.
Linearity in g	$\int f\, d(c_1 g_1 + c_2 g_2) = c_1 \int f\, dg_1 + c_2 \int f\, dg_2$	If the integrator is a sum of functions, the integral can be decomposed into a sum of integrals.

2. Integration by Parts

One of the most powerful features of this integral is its symmetry:

$$\int_a^b f(x)\,dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,df(x)$$

If one side exists, the other side exists as well. This effectively allows the integrand and integrator to "swap" roles.

---

3. Bounded Variation (BV) Implications

The existence and behavior of the integral are heavily dependent on whether the integrator g is of **Bounded Variation**.

- **Existence Theorem:** If f is continuous on $[a, b]$ and g is of bounded variation on $[a, b]$, then the Riemann–Stieltjes integral $\int_a^b f\,dg$ is guaranteed to exist.
- **Total Variation Bound:** The absolute value of the integral is bounded by the maximum value of f and the total variation $V_a^b(g)$:

$$\left| \int_a^b f\,dg \right| \leq \max_{x \in [a,b]} |f(x)| \cdot V_a^b(g)$$

- **Monotonicity:** If g is non-decreasing (a specific type of BV), then if $f_1 \leq f_2$, it follows that $\int f_1\,dg \leq \int f_2\,dg$.

Fill in the blank: If the integrator g(x)is of bounded variation, the Riemann–Stieltjes integral exists for every __________ function.

3.2.2 Integrability conditions

For the Riemann–Stieltjes integral abf(x) dg(x)to exist, certain conditions must be satisfied:

If f(x)is continuous and g(x)is of bounded variation, the integral exists.

If f(x)has only a finite number of discontinuities and g(x)is monotonic, the integral also exists.

If both fand gare continuous, integrability is guaranteed.

These conditions extend the scope of integration beyond the classical Riemann integral, allowing more general functions to be integrated.

Integrability conditions for Riemann–Stieltjes integrals

Condition on f(x)	Condition on g(x)	Integrability
Continuous	Bounded variation	Exists
Finite discontinuities	Monotonic	Exists
Continuous	Continuous	Guaranteed

Fill in the blank: If f(x)is continuous and g(x)is of bounded variation, the Riemann–Stieltjes integral __________.

Pilot Questions 3.2

The Riemann–Stieltjes integral is linear, meaning: A. It respects addition and scalar multiplication B. It ignores constants C. It only applies to monotonic functions D. None of the above (Linked to SLO 3.2)

If the integrator g(x)is of bounded variation, the integral exists for: A. Discontinuous functions B. Continuous functions C. Non-integrable functions D. None of the above (Linked to SLO 3.2)

If f(x)has finite discontinuities and g(x)is monotonic, the integral: A. Does not exist B. Exists C. Is infinite D. None of the above (Linked to SLO 3.3)

Integrability is guaranteed when: A. Both fand gare continuous B. Both fand gare discontinuous C. fis discontinuous and gis monotonic D. None of the above (Linked to SLO 3.3)

The property of bounded variation ensures that increments of g(x)are: A. Infinite B. Controlled and finite C. Undefined D. None of the above (Linked to SLO 3.2)

3.3 Applications of the Riemann–Stieltjes Integral

The Riemann–Stieltjes integral is a powerful tool because it extends integration beyond the classical Riemann framework. By allowing integration with respect to another function g(x), it finds applications in diverse areas such as probability theory and Fourier analysis.

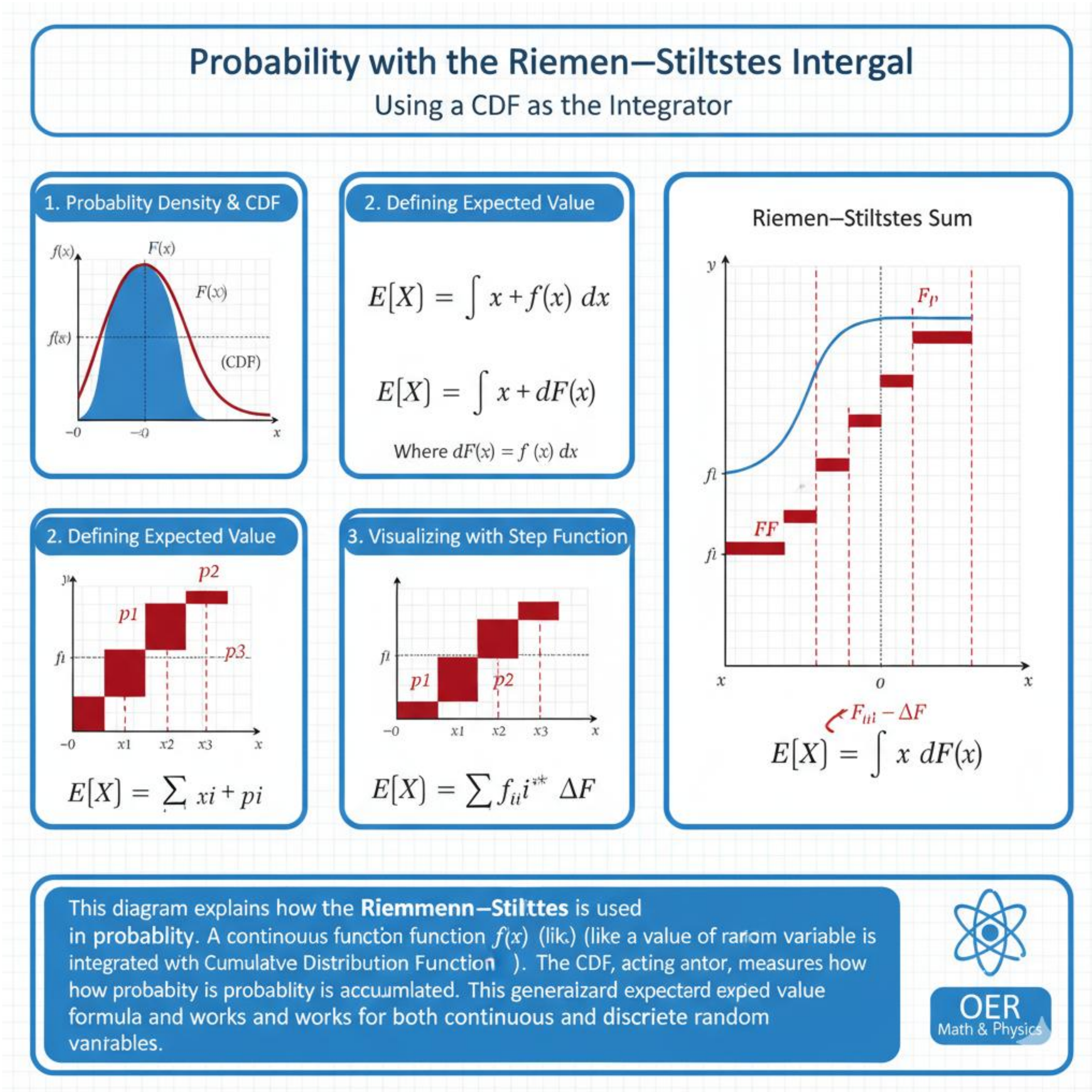
Fill in the blank: The Riemann–Stieltjes integral extends integration by allowing integration with respect to another __________.

3.3.1 Probability theory and distribution functions

In probability theory, the Riemann–Stieltjes integral is used to define expectations of random variables. If Xis a random variable with cumulative distribution function (CDF) F(x), then the expected value of a function f(X)is given by:

$$E[f(X)]=-f(x)\ dF(x)$$

Here, F(x)acts as the integrator. This formulation is particularly useful when dealing with discrete or mixed distributions, where the classical Riemann integral may not apply directly.



Using a CDF as an integrator in probability theory

Fill in the blank: In probability theory, the expected value of a function f(X)is given by f(x) _________.

3.3.2 Connections with Fourier analysis

The Riemann–Stieltjes integral also plays a role in Fourier analysis. When defining Fourier transforms or Fourier series, integrals often involve oscillatory functions. By integrating with respect to functions of bounded variation, the Riemann–Stieltjes integral provides a rigorous framework for handling these cases.

For example, the Fourier transform of a function f(x)can be expressed in terms of a Riemann–Stieltjes integral when the integrator captures oscillatory behaviour. This allows analysts to extend Fourier techniques to functions that are not strictly Riemann integrable.

Applications of the Riemann–Stieltjes integral Content:

• Probability theory: Expected values using cumulative distribution functions

. • Fourier analysis: Handling oscillatory functions rigorously

. • Extends integration beyond classical Riemann methods. Fill in the blank:

Applications of the Riemann–Stieltjes Integral

The Riemann–Stieltjes integral is a versatile tool that allows mathematicians to treat discrete and continuous scenarios under a single unified framework. By choosing different types of "integrators" ($g(x)$), we can model everything from smooth physical systems to "jumpy" statistical data.

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1. Probability Theory

In probability, the Riemann–Stieltjes integral is the standard way to define the **Expected Value** ($E[X]$) of a random variable, regardless of whether it is discrete, continuous, or a mix of both.

2. Fourier Analysis

In Fourier analysis, the Riemann–Stieltjes integral is essential for dealing with signals that contain both continuous frequencies and "pure tones" (discrete frequencies).

- **Fourier–Stieltjes Transform:** This generalizes the standard Fourier Transform. It allows us to analyze the frequency content of a **measure** or a function of bounded variation.

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{-itx}\, d\mu(x)$$

- **Signal Processing:** It is used to represent signals that have "shocks" or sudden changes (modeled by step functions in the integrator) alongside smooth oscillations.
- **Parseval's Theorem:** In its generalized form, it relates the energy of a signal in the time domain to its representation in the frequency domain using Stieltjes integration.

The Riemann–Stieltjes integral provides a rigorous framework for handling __________ functions in Fourier analysis.

Pilot Questions 3.3

In probability theory, the Riemann–Stieltjes integral is used to define: A. Derivatives B. Expected values C. Variance only D. None of the above (Linked to SLO 3.4)

The expected value of a function f(X)is given by: A. ∫f(x)dxB. ∫f(x)dF(x)C. ∫F(x)dxD. None of the above (Linked to SLO 3.4)

In Fourier analysis, the Riemann–Stieltjes integral helps handle: A. Constant functions B. Oscillatory functions C. Linear functions D. None of the above (Linked to SLO 3.4)

The integrator in probability applications of the Riemann–Stieltjes integral is typically: A. A derivative B. A cumulative distribution function C. A Fourier series D. None of the above (Linked to SLO 3.4)

The Riemann–Stieltjes integral extends integration beyond: A. Classical Riemann methods B. Differentiation C. Algebraic manipulation D. None of the above (Linked to SLO 3.4)

3.4 Problem Sets and Worked Examples

The Riemann–Stieltjes integral can feel abstract at first, but worked examples and practice problems make its application clearer. By evaluating simple cases, you will see how the integrator function g(x)influences the integral and how this generalisation extends beyond the classical Riemann framework.

Fill in the blank: Worked examples help clarify how the integrator function __________ influences the Riemann–Stieltjes integral.

3.4.1 Evaluation of simple Riemann–Stieltjes integrals

Example 1: Linear integrator Let f(x)=xand g(x)=xon 1. Then:

$$01f(x)\ dg(x)=01x\ dx=12$$

This reduces to the classical Riemann integral.

Example 2: Non-linear integrator Let f(x)=1and g(x)=x2on 1. Then:

$$011\ dg(x)=g(1)-g(0)=12-02=1$$

Here, the integral depends entirely on the increments of g(x).

Example of a Riemann–Stieltjes integral with non-linear integrator

Riemann–Stieltjes Integral: $\int_0^1 1\,d(x^2)$

The diagram above illustrates the Riemann–Stieltjes integral where the integrand is a constant function $f(x) = 1$ and the integrator is a quadratic function $g(x) = x^2$.

Key Concepts:

- **The Integral Setup:** We are evaluating $\int_a^b f(x)\,dg(x)$. Unlike the standard Riemann integral which measures area with respect to the x-axis (Δx), the Stieltjes integral measures "weight" or "area" with respect to the change in $g(x)$ (Δg).
- **Evaluation:**

$$\int_0^1 1\,d(x^2) = [g(x)]_0^1 = g(1) - g(0) = 1^2 - 0^2 = 1$$

”

- **Visualization (Right Panel):** By plotting $f(x)$ against the values of $g(x)$, the integral becomes the area under the curve in this transformed "Stieltjes space." Since $f(x) = 1$ is constant, the area is simply a rectangle with height 1 and width equal to the total variation of g on the interval, which is 1.
- **Summation Interpretation:** In a Riemann–Stieltjes sum, we calculate $\sum f(x_i^*)[g(x_i) - g(x_{i-1})]$. Because $g(x) = x^2$ increases more rapidly as x increases, the "width" of the sub-rectangles in the Stieltjes space (the Δg increments) grows larger for intervals closer to $x = 1$, even if the Δx intervals are uniform.

Fill in the blank: If f(x)=1and g(x)=x2on 1, then the Riemann–Stieltjes integral equals __________.

3.4.2 Practice exercises

Try solving the following exercises to consolidate your understanding:

Evaluate 01x d(x2).

Show that 011 d(x3)=1.

Compare 01x dxwith 01x d(x2).

Explain why the Riemann–Stieltjes integral is useful in probability when g(x)is a cumulative distribution function.

Practice exercises on Riemann–Stieltjes integrals Content:

• Evaluate 01x d(x2). • Show that 011 d(x3)=1.

• Compare 01x dxwith 01x d(x2).

• Discuss applications in probability.

Practice Exercises: Riemann–Stieltjes Integrals

These exercises transition from basic mechanical evaluations to more abstract applications in probability and analysis.

Section 1: Basic Evaluation

1. **Constant Integrand:** Evaluate $\int_0^3 5\, d(x^3)$.
   - *Hint:* For a constant c, $\int f\, dg = c[g(b) - g(a)]$.
2. **Smooth Integrator:** Calculate $\int_0^\pi \sin(x)\, d(\cos(x))$.
   - *Hint:* Use the substitution $dg = g'(x)dx$.
3. **Step Function Integrator:** Evaluate $\int_0^2 x^2\, dg(x)$ where $g(x)$ is defined as:

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 4 & \text{if } 1 \leq x \leq 2 \end{cases}$$

$4 \quad \text{if } 1 \leq x \leq 2$

   - *Hint:* The integral of a continuous function against a step function is the function's value at the jump point multiplied by the height of the jump.

Section 2: Integration by Parts

4. **Symmetry Application:** Given that $\int_0^1 x\, de^x = 1$, use the Integration by Parts formula for Riemann–Stieltjes integrals to find the value of $\int_0^1 e^x\, dx$.
   - *Formula:* $\int f\, dg = fg|_a^b - \int g\, df$.
5. **Monotonicity & Bounds:** Let $f(x) = x$ and $g(x) = \sin(x)$ on $[0, \pi/2]$. Without evaluating fully, prove that $0 \leq \int_0^{\pi/2} f\, dg \leq \pi/2$.

Section 3: Probability & Advanced Applications

6. **Expected Value (Discrete):** A discrete random variable X has a CDF $F(x)$ that jumps at $x = 1$ (height 0.3) and $x = 2$ (height 0.7). Set up and solve the Riemann–Stieltjes integral $\int x\,dF(x)$ to find $E[X]$.
7. **Total Variation Bound:** If $f(x)$ is continuous and $|f(x)| \leq 10$, and $g(x) = \cos(x)$ on $[0, 2\pi]$, what is the maximum possible absolute value of $\int_0^{2\pi} f\,dg$?
   - *Hint:* Use the property $|\int f\,dg| \leq \max|f| \cdot V_a^b(g)$. Note that $V_0^{2\pi}(\cos x) = 4$.

Quick Self-Check

- Do you know when to use $dg = g'(x)dx$ versus summing jumps?
- Can you explain why $\int f\,dg$ might not exist if f and g share a discontinuity at the same point?

Fill in the blank: The integral 011 d(x3)equals __________.

Pilot Questions 3.4

The integral 01x dxequals: A. 12B. 1 C. 0 D. None of the above (Linked to SLO 3.5)

The integral 011 d(x2)equals: A. 0 B. 1 C. 12D. None of the above (Linked to SLO 3.5)

The integral 011 d(x3)equals: A. 0 B. 1 C. 13D. None of the above (Linked to SLO 3.5)

Comparing 01x dxand 01x d(x2), the difference arises because: A. The integrator changes B. The function changes C. Both change D. None of the above (Linked to SLO 3.5)

The Riemann–Stieltjes integral is useful in probability because: A. It uses derivatives B. It integrates with respect to cumulative distribution functions C. It ignores discontinuities D. None of the above (Linked to SLO 3.5)

Series Summary

This Series introduced the Riemann–Stieltjes integral, a generalisation of the Riemann integral that allows integration with respect to another function g(x). We began by defining the integral formally and comparing it with the classical Riemann integral. We then explored its properties, including linearity and the role of bounded variation, and outlined integrability conditions. Next, we examined applications in probability theory (using cumulative distribution functions to define expectations) and Fourier analysis (handling oscillatory functions). Finally, we worked through examples and practice problems to demonstrate how the integrator function influences the integral.

By completing this Series, you should now be able to define the Riemann–Stieltjes integral (SLO 3.1), explain its properties (SLO 3.2), describe integrability conditions (SLO 3.3), apply it in probability and Fourier analysis (SLO 3.4), and evaluate simple Riemann–Stieltjes integrals (SLO 3.5).

Glossary of Terms

Riemann–Stieltjes integral: A generalisation of the Riemann integral where integration is taken with respect to another function g(x).

Riemann–Stieltjes sum: A sum of the form $\sum f(i)(g(xi)\text{-}g(xi\text{-}1))$, used to define the integral.

Integrator function: The function g(x)with respect to which integration is performed.

Bounded variation: A property of g(x)ensuring finite increments, which guarantees integrability.

Cumulative distribution function (CDF): A probability function used as an integrator in defining expectations.

Concept Check Questions 3

Objective questions

The Riemann–Stieltjes integral reduces to the Riemann integral when \(g(x) = \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\. (Linked to SLO 3.1)

If g(x)is of bounded variation, the Riemann–Stieltjes integral exists for every __________ function. (Linked to SLO 3.2)

In probability theory, the expected value of f(X)is given by …………… (Linked to SLO 3.4)

Short-answer questions

Define the Riemann–Stieltjes sum. (Linked to SLO 3.1)

State one condition under which the Riemann–Stieltjes integral exists. (Linked to SLO 3.3)

Explain why bounded variation of g(x)is important. (Linked to SLO 3.2)

Long-answer questions

Compare the Riemann and Riemann–Stieltjes integrals. (Linked to SLO 3.1)

Basic response: State definitions and highlight differences.

Value-added response: Provide examples showing how the integrator function changes the integral.

Apply the Riemann–Stieltjes integral to probability theory. (Linked to SLO 3.4)

Basic response: State that expectations can be defined using CDFs.

Value-added response: Provide an example of calculating an expectation using a CDF.

Evaluate 011 d(x2)and explain why the result depends on the integrator. (Linked to SLO 3.5)

Basic response: State the result.

Value-added response: Show the calculation and explain the role of g(x).

Answers to Concept Check Questions 3

g(x)=x

Continuous

dF(x)(CDF)

The Riemann–Stieltjes sum is ∑f(i)(g(xi)-g(xi-1)).

If f(x)is continuous and g(x)is of bounded variation.

Bounded variation ensures increments of g(x)are finite, guaranteeing convergence of sums.

Basic response: Riemann integrates with respect to x, Riemann–Stieltjes integrates with respect to g(x). Value-added response: Example: 011 d(x2)=1, which differs from 011 dx=1.

Basic response: Expectations are defined as ∫f(x) dF(x). Value-added response: Example: For uniform distribution on 1, E[X]=01x dF(x)=12.

Basic response: 011 d(x2)=1. Value-added response: The result depends on increments of g(x)=x2, not just the length of the interval.

Answers to Pilot Questions 3

Part 3.1: 1-B, 2-B, 3-A, 4-B, 5-B Part 3.2: 1-A, 2-B, 3-B, 4-A, 5-B Part 3.3: 1-B, 2-B, 3-B, 4-B, 5-A Part 3.4: 1-A, 2-B, 3-B, 4-A, 5-B

References and Attributions 3

Apostol, T. M. (1974). *Mathematical Analysis*. Addison-Wesley.

Bartle, R. G., & Sherbert, D. R. (2011). *Introduction to Real Analysis*. Wiley.

Open Educational Resources (OER) diagrams and examples on Riemann–Stieltjes integrals.

Illustration prompts and sources:

Figure 3.1 Partitioning an interval for Riemann–Stieltjes sums

AI prompt: “Generate a labelled diagram showing partitioning of an interval with increments of g(x) used in Riemann–Stieltjes sums.”

Search string: “Riemann–Stieltjes sum diagram OER”

Table 3.1 Comparison between Riemann and Riemann–Stieltjes integrals

AI prompt: “Create a comparison table between Riemann and Riemann–Stieltjes integrals with features, definitions, scope, and applications.”

Search string: “Riemann vs Riemann–Stieltjes integral comparison OER”

Figure 3.3 Using a CDF as an integrator in probability theory

AI prompt: “Generate a labelled diagram showing a cumulative distribution function used as an integrator in the Riemann–Stieltjes integral.”

Search string: “Riemann–Stieltjes integral probability CDF OER”

Figure 3.4 Example of a Riemann–Stieltjes integral with non-linear integrator

AI prompt: “Generate a graph showing f(x) = 1 with integrator g(x) = x^2 on [0,1].”

Search string: “Riemann–Stieltjes integral example OER”

Box 3.1 Linearity and bounded variation in Riemann–Stieltjes integrals

Box 3.3 Applications of the Riemann–Stieltjes integral

Box 3.4 Practice exercises on Riemann–Stieltjes integrals

Series 4: Convergence of Sequences and Series of Functions

Series Outline

Part 4.1: Convergence of Sequences of Functions

Sub-Part 4.1.1: Pointwise convergence

Sub-Part 4.1.2: Uniform convergence

Part 4.2: Convergence of Series of Functions

Sub-Part 4.2.1: Definition and examples

Sub-Part 4.2.2: Tests for convergence

Part 4.3: Implications of Convergence

Sub-Part 4.3.1: Continuity and integrability

Sub-Part 4.3.2: Differentiability

Part 4.4: Applications and Problem Sets

Sub-Part 4.4.1: Worked examples

Sub-Part 4.4.2: Practice exercises

Introduction

In the previous Series, we studied the Riemann–Stieltjes integral, which extended integration to functions of bounded variation and cumulative

distribution functions. Now, we turn to the convergence of sequences and series of functions, a fundamental concept in analysis.

Convergence determines whether a sequence or series of functions approaches a limiting function as the number of terms increases. Understanding the type of convergence—pointwise or uniform—is crucial because it affects whether properties such as continuity, integrability, and differentiability are preserved in the limit.

This Series will begin with the convergence of sequences of functions, distinguishing between pointwise and uniform convergence. Next, we will study series of functions and the tests used to determine their convergence. We will then explore the implications of convergence for continuity, integrability, and differentiability. Finally, we will apply these concepts through worked examples and problem sets.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 4.1 define and distinguish between pointwise and uniform convergence of sequences of functions,**
- **SLO 4.2 explain the convergence of series of functions and apply convergence tests,**
- **SLO 4.3 analyse the implications of convergence for continuity, integrability, and differentiability, and**
- **SLO 4.4 apply convergence concepts to solve practical problems involving sequences and series of functions.**

4.1 Convergence of Sequences of Functions

A **sequence of functions** is a list of functions $f_n(x)$ defined on a common domain. The study of convergence asks whether these functions approach a limiting function $f(x)$ as $n\to$. There are two main types of convergence: **pointwise** and **uniform**.

Fill in the blank: A sequence of functions $f_n(x)$ converges to a limiting function $f(x)$ as $n\to$ if the functions approach __________.

4.1.1 Pointwise convergence

A sequence of functions $f_n(x)$ converges **pointwise** to a function $f(x)$ on a domain D if, for every $x\in D$:

$$\lim_{n\to} f_n(x)=f(x)$$

This means that for each fixed x, the sequence of real numbers $fn(x)$converges to $f(x)$.

Example: Let $fn(x)=xnon\ 1$. Then:

$$\lim n\to fn(x)=0$$

So, $fn(x)$converges pointwise to the zero function.

[Insert Figure here] Short description: Graph showing $fn(x)=xn$converging pointwise to 0. Caption: Figure 4.1 Pointwise convergence of $fn(x)=xn$to 0 AI prompt: “Generate a graph showing f_n(x) = x/n on [0,1] converging pointwise to 0.” Search string: “pointwise convergence example OER”

Fill in the blank: Pointwise convergence requires that for each fixed x, \lim_{n \to \infty} f_n(x) = __________.

4.1.2 Uniform convergence

A sequence of functions $fn(x)$converges **uniformly** to a function $f(x)$on a domain Dif:

$$\lim n\to \sup x\in D|fn(x)-f(x)|=0$$

This means the convergence is not just pointwise but occurs at the same rate across the entire domain.

Example: Let $fn(x)=xnon\ 1$. Then:

$$\sup x\in[0,1]|xn-0|=1n\to 0$$

So, $fn(x)$converges uniformly to 0.

[Insert Box here] Caption: Box 4.1 Difference between pointwise and uniform convergence Content: • Pointwise: Convergence at each individual point. • Uniform: Convergence occurs uniformly across the entire domain. • Uniform convergence preserves continuity and integrability. AI prompt: “Create a summary box comparing pointwise and uniform convergence with examples.” Search string: “pointwise vs uniform convergence OER”

Fill in the blank: Uniform convergence requires that \sup_{x \in D} |f_n(x) - f(x)| \to __________.

Pilot Questions 4.1

Pointwise convergence means: A. Convergence at each individual point B. Convergence uniformly across the domain C. Convergence only at endpoints D. None of the above (Linked to SLO 4.1)

Uniform convergence means: A. Convergence at each point separately B. Convergence occurs at the same rate across the domain C. Convergence only for continuous functions D. None of the above (Linked to SLO 4.1)

The sequence $fn(x)=xnon\ 1$converges: A. Pointwise to 0 B. Uniformly to 0 C. Both pointwise and uniformly to 0 D. None of the above (Linked to SLO 4.1)

Uniform convergence preserves: A. Continuity and integrability B. Differentiability only C. Oscillations D. None of the above (Linked to SLO 4.1)

Pointwise convergence does not necessarily preserve: A. Continuity B. Integrability C. Differentiability D. All of the above (Linked to SLO 4.1)

4.2 Convergence of Series of Functions

A **series of functions** is formed by summing a sequence of functions:

$$n=1fn(x)$$

The study of convergence asks whether this infinite sum approaches a limiting function $f(x)$. Just as with sequences, convergence can be **pointwise** or **uniform**, and the type of convergence determines whether properties like continuity and integrability are preserved.

Fill in the blank: A series of functions is written as \sum_{n=1}^{\infty} __________.

4.2.1 Definition and examples

Pointwise convergence of a series: The series $\sum fn(x)$converges pointwise to $f(x)$if, for each fixed x, the sequence of partial sums converges to $f(x)$.

Uniform convergence of a series: The series converges uniformly if the sequence of partial sums converges uniformly to $f(x)$.

Example: Consider the series

$$n=1xnn2,x\in[0,1]$$

This series converges pointwise for all $x\in[0,1]$. It also converges uniformly on 1because the terms decrease sufficiently fast.

[Insert Figure here] Short description: Graph showing partial sums of $\sum xnn2$converging to a limiting function. Caption: Figure 4.2 Convergence of a series of functions AI prompt: “Generate a graph showing partial sums of the series Σ $x^{n/n}$2 converging to a limiting function on [0,1].” Search string: “series of functions convergence example OER”

Fill in the blank: The series $\sum xnn2$converges __________ on 1.

4.2.2 Tests for convergence

Several tests help determine whether a series of functions converges:

Weierstrass M-test: If $|fn(x)|Mn$for all $x$and $\sum Mn$converges, then $\sum fn(x)$converges uniformly.

Comparison test: Compare the series with another known convergent series.

Ratio test: Useful for power series, checking the ratio of successive terms.

[Insert Box here] Caption: Box 4.2 Convergence tests for series of functions Content: • Weierstrass M-test: Ensures uniform convergence. • Comparison test: Relates to known convergent series. • Ratio test: Useful for power series. AI prompt: “Create a summary box listing convergence tests for series of functions with short explanations.” Search string: “convergence tests series of functions OER”

Fill in the blank: The Weierstrass M-test ensures _________ convergence of a series of functions.

Pilot Questions 4.2

A series of functions is formed by: A. Adding derivatives B. Summing a sequence of functions C. Multiplying functions D. None of the above (Linked to SLO 4.2)

The series $\sum \frac{x^n}{n^2}$on 1converges: A. Pointwise only B. Uniformly C. Diverges D. None of the above (Linked to SLO 4.2)

The Weierstrass M-test ensures: A. Pointwise convergence B. Uniform convergence C. Divergence D. None of the above (Linked to SLO 4.2)

The ratio test is particularly useful for: A. Power series B. Trigonometric series C. Polynomial functions D. None of the above (Linked to SLO 4.2)

Uniform convergence of a series preserves: A. Continuity and integrability B. Differentiability only C. Oscillations D. None of the above (Linked to SLO 4.2)

4.3 Implications of Convergence

The type of convergence—pointwise or uniform—has important consequences for the properties of the limiting function. While pointwise convergence only guarantees that the sequence or series approaches the limit at each individual point, uniform convergence ensures stronger preservation of properties such as continuity, integrability, and differentiability.

Fill in the blank: Uniform convergence is stronger than pointwise convergence because it preserves _________ of the limiting function.

4.3.1 Continuity and integrability

Continuity: If a sequence of continuous functions converges uniformly to a function $f(x)$, then $f(x)$is also continuous. Pointwise convergence alone does not guarantee continuity.

Integrability: If a sequence of integrable functions converges uniformly to $f(x)$, then $f(x)$is integrable, and the integral of the limit equals the limit of the integrals:

$$\lim_{n\to\infty}\int f_n(x)dx=\int \lim_{n\to\infty} f_n(x)dx$$

Example: Let $f_n(x)=\frac{x}{n}$on 1. The sequence converges uniformly to 0, which is continuous and integrable.

[Insert Box here] Caption: Box 4.3 Continuity and integrability under uniform convergence Content: • Uniform convergence preserves continuity. • Uniform convergence preserves integrability. • Pointwise convergence may fail to preserve these properties. AI prompt: “Create a summary box explaining continuity and integrability under uniform convergence with examples.” Search string: “uniform convergence continuity integrability OER”

Fill in the blank: Uniform convergence ensures that the limit of integrals equals the integral of the _________.

4.3.2 Differentiability

Differentiability is more delicate:

- If $fn(x)$are differentiable and converge uniformly to $f(x)$, this does not automatically guarantee that $f(x)$is differentiable.
- However, if the derivatives $fn'(x)$converge uniformly to a function $g(x)$, then $f(x)$is differentiable and $f'(x)=g(x)$.

Example: Consider $fn(x)=\sin(nx)non\ 2\pi$. The sequence converges uniformly to 0, but the derivatives $fn'(x)=\cos(nx)$do not converge uniformly. Hence, differentiability of the limit is not guaranteed.

[Insert Figure here] Short description: Graph showing $fn(x)=\sin(nx)/n$converging uniformly to 0, but derivatives oscillating. Caption: Figure 4.3 Differentiability issues under uniform convergence AI prompt: “Generate a graph showing f_n(x) = sin(nx)/n converging uniformly to 0, with derivatives cos(nx) oscillating.” Search string: “uniform convergence differentiability example OER”

Fill in the blank: Differentiability of the limit function requires uniform convergence of the _________.

Pilot Questions 4.3

- Uniform convergence preserves: A. Continuity and integrability B. Differentiability only C. Oscillations D. None of the above (Linked to SLO 4.3)
- Pointwise convergence does not necessarily preserve: A. Continuity B. Integrability C. Differentiability D. All of the above (Linked to SLO 4.3)
- If $fn(x)$are continuous and converge uniformly to $f(x)$, then $f(x)$is: A. Continuous B. Discontinuous C. Non-integrable D. None of the above (Linked to SLO 4.3)
- Differentiability of the limit function requires uniform convergence of: A. The derivatives B. The integrals C. The oscillations D. None of the above (Linked to SLO 4.3)
- The sequence $fn(x)=\sin(nx)/n$converges uniformly to: A. 0 B. $\sin(x)$C. $\cos(x)$D. None of the above (Linked to SLO 4.3)

4.4 Applications and Problem Sets

The study of convergence of sequences and series of functions is not purely theoretical—it has practical implications in analysis, approximation theory, and applied mathematics. By working through examples and exercises, you can see how convergence affects continuity, integrability, and differentiability, and how uniform convergence provides stronger guarantees than pointwise convergence.

Fill in the blank: Uniform convergence provides stronger guarantees than pointwise convergence because it preserves __________ and integrability.

4.4.1 Worked examples

Example 1: Pointwise but not uniform convergence Let fn(x)=xnon 1.

For each fixed x∈[0,1), lim n→fn(x)=0.

At x=1, lim n→fn(1)=1. Thus, the sequence converges pointwise to a function that is 0 on [0,1)and 1 at x=1. However, convergence is not uniform because the rate of convergence depends on x.

Example 2: Uniform convergence Let fn(x)=xnon 1.

For all x, lim n→fn(x)=0.

The supremum difference is sup x∈[0,1]|fn(x)-0|=1n→0. Thus, convergence is uniform to the zero function.

[Insert Figure here] Short description: Graph showing fn(x)=xnconverging pointwise but not uniformly, and fn(x)=x/nconverging uniformly. Caption: Figure 4.4 Examples of pointwise vs uniform convergence AI prompt: "Generate a graph showing f_n(x) = x^n converging pointwise but not uniformly, and f_n(x) = x/n converging uniformly." Search string: "pointwise vs uniform convergence examples OER"

Fill in the blank: The sequence fn(x)=xnon 1converges __________ but not uniformly.

4.4.2 Practice exercises

Try solving the following exercises to consolidate your understanding:

Show that fn(x)=xnconverges uniformly to 0 on 1.

Determine whether fn(x)=xnconverges uniformly on 1.

Apply the Weierstrass M-test to prove uniform convergence of ∑xnn2on 1.

Explain why uniform convergence preserves continuity but pointwise convergence may not.

Give an example of a sequence of differentiable functions whose limit is not differentiable.

[Insert Box here] Caption: Box 4.4 Practice exercises on convergence of sequences and series of functions Content: • Show uniform convergence of fn(x)=x/n. • Test uniform convergence of fn(x)=xn. • Apply Weierstrass M-test to ∑xn/n2. • Explain preservation of continuity under uniform convergence. • Example of differentiable functions with non-differentiable limit. AI prompt: “Create a summary box listing practice exercises on convergence of sequences and series of functions.” Search string: “practice problems convergence of functions OER”

Fill in the blank: The Weierstrass M-test is used to prove __________ convergence of a series of functions.

Pilot Questions 4.4

The sequence $f_n(x)=x^n$ on 1converges: A. Pointwise but not uniformly B. Uniformly C. Diverges D. None of the above (Linked to SLO 4.4)

The sequence $f_n(x)=x/n$ on 1converges: A. Pointwise only B. Uniformly to 0 C. Diverges D. None of the above (Linked to SLO 4.4)

The Weierstrass M-test is used to prove: A. Pointwise convergence B. Uniform convergence C. Divergence D. None of the above (Linked to SLO 4.4)

Uniform convergence preserves: A. Continuity and integrability B. Differentiability only C. Oscillations D. None of the above (Linked to SLO 4.4)

Pointwise convergence may fail to preserve: A. Continuity B. Integrability C. Differentiability D. All of the above (Linked to SLO 4.4)

Series Summary

This Series focused on the **convergence of sequences and series of functions**, a cornerstone of real analysis. We began by distinguishing **pointwise convergence** (convergence at each individual point) from **uniform convergence** (convergence occurring at the same rate across the domain). We then extended these ideas to **series of functions**, introducing convergence tests such as the **Weierstrass M-test**. Next, we explored the **implications of convergence** for continuity, integrability, and differentiability, highlighting how uniform convergence preserves these properties while pointwise convergence may fail. Finally, we applied these concepts through worked examples and practice exercises, showing how convergence affects the behaviour of limiting functions.

By completing this Series, you should now be able to **define and distinguish** pointwise and uniform convergence (SLO 4.1), **apply convergence tests** to series of functions (SLO 4.2), **analyse implications** for continuity, integrability, and differentiability (SLO 4.3), and **solve problems** involving convergence of sequences and series of functions (SLO 4.4).

Glossary of Terms

Pointwise convergence: A sequence of functions $f_n(x)$converges pointwise to $f(x)$if $\lim_{n\to} f_n(x)=f(x)$for each fixed x.

Uniform convergence: A sequence of functions converges uniformly to $f(x)$if $\sup_{x\in D}|f_n(x)-f(x)|\to 0$.

Series of functions: An infinite sum of functions, $\sum f_n(x)$, whose convergence can be pointwise or uniform.

Weierstrass M-test: A test ensuring uniform convergence if $|f_n(x)| M_n$and $\sum M_n$converges.

Partial sums: The finite sums of the first $n$terms of a series, used to study convergence.

Concept Check Questions 4

Objective questions

Pointwise convergence requires that for each fixed x, \(\lim_{n \to \infty} f_n(x) = \\. (Linked to SLO 4.1)

Uniform convergence requires that \(\sup_{x \in D} |f_n(x) - f(x)| \to \\. (Linked to SLO 4.1)

The Weierstrass M-test ensures \\ convergence of a series of functions. (Linked to SLO 4.2)

Short-answer questions

Define pointwise convergence of a sequence of functions. (Linked to SLO 4.1)

State one condition under which uniform convergence preserves integrability. (Linked to SLO 4.3)

Give an example of a sequence of functions that converges pointwise but not uniformly. (Linked to SLO 4.4)

Long-answer questions

Compare pointwise and uniform convergence. (Linked to SLO 4.1)

- **Basic response**: State definitions and differences.
- **Value-added response**: Provide examples and explain implications for continuity.

Apply the Weierstrass M-test to prove uniform convergence of $\sum \frac{x^n}{n^2}$on 1. (Linked to SLO 4.2)

- **Basic response**: State the inequality and convergence of $\sum 1/n^2$.
- **Value-added response**: Show how the bound ensures uniform convergence.

Analyse why uniform convergence is necessary to interchange limits and integrals. (Linked to SLO 4.3)

- **Basic response**: State the condition.

Value-added response: Provide an example where pointwise convergence fails to preserve integrability.

Answers to Concept Check Questions 4

$f(x)$

0

Uniform

Pointwise convergence means $\lim n \to f_n(x) = f(x)$ for each fixed x.

If $f_n(x)$ are integrable and converge uniformly to $f(x)$, then $f(x)$ is integrable and $\lim \int f_n = \int f$.

Example: $f_n(x) = x^n$ on 1.

Basic response: Pointwise convergence is at each point, uniform convergence is across the domain. **Value-added response**: Example: $f_n(x) = x^n$ converges pointwise but not uniformly; $f_n(x) = x/n$ converges uniformly.

Basic response: Since $|x^n/n^2| \leq 1/n^2$ and $\sum 1/n^2$ converges, the series converges uniformly. **Value-added response**: The bound applies uniformly across 1, ensuring uniform convergence.

Basic response: Uniform convergence allows interchange of limit and integral. **Value-added response**: Example: A pointwise convergent sequence may yield a discontinuous limit, breaking integrability.

Answers to Pilot Questions 4

Part 4.1: 1-A, 2-B, 3-C, 4-A, 5-D **Part 4.2**: 1-B, 2-B, 3-B, 4-A, 5-A **Part 4.3**: 1-A, 2-D, 3-A, 4-A, 5-A **Part 4.4**: 1-A, 2-B, 3-B, 4-A, 5-D

Series 5: Continuous Differentiable Functions and Power Series

Series Outline

Part 5.1: Continuous and Differentiable Functions

Sub-Part 5.1.1: Definition of continuity

Sub-Part 5.1.2: Definition of differentiability

Sub-Part 5.1.3: Relationship between continuity and differentiability

Part 5.2: Properties of Continuous Differentiable Functions

Sub-Part 5.2.1: Mean Value Theorem

Sub-Part 5.2.2: Rolle's Theorem

Sub-Part 5.2.3: Implications for analysis

Part 5.3: Power Series

Sub-Part 5.3.1: Definition and examples

Sub-Part 5.3.2: Radius and interval of convergence

Sub-Part 5.3.3: Differentiation and integration of power series

Part 5.4: Applications and Problem Sets

Sub-Part 5.4.1: Approximating functions with Taylor and Maclaurin series

Sub-Part 5.4.2: Practice exercises

Introduction

This Series unites two central themes in analysis: continuous differentiable functions and power series. Continuous differentiable functions form the backbone of calculus, ensuring smooth behaviour and allowing the use of powerful theorems like the Mean Value Theorem. Power series, on the other hand, provide a way to represent functions as infinite polynomials, enabling approximation, analysis, and applications in solving differential equations.

We will begin by defining continuity and differentiability, then explore their properties and implications. Next, we will introduce power series, study their convergence, and examine how they can be differentiated and integrated term by term. Finally, we will apply these concepts through worked examples and problem sets, focusing on Taylor and Maclaurin series expansions.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 5.1 define continuous and differentiable functions and explain their relationship,

SLO 5.2 apply key theorems such as the Mean Value Theorem and Rolle's Theorem, SLO 5.3 define power series, determine their radius and interval of convergence, and apply differentiation and integration,

SLO 5.4 use Taylor and Maclaurin series to approximate functions and solve practice problems.

5.1 Continuous and Differentiable Functions

In this section, we establish the foundations of continuous differentiable functions, which are central to calculus and analysis. These functions are smooth enough to allow differentiation, and their continuity ensures no breaks or jumps in behaviour.

5.1.1 Definition of Continuity

A function $f(x)$ is continuous at a point c if:

$$\lim_{x \to c} f(x) = f(c)$$

This means the value of the function at c matches the limit of the function as x approaches c.

Intuition: You can draw the graph of a continuous function without lifting your pen.

Example: $f(x) = x^2$ is continuous everywhere on R.

Fill in the blank: A function is continuous at c if \(\lim_{x \to c} f(x) = \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\.

5.1.2 Definition of Differentiability

A function $f(x)$ is differentiable at a point c if the derivative exists:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

This means the function has a well-defined tangent line at c.

Intuition: Differentiability requires smoothness—no sharp corners or cusps.

Example: $f(x) = x^2$ is differentiable everywhere, with derivative $f'(x) = 2x$.

Fill in the blank: A function is differentiable at c if the derivative $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$__________.

5.1.3 Relationship between Continuity and Differentiability

Differentiability implies continuity: If f(x)is differentiable at c, then it is also continuous at c.

Continuity does not imply differentiability: A function can be continuous but not differentiable (e.g., f(x)=|x|at x=0).

Relationship between continuity and differentiability Content:

• Differentiability ⇒ Continuity

• Continuity ⇏ Differentiability

• Example: f(x)=|x|is continuous everywhere but not differentiable at 0.

Continuity vs. Differentiability

In calculus, the relationship between continuity and differentiability is a "one-way street." Understanding this hierarchy is crucial for analyzing the behavior of functions.

The Core Relationship

> **The Golden Rule:** If a function is **differentiable** at a point, it must be **continuous** there. However, being **continuous** does not guarantee that the function is **differentiable**.

Property	Definition	Visual Intuition
Continuity	$\lim_{x \to c} f(x) = f(c)$	The graph can be drawn without lifting your pen; there are no holes or jumps.
Differentiability	$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ exists.	The graph is "smooth"; it has a well-defined tangent line at that point.

Fill in the blank: Differentiability implies __________, but continuity does not necessarily imply differentiability.

Pilot Questions 5.1

A function is continuous at cif: A. lim x→cf(x)=f(c)B. lim x→cf(x)f(c)C. The derivative exists D. None of the above (Linked to SLO 5.1)

A function is differentiable at cif: A. The derivative exists B. The function is continuous only C. The function has a jump discontinuity D. None of the above (Linked to SLO 5.1)

Differentiability implies: A. Continuity B. Discontinuity C. Integrability only D. None of the above (Linked to SLO 5.1)

Continuity does not necessarily imply: A. Differentiability B. Integrability C. Existence of limits D. None of the above (Linked to SLO 5.1)

The function f(x)=|x|is: A. Continuous everywhere but not differentiable at 0 B. Differentiable everywhere C. Discontinuous at 0 D. None of the above (Linked to SLO 5.1)

5.2 Properties of Continuous Differentiable Functions

Once we establish continuity and differentiability, we can explore the powerful theorems that govern these functions. These results form the backbone of analysis and calculus, providing deep insights into the behaviour of smooth functions.

5.2.1 Mean Value Theorem (MVT)

The Mean Value Theorem states: If f(x)is continuous on band differentiable on b, then there exists a point c∈(a,b)such that:

$$f'(c)=f(b)-f(a)b-a$$

Interpretation: There is at least one point where the instantaneous rate of change equals the average rate of change.

Example: For f(x)=x2on 3,

$$f(3)-f(1)3-1=9-12=4$$

Since f'(x)=2x, the point c=2satisfies f'(c)=4.

Fill in the blank: The Mean Value Theorem guarantees a point cwhere the instantaneous rate of change equals the __________ rate of change.

5.2.2 Rolle's Theorem

Rolle's Theorem is a special case of the MVT: If f(x)is continuous on b, differentiable on b, and $f(a)=f(b)$, then there exists a point $c\in(a,b)$such that:

$$f'(c)=0$$

Interpretation: If a function starts and ends at the same value, then somewhere in between it has a horizontal tangent.

Example: For $f(x)=x2-1$on -1,1, we have $f(-1)=f(1)=0$. Rolle's Theorem guarantees a point cwhere $f'(c)=0$. Since $f'(x)=2x$, $c=0$works.

Fill in the blank: Rolle's Theorem guarantees a point cwhere the derivative equals __________.

5.2.3 Implications for Analysis

These theorems are fundamental in proving other results, such as Taylor's theorem and the existence of roots.

They ensure that continuous differentiable functions behave predictably, with smooth transitions and well-defined slopes.

They also underpin numerical methods, such as root-finding algorithms.

Properties of continuous differentiable functions Content:

- Mean Value Theorem: Instantaneous = average rate of change.
- Rolle's Theorem: Guarantees a point with zero slope.
- Applications: Root-finding, Taylor expansions, numerical analysis.

Rolle's Theorem & The Mean Value Theorem (MVT)

These two theorems form the backbone of differential calculus, bridging the gap between a function's local rate of change (derivative) and its global behavior over an interval.

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1. Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem. It states that if a function "comes back" to its starting height, it must have "turned around" at least once.

Requirement	Description
Continuity	$f(x)$ is continuous on the closed interval $[a, b]$.
Differentiability	$f(x)$ is differentiable on the open interval (a, b).
Equal Endpoints	$f(a) = f(b)$.
The Result	There exists at least one point $c \in (a, b)$ where $f'(c) = 0$.

> **Example:** $f(x) = x^2 - 4$ on $[-2, 2]$. Since $f(-2) = 0$ and $f(2) = 0$, Rolle's Theorem guarantees a point where the slope is zero. Indeed, $f'(x) = 2x$, which is 0 at $x = 0$.

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2. The Mean Value Theorem (MVT)

The MVT is a more general version of Rolle's. it states that at some point, the **instantaneous rate of change** (tangent slope) must equal the **average rate of change** (secant slope).

Requirement	Description
Continuity	$f(x)$ is continuous on $[a, b]$.
Differentiability	$f(x)$ is differentiable on (a, b).
The Result	There exists at least one point $c \in (a, b)$ such that: $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Pilot Questions 5.2

The Mean Value Theorem requires: A. Continuity on band differentiability on bB. Differentiability only C. Continuity only D. None of the above (Linked to SLO 5.2)

The Mean Value Theorem guarantees a point where: A. The derivative equals the average rate of change B. The derivative equals zero C. The function is discontinuous D. None of the above (Linked to SLO 5.2)

Rolle's Theorem requires that: A. f(a)=f(b)B. f(a)f(b)C. The function is discontinuous D. None of the above (Linked to SLO 5.2)

Rolle's Theorem guarantees a point where: A. The derivative equals zero B. The derivative equals the average rate of change C. The function is discontinuous D. None of the above (Linked to SLO 5.2)

Both MVT and Rolle's Theorem apply to functions that are: A. Continuous and differentiable B. Discontinuous C. Non-differentiable D. None of the above (Linked to SLO 5.2)

5.3 Power Series

Power series are infinite polynomial expansions that allow us to represent functions in a flexible and powerful way. They are central to analysis because they connect differentiability, convergence, and approximation.

5.3.1 Definition and Examples

A power series centered at ais defined as:

$$n=0cn(x-a)n$$

where cnare coefficients and ais the center.

Example 1 (Maclaurin series):

$$ex=n=0xnn!$$

Example 2 (Maclaurin series for sine):

$$\sin(x)=n=0(1)n(2n+1)!x2n+1$$

Fill in the blank: A power series centered at ais written as ∑cn(x-_________________)n.

5.3.2 Radius and Interval of Convergence

The radius of convergence Rdetermines the interval around awhere the series converges.

Found using the ratio test:

$$R=\lim n\to|cncn+1|$$

The interval of convergence is a-R,a+R, possibly including endpoints depending on convergence tests.

Example: For ∑xnn, the radius of convergence is 1, so the interval is (-1,1].

Fill in the blank: The radius of convergence Rdetermines the interval a-R,a+Rwhere the series __________.

5.3.3 Differentiation and Integration of Power Series

Within the interval of convergence, power series can be differentiated and integrated term by term:

Differentiation:

$$ddxn=0cn(x-a)n=n=1ncn(x-a)n-1$$

Integration:

$$\int n=0cn(x\text{-}a)ndx=n=0cnn+1(x\text{-}a)n+1+C$$

This property makes power series especially useful for approximating functions and solving differential equations.

Properties of power series Content:

- Radius of convergence defines validity region.
- Differentiation and integration can be done term by term.
- Examples include exponential and trigonometric functions.

Power Series: Convergence and Calculus

A power series is a function defined as an infinite sum of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$. These series are essential in analysis because they allow us to represent complex functions as infinite polynomials.

1. Radius of Convergence (R)

The radius of convergence defines the "safe zone" around the center c where the series converges to a finite value.

- **The Three Possibilities:**
  1. The series converges only at $x = c$ ($R = 0$).
  2. The series converges for all real x ($R = \infty$).
  3. There exists a number $R > 0$ such that the series converges if $|x - c| < R$ and diverges if $|x - c| > R$.
- **Finding R:** Usually determined using the **Ratio Test**:

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \implies R = \frac{1}{L}$$

- **The Interval of Convergence:** This is the set of all x-values for which the series converges. It includes the open interval $(c - R, c + R)$ and potentially the endpoints $c - R$ and $c + R$ (which must be tested separately).

2. Term-by-Term Differentiation

Inside the interval of convergence ($|x - c| < R$), a power series is infinitely differentiable. You can find the derivative by differentiating each term individually.

- **Formula:** If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, then:

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$

- **Key Property:** The **radius of convergence remains the same** after differentiation, though convergence at the endpoints may change.

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3. Term-by-Term Integration

Similarly, power series can be integrated term-by-term within the radius of convergence.

- **Formula:**

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

- **Key Property:** Just like differentiation, the **radius of convergence R does not change**, but the behavior at the endpoints might improve (a series that diverged at an endpoint might converge after integration).

Fill in the blank: Power series can be differentiated and integrated __________ by term within their interval of convergence.

Pilot Questions 5.3

A power series centered at ais written as: A. ∑cn(x-a)nB. ∑cnxnonly C. ∑cn(x+a)nD. None of the above (Linked to SLO 5.3)

The radius of convergence determines: A. Where the series converges B. Where the series diverges C. The coefficients of the series D. None of the above (Linked to SLO 5.3)

The ratio test is used to find: A. Radius of convergence B. Interval length C. Derivative of the series D. None of the above (Linked to SLO 5.3)

Power series can be differentiated: A. Term by term within the interval of convergence B. Only at the center C. Only for exponential functions D. None of the above (Linked to SLO 5.3)

The Maclaurin series for exis: A. ∑xnn!B. ∑xnnC. ∑xnD. None of the above (Linked to SLO 5.3)

5.4 Applications and Problem Sets

This section highlights how power series are applied in practice and provides exercises to reinforce the concepts.

5.4.1 Approximating Functions with Taylor and Maclaurin Series

Taylor Series: Expansion of a function f(x)around a point a:

f(x)=n=0fn(a)n!(x-a)n

Maclaurin Series: Special case of Taylor series centered at a=0.

Examples:

Exponential function:

ex=n=0xnn!

Sine function:

sin(x)=n=0(1)n(2n+1)!x2n+1

Cosine function:

cos(x)=n=0(1)n(2n)!x2n

Fill in the blank: The Maclaurin series is a Taylor series centered at __________.

5.4.2 Practice Exercises

Try solving the following to consolidate your understanding:

Show that f(x)=excan be approximated by its Maclaurin series up to the 3rd term.

Derive the Maclaurin series for cos(x).

Use the ratio test to find the radius of convergence of ∑xnn.

Explain why term-by-term differentiation is valid within the interval of convergence.

Approximate sin(0.1)using the first two terms of its Maclaurin series.

Practice exercises on power series Content:

• Approximate exwith first three terms.

• Derive Maclaurin series for cos(x).

• Apply ratio test to ∑xn/n.

• Explain term-by-term differentiation. • Approximate sin(0.1)with first two terms.

Practice Exercises: Power Series

These exercises cover the identification of convergence zones and the application of calculus to infinite series.

Section 1: Radius and Interval of Convergence

1. **Ratio Test Application:** Find the radius of convergence (R) and the interval of convergence (I) for the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n \cdot 3^n}$$

- *Hint:* Don't forget to check the endpoints $x = -1$ and $x = 5$ separately.

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2. **Factorials in Series:** Determine the radius of convergence for:

$$\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$$

3. **Visualizing Convergence:** Given a power series centered at $c = 0$ that converges at $x = 4$ and diverges at $x = -6$, what can you conclude about the convergence at $x = 3$ and $x = 7$?

Section 2: Term-by-Term Differentiation & Integration

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4. **Deriving New Series:** Start with the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.
   - Differentiate term-by-term to find a power series for $\frac{1}{(1-x)^2}$.

5. **Integrating for Logarithms:** Use the series $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ to find the power series expansion for $f(x) = \ln(1+x)$ by integrating.
   - *Challenge:* What is the interval of convergence for this new series?

6. **Finding the Sum:** Use term-by-term differentiation on a known series to find the exact sum of:

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

Section 2: Term-by-Term Differentiation & Integration

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   - *Challenge:* What is the interval of convergence for this new series?
6. **Finding the Sum:** Use term-by-term differentiation on a known series to find the exact sum of:

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

Fill in the blank: The ratio test is used to find the __________ of convergence of a power series.

Pilot Questions 5.4

The Maclaurin series is a Taylor series centered at: A. 0 B. 1 C. D. None of the above (Linked to SLO 5.4)

Power series can be differentiated: A. Term by term within the interval of convergence B. Only at the center C. Only for exponential functions D. None of the above (Linked to SLO 5.4)

The ratio test is used to find: A. Radius of convergence B. Derivative of the series C. Interval length only D. None of the above (Linked to SLO 5.4)

The Maclaurin series for exis: A. ∑xnn!B. ∑xnnC. ∑xnD. None of the above (Linked to SLO 5.4)

Applications of power series include: A. Function approximation and solving differential equations B. Only graphing functions C. Only numerical integration D. None of the above (Linked to SLO 5.4)

Series 5 Summary

This Series combined the study of continuous differentiable functions with the theory of power series. We began by defining continuity and differentiability, emphasizing their relationship: differentiability implies continuity, but continuity does not necessarily imply differentiability. We then explored key theorems such as the Mean Value Theorem and Rolle’s Theorem, which provide fundamental insights into the behaviour of smooth functions.

Next, we introduced power series, defined their general form, and studied their convergence through the concepts of radius and interval of convergence. We highlighted the remarkable property that power series can be differentiated and integrated term by term within their interval of convergence. Finally, we applied these ideas through Taylor and Maclaurin series, showing how functions like ex, $\sin(x)$, and $\cos(x)$can be approximated by infinite polynomial expansions.

By completing this Series, you should now be able to define continuous and differentiable functions (SLO 5.1), apply the Mean Value and Rolle’s Theorems (SLO 5.2), analyze power series and their convergence (SLO 5.3), and use Taylor and Maclaurin series to approximate functions (SLO 5.4).

Glossary of Terms

Continuous function: A function with no breaks, jumps, or holes; $\lim x \to c f(x)=f(c)$.

Differentiable function: A function with a well-defined derivative at a point.

Mean Value Theorem (MVT): Guarantees a point where the instantaneous rate of change equals the average rate of change.

Rolle’s Theorem: Guarantees a point with zero slope if $f(a)=f(b)$.

Power series: An infinite polynomial expansion of the form $\sum cn(x-a)n$.

Radius of convergence: The distance from the center within which the series converges.

Taylor series: Expansion of a function around a point a.

Maclaurin series: Taylor series centered at $a=0$.

Concept Check Questions 5

Objective questions

A function is continuous at cif \(\lim_{x \to c} f(x) ----------- (Linked to SLO 5.1)

Differentiability implies --------------- (Linked to SLO 5.1)

The Mean Value Theorem guarantees a point where the derivative equals the ----------------(Linked to SLO 5.2)

A power series is written as ∑cn(x-a)n. The number ais called the ------------ (Linked to SLO 5.3)

The Maclaurin series is a Taylor series centered at ----------------- (Linked to SLO 5.4)

Short-answer questions

Define differentiability at a point. (Linked to SLO 5.1)

State Rolle's Theorem. (Linked to SLO 5.2)

Explain the radius of convergence of a power series. (Linked to SLO 5.3)

Long-answer questions

Compare continuity and differentiability. (Linked to SLO 5.1)

Basic response: State definitions and relationship.

Value-added response: Provide examples where continuity does not imply differentiability.

Apply the Mean Value Theorem to f(x)=x2on 3. (Linked to SLO 5.2)

Basic response: State the average rate of change.

Value-added response: Identify the point cwhere f'(c)equals this rate.

Derive the Maclaurin series for sin(x). (Linked to SLO 5.4)

Basic response: State the formula.

Value-added response: Show the derivation using derivatives at 0.

Answers to Concept Check Questions 5

f(c)

Continuity

Average rate of change

Center

0

Differentiability at cmeans the derivative exists: $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$.

Rolle's Theorem: If f(a)=f(b), with continuity on band differentiability on b, then there exists cwith $f'(c)=0$.

Radius of convergence is the distance from the center within which the series converges.

Basic response: Differentiability implies continuity, but not vice versa. Value-added response: Example: $f(x)=|x|$is continuous but not differentiable at 0.

Basic response: Average rate of change = 4. Value-added response: $f'(x)=2x$, so $c=2$satisfies $f'(c)=4$.

Basic response: $\sin(x)=\sum \frac{(1)^n}{(2n+1)!} x^{2n+1}$. Value-added response: Derived by computing derivatives of $\sin(x)$at 0 and substituting into the Taylor formula.

Answers to Pilot Questions 5

Part 5.1: 1-A, 2-A, 3-A, 4-A, 5-A Part 5.2: 1-A, 2-A, 3-A, 4-A, 5-A Part 5.3: 1-A, 2-A, 3-A, 4-A, 5-A Part 5.4: 1-A, 2-A, 3-A, 4-A, 5-A