La-Net

MTH207

REAL ANALYSIS

Series 1: Bounds and Convergence of Sequences

Series Outline

Part 1.1 Bounds of Real Numbers

- 1.1.1 Definition of bounds
- 1.1.2 Upper and lower bounds
- 1.1.3 Supremum and infimum

Part 1.2 Convergence of Sequences

- 1.2.1 Definition of sequence and limit
- 1.2.2 Convergent and divergent sequences
- 1.2.3 Examples of convergent sequences

Part 1.3 Properties of Convergent Sequences

- 1.3.1 Uniqueness of limits
- 1.3.2 Algebra of limits
- 1.3.3 Order properties of limits

Part 1.4 Applications of Convergence

- 1.4.1 Approximation of real numbers
- 1.4.2 Use in analysis and calculus

Introduction

Real numbers form the foundation of analysis, and understanding how sequences behave is one of the first steps in building this knowledge. Sequences appear naturally in mathematics, for example when approximating irrational numbers or studying infinite processes. The ideas of bounds and convergence help us to determine whether a sequence stabilises towards a particular value or continues without limit. This Series will therefore provide you with the essential tools to study sequences rigorously, which is crucial for later topics in Real Analysis.

The aim of this Series is to equip you with the ability to define bounds of real numbers, explain the concept of convergence, and apply the properties of convergent sequences to practical problems in analysis.

We shall commence this Series by examining the bounds of real numbers, including upper and lower bounds, as well as the important notions of supremum and infimum. Next, we will move on to the convergence of sequences, where we will define limits and distinguish between convergent and divergent sequences with examples. Subsequently, we will consider the properties of convergent sequences, such as uniqueness of limits, algebra of limits, and order properties. Finally, we will conclude by exploring applications of convergence, particularly in approximating real numbers and in laying the groundwork for calculus.

Series Learning Outcomes

By the end of **Series 1: Bounds and Convergence of Sequences**, you should be able to:

- **SLO 1.1** define bounds of real numbers and distinguish between upper bounds, lower bounds, supremum, and infimum,
- **SLO 1.2** explain the concept of a sequence and its limit, and differentiate between convergent and divergent sequences,
- **SLO 1.3** analyse the properties of convergent sequences, including uniqueness of limits, algebra of limits, and order properties,
- **SLO 1.4** apply the concept of convergence to approximate real numbers and to support further study in analysis and calculus.

1.1 Bounds Of Real Numbers

1.1.1 Definition of bounds

A **bound** is a value that restricts how large or small the elements of a set or sequence can be. If a set of numbers does not exceed a certain value, we say that value is a bound. Bounds are important because they help us describe the limits within which numbers lie, and they provide a foundation for understanding convergence.

Fill in the blank: A value that restricts how large or small the elements of a set can be is called a __________________.

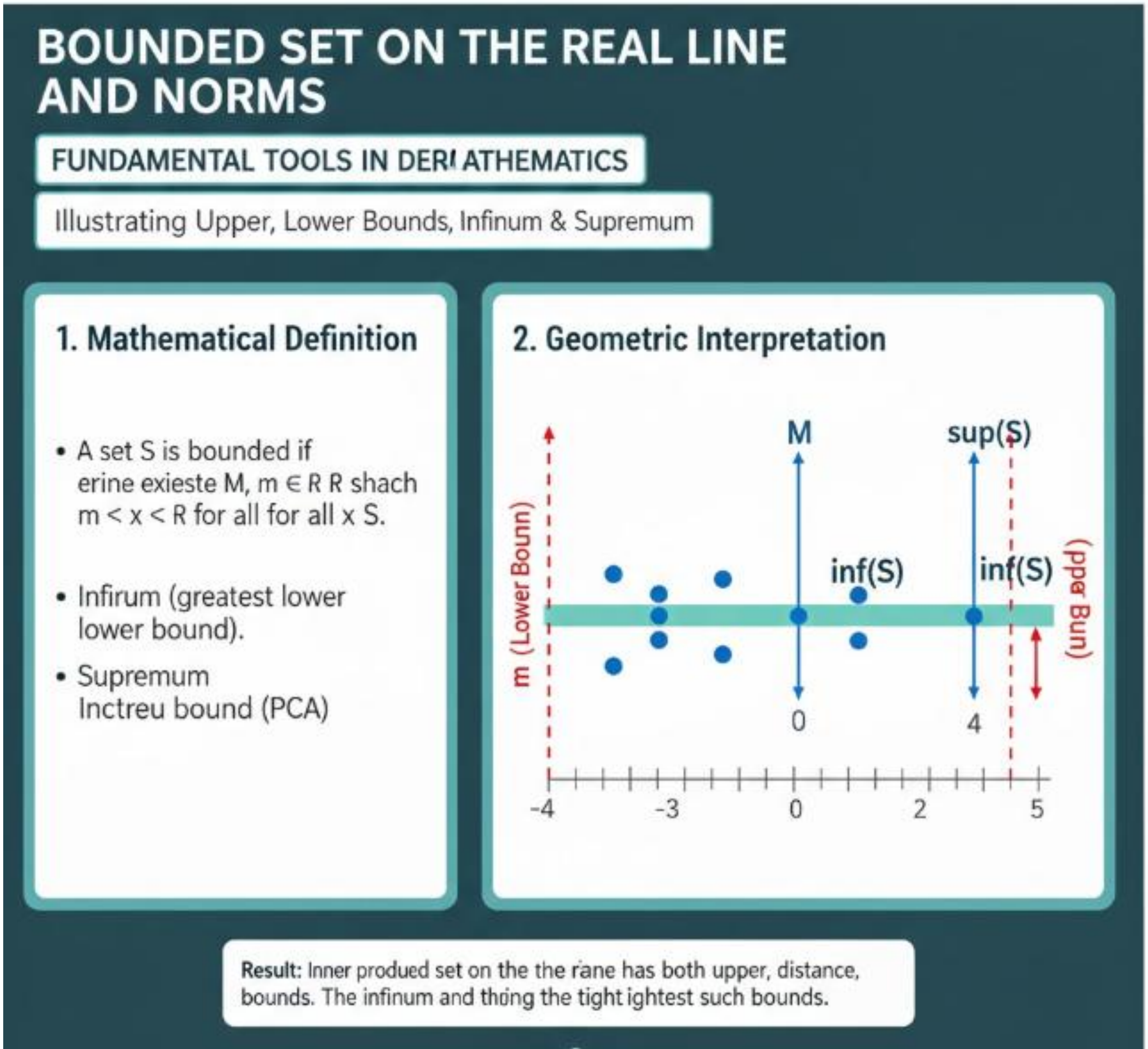


Illustration of a bounded set on the real number line

1.1.2 Upper and lower bounds

An **upper bound** of a set is a number that is greater than or equal to every element in the set. A **lower bound** is a number that is less than or equal to every element in the set. For example, in the set 3, the number 3 is an upper bound, while 1 is a lower bound.

Fill in the blank: A number that is greater than or equal to every element in a set is called an ___________________.

1.1.3 Supremum and infimum

The **supremum** of a set is the least upper bound, meaning it is the smallest number that is greater than or equal to all elements of the set. The **infimum** is the greatest lower bound, meaning it is the largest number that is less than or equal to all elements of the set. These concepts are crucial because not all sets have maximum or minimum values, but they always have a supremum and infimum if they are bounded.

Fill in the blank: The least upper bound of a set is called the ____________________.

Key differences between maximum/minimum and supremum/infimum

Feature	Maximum / Minimum	Supremum / Infimum
Definition	The largest (max) or smallest (min) value **within** the set.	The **least upper bound** (sup) or **greatest lower bound** (inf).
Membership	**Must** be an element of the set.	**May or may not** be an element of the set.
Existence	Does not always exist, even for bounded sets (e.g., open intervals).	Always exists for any non-empty set bounded above/below (Completeness Axiom).
Relationship	If a maximum exists, it is also the supremum.	If the supremum is in the set, it is also the maximum.

Pilot Questions 1.1

A value that restricts how large or small the elements of a set can be is called a: A. Limit B. Bound C. Sequence D. Supremum (Linked to SLO 1.1)

A number that is greater than or equal to every element in a set is called a: A. Lower bound B. Supremum C. Upper bound D. Infimum (Linked to SLO 1.1)

The least upper bound of a set is known as the: A. Maximum B. Supremum C. Limit D. Bound (Linked to SLO 1.1)

The greatest lower bound of a set is called the: A. Infimum B. Minimum C. Bound D. Upper bound (Linked to SLO 1.1)

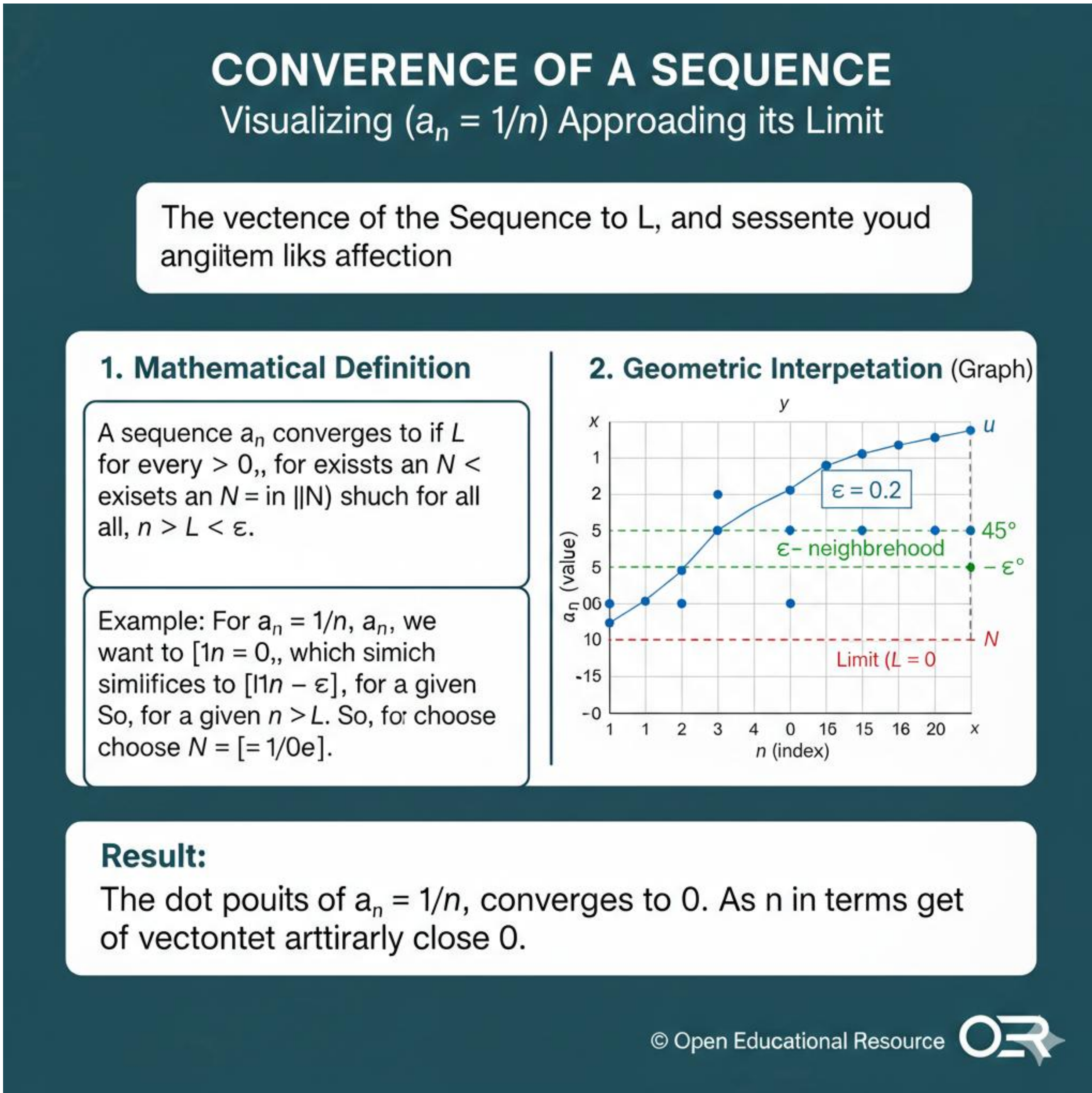
Which of the following statements is true? A. Every set has a maximum and minimum. B. Every bounded set has a supremum and infimum. C. Supremum and maximum are always the same. D. Infimum and minimum are always the same. (Linked to SLO 1.1)

1.2 Convergence Of Sequences

1.2.1 Definition of sequence and limit

A **sequence** is an ordered list of numbers, often defined by a rule or formula. For example, the sequence an=1nproduces the terms 11,12,13,.... A **limit** of a sequence is the value that the terms of the sequence approach as nbecomes very large. If the terms get closer and closer to a single number, we say the sequence converges to that limit.

Fill in the blank: A sequence that approaches a single value as nbecomes very large is said to __________________.

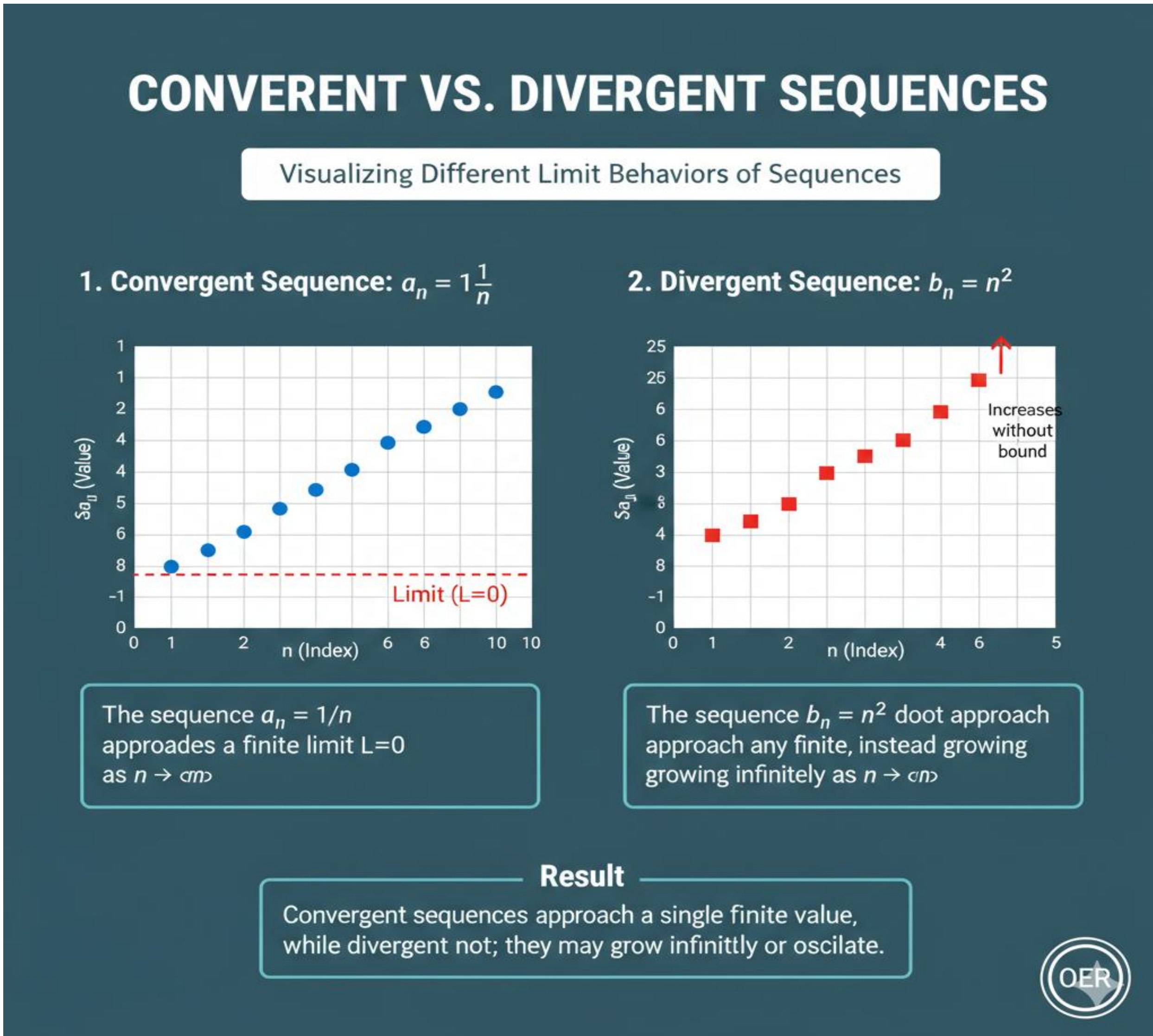


Graph of the sequence an=1napproaching the limit 0 AI prompt:

1.2.2 Convergent and divergent sequences

A **convergent sequence** is one whose terms approach a finite limit. For example the sequence an=1nconverges to 0. A **divergent sequence** is one that does not approach any finite value. For instance, the sequence an=ndiverges because its terms grow without bound.

Fill in the blank: A sequence whose terms grow without bound is called a ___________________ sequence.



comparison of a convergent sequence and a divergent sequence

1.2.3 Examples of convergent sequences

Examples help to clarify the concept of convergence. Consider the sequence an=12n. As nincreases, the terms become smaller and approach 0. Another example is an=nn+1, which approaches 1 as ngrows. These examples show that convergence can occur towards different values, not just 0.

Fill in the blank: The sequence an=nn+1converges to _________________.

Pilot Questions 1.2

A sequence that approaches a single value as nbecomes very large is said to: A. Diverge B. Converge C. Oscillate D. Expand (Linked to SLO 1.2)

A sequence whose terms grow without bound is called a: A. Convergent sequence B. Divergent sequence C. Bounded sequence D. Supremum sequence (Linked to SLO 1.2)

The sequence an=1nconverges to: A. 1 B. 0 C. Infinity D. 12(Linked to SLO 1.2)

The sequence an=nis: A. Convergent B. Divergent C. Bounded D. Supremum (Linked to SLO 1.2)

The sequence an=nn+1converges to: A. 0 B. 1 C. Infinity D. 12(Linked to SLO 1.2)

.3 Properties Of Convergent Sequences

1.3.1 Uniqueness of limits

One of the most important properties of a **convergent sequence** is that its limit is unique. This means that if a sequence converges, it can only approach one value. For example, the sequence an=1nconverges to 0, and no other number. If a sequence were to converge to two different values, those values would have to be arbitrarily close, which is impossible.

Fill in the blank: A convergent sequence can only approach one value, which means its limit is __________________.

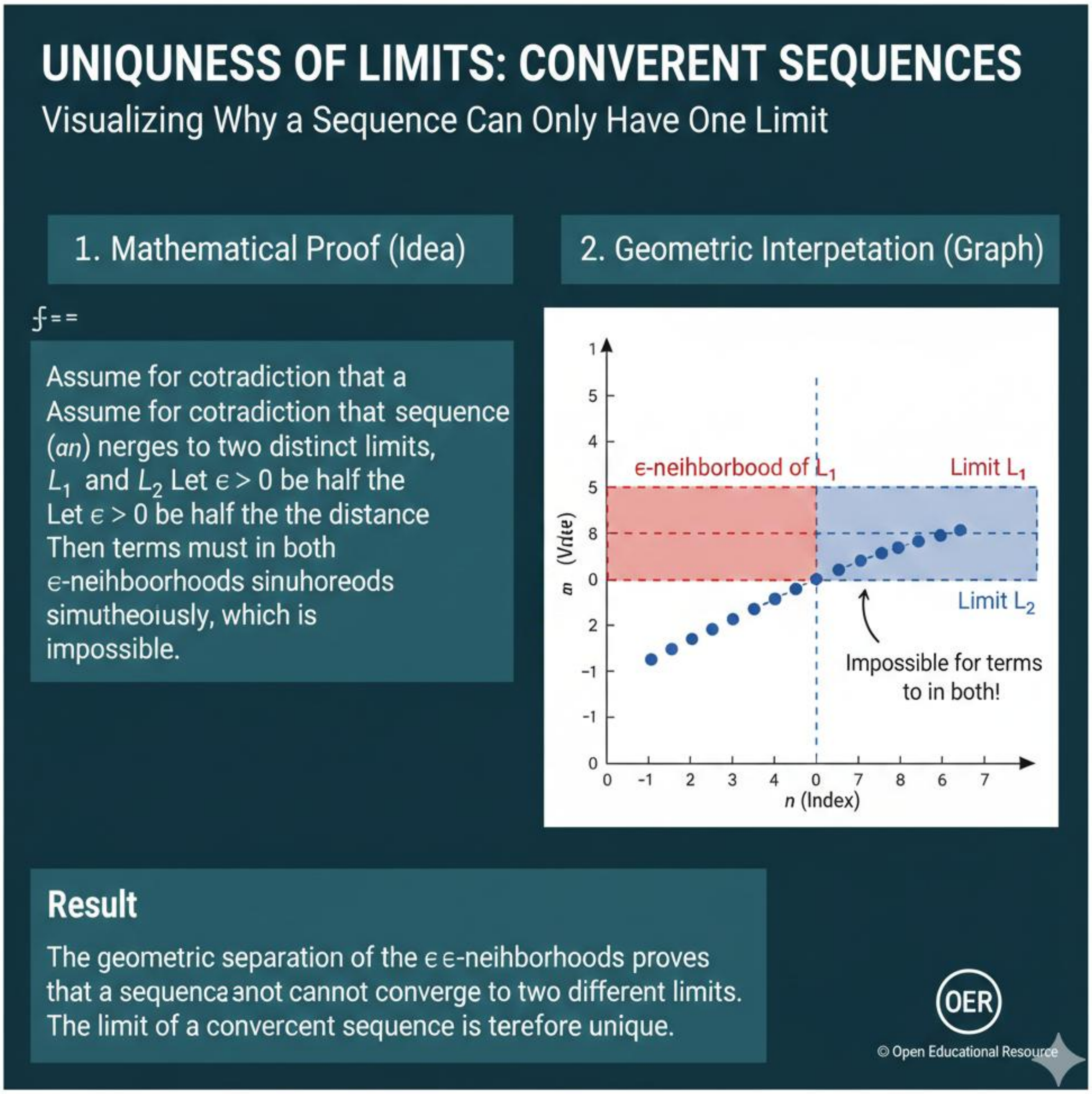


Illustration showing a sequence approaching a single limit

1.3.2 Algebra of limits

The **algebra of limits** refers to the rules that govern how limits behave under addition, subtraction, multiplication, and division. For example, if $a_n \to A$ and $b_n \to B$, then $a_n + b_n \to A + B$. Similarly, the product of two convergent sequences converges to the product of their limits. These rules allow us to combine sequences and still determine their limits.

The Algebra of Limits for Convergent Sequences

If we have two convergent sequences, (a_n) and (b_n), such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, the following rules apply:

Rule	Mathematical Expression	Description
Sum Rule	$\lim_{n\to\infty}(a_n + b_n) = A + B$	The limit of a sum is the sum of the limits.
Difference Rule	$\lim_{n\to\infty}(a_n - b_n) = A - B$	The limit of a difference is the difference of the limits.
Product Rule	$\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$	The limit of a product is the product of the limits.
Quotient Rule	$\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{A}{B}$	Valid only if $b_n \neq 0$ and $B \neq 0$.
Constant Multiple	$\lim_{n\to\infty}(k \cdot a_n) = k \cdot A$	A constant factor k can be moved outside the limit.
Power Rule	$\lim_{n\to\infty}(a_n)^p = A^p$	Valid if A^p is defined.

Fill in the blank: If an→Aand bn→B, then an+bnconverges to ____________________.

1.3.3 Order properties of limits

The **order properties of limits** state that if one sequence is always less than another, then their limits will preserve this order. For example, if anbnfor all n, and both sequences converge, then lim⁡anlim⁡bn. This property is useful when comparing sequences and proving inequalities in analysis.

Fill in the blank: If anbnfor all n, then \lim a_n \leq \lim ____________________.

Table 1.2: Examples of order properties of limits

Order Properties of Convergent Sequences

If $(a_n) \to A$ and $(b_n) \to B$ are convergent sequences:

Property	Condition	Result
Non-negativity	If $a_n \geq 0$ for all n	$A \geq 0$
Comparison	If $a_n \leq b_n$ for all n	$A \leq B$
Squeeze Theorem	If $a_n \leq c_n \leq b_n$ and $A = B$	$(c_n) \to A$

Pilot Questions 1.3

A convergent sequence can only approach one value. This property is called: A. Divergence B. Uniqueness of limits C. Supremum D. Infimum (Linked to SLO 1.3)

If an→Aand bn→B, then an+bnconverges to: A. A+BB. A-BC. ABD. A/B(Linked to SLO 1.3)

The product of two convergent sequences converges to: A. The sum of their limits B. The product of their limits C. The difference of their limits D. The ratio of their limits (Linked to SLO 1.3)

If anbnfor all n, then: A. limanlimbnB. limanlimbnC. liman=limbnD. an diverges (Linked to SLO 1.3)

Which of the following statements is true? A. A convergent sequence can have more than one limit. B. The sum of two convergent sequences is always divergent. C. The order of sequences is preserved in their limits. D. The product of two convergent sequences is undefined. (Linked to SLO 1.3)

1.4 Applications of Convergence

1.4.1 Approximation of real numbers

One of the most practical uses of **convergence** is in approximating real numbers. For example, the decimal expansion of irrational numbers such as or ecan be expressed as convergent sequences of fractions or series. Each successive term provides a closer approximation to the actual value. This principle is widely used in numerical methods, where sequences are employed to approximate solutions to equations or values of functions.

Fill in the blank: The decimal expansion of can be expressed as a __________________ sequence of fractions.

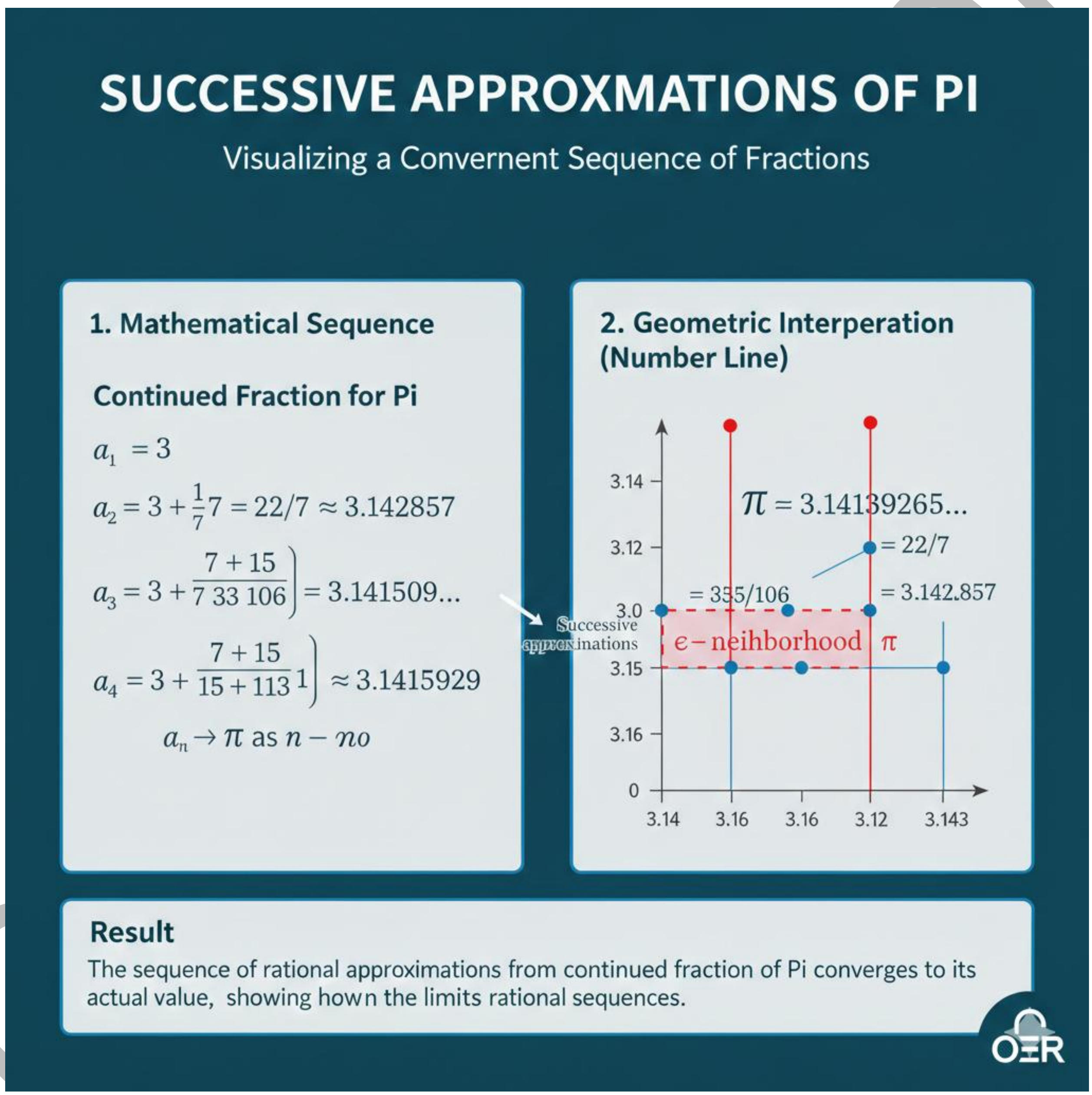
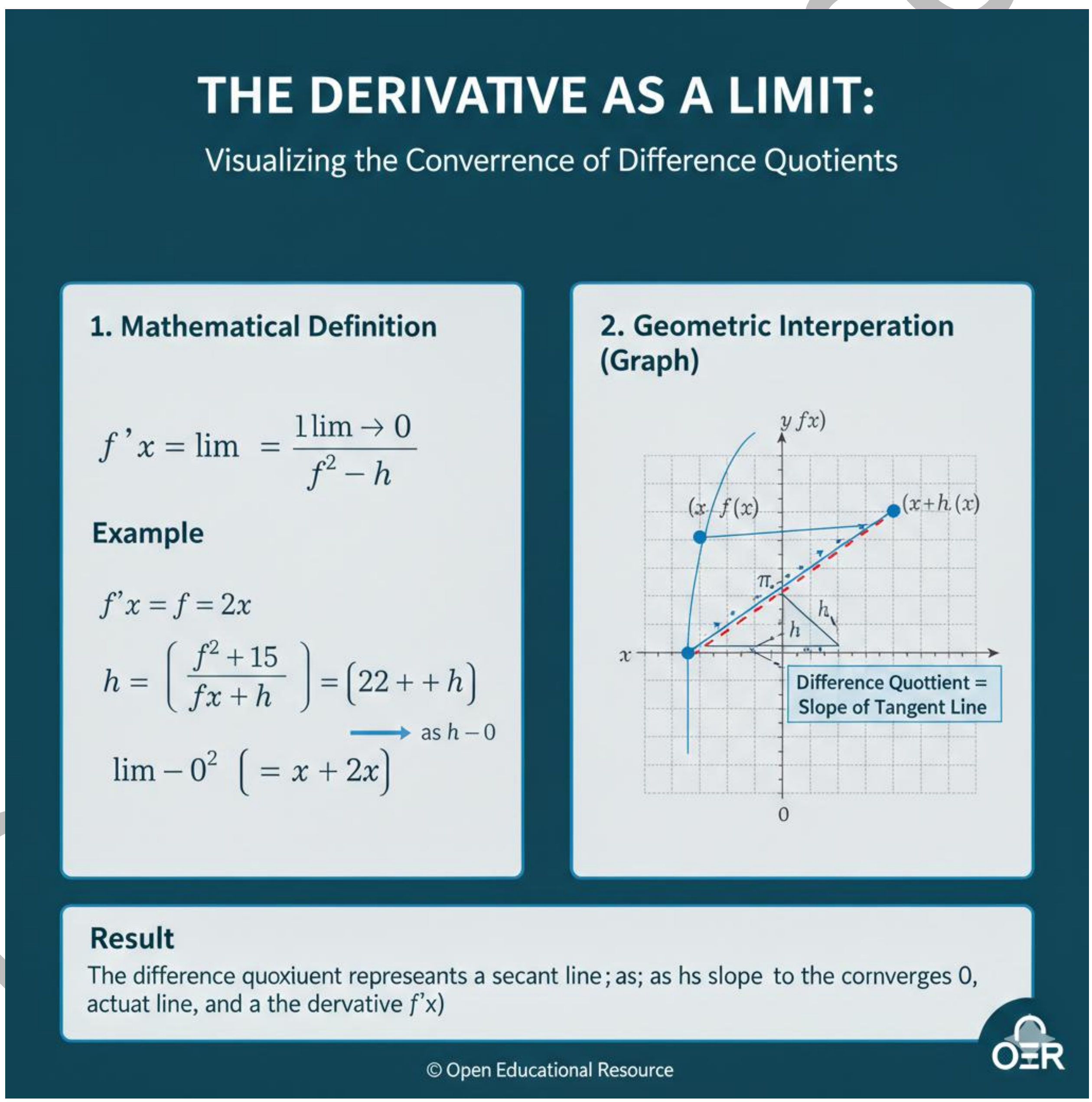


Illustration of successive approximations of using a convergent sequence **1.4.2**

Use in analysis and calculus

Convergence plays a central role in **analysis** and **calculus**. Limits of sequences underpin the definition of continuity, differentiability, and integrals. For example, the derivative of a function is defined using the limit of a sequence of difference quotients. Similarly, integrals can be understood as limits of sums. Without convergence, these fundamental concepts would not be well defined.

Fill in the blank: The derivative of a function is defined using the limit of a sequence of __________________ quotients.



Visual representation of convergence applied in calculus (difference quotients approaching derivative)

Pilot Questions 1.4

Convergence is used to approximate real numbers such as: A. Rational numbers B. Irrational numbers C. Integers D. Prime numbers (Linked to SLO 1.4)

The decimal expansion of can be expressed as: A. A divergent sequence B. A convergent sequence C. A bounded sequence D. A supremum sequence (Linked to SLO 1.4)

The derivative of a function is defined using the limit of: A. Difference quotients B. Supremum values C. Divergent sequences D. Upper bounds (Linked to SLO 1.4)

Integrals can be understood as limits of: A. Bounds B. Sums C. Divergent series D. Infimum values (Linked to SLO 1.4)

Which of the following best describes the role of convergence in calculus? A. It defines continuity, differentiability, and integrals. B. It only applies to bounded sets. C. It is unrelated to calculus. D. It defines supremum and infimum. (Linked to SLO 1.4)

Series Summary

This Series has introduced you to the essential concepts of **bounds and convergence of sequences**, which form the foundation of Real Analysis. We began by exploring bounds of real numbers, including upper and lower bounds, and the important notions of supremum and infimum. We then examined the convergence of sequences, distinguishing between convergent and divergent sequences with clear examples. Following that, we studied the properties of convergent sequences, such as uniqueness of limits, algebra of limits, and order properties. Finally, we considered applications of convergence, particularly in approximating real numbers and in supporting the development of calculus.

By completing this Series, you should now be able to define bounds of real numbers, explain the concept of convergence, analyse the properties of convergent sequences, and apply these ideas to practical problems in analysis. These abilities directly reflect the Series Learning Outcomes (SLOs) and prepare you for deeper topics in Real Analysis.

Glossary of Terms

Algebra of limits: Rules describing how limits behave under addition, subtraction, multiplication, and division.

Bound: A value that restricts how large or small the elements of a set or sequence can be.

Convergence: The property of a sequence approaching a specific value as the number of terms increases.

Divergence: The property of a sequence not approaching any finite value as the number of terms increase.

Infimum: The greatest lower bound of a set.

Limit: The value that a sequence approaches as the number of terms tends to infinity.

Lower bound: A value that is less than or equal to every element of a set.

Sequence: An ordered list of numbers, often defined by a rule or formula.

Supremum: The least upper bound of a set.

Upper bound: A value that is greater than or equal to every element of a set.

Concept Check Questions 1

Objective questions

The least upper bound of a set is called the: A. Infimum B. Supremum C. Limit D. Bound (Linked to SLO 1.1)

A sequence that approaches a finite value as the number of terms increases is said to: A. Diverge B. Converge C. Oscillate D. Expand (Linked to SLO 1.2)

Short-answer questions

Define a lower bound and give an example. (Linked to SLO 1.1)

Explain why the limit of a convergent sequence is unique. (Linked to SLO 1.3)

Long-answer questions

Discuss the properties of convergent sequences, including algebra of limits and order properties. (Linked to SLO 1.3)

Basic response: Describe the rules for addition and multiplication of limits.

Value-added response: Provide detailed examples showing how these properties are applied in proofs and problem-solving.

Analyse how convergence of sequences can be used to approximate real numbers. (Linked to SLO 1.4)

Basic response: Explain with a simple example, such as approximating .

Value-added response: Extend the discussion to applications in calculus, such as evaluating infinite series or limits of functions.

Answers to Concept Check Questions 1

B. Supremum

B. Converge

A lower bound is a value less than or equal to every element of a set. For example, 0 is a lower bound of the set of positive integers.

The limit of a convergent sequence is unique because if a sequence approached two different values, the distance between them would eventually become arbitrarily small, which is impossible.

Basic response: The algebra of limits states that if two sequences converge, their sum and product also converge to the sum and product of their limits.

Value-added response: For example, if an→Aand bn→B, then an+bn→A+B. This property is used in proofs to establish convergence of more complex sequences.

Basic response: Convergence can approximate real numbers, such as using the sequence of partial sums of a series to approximate .

Value-added response: In calculus, convergence is used to evaluate infinite series, such as the Taylor series expansion of functions, which provides approximations of values like ex.

Answers to Pilot Questions 1

Bound

Upper bound

Supremum

Infimum

Every bounded set has a supremum and infimum.

Series 2: Monotone Sequences and Nested Interval Theorem

Series Outline

Part 2.1 Monotone Sequences

2.1.1 Definition of monotone sequences

2.1.2 Increasing and decreasing sequences

2.1.3 Examples of monotone sequences

Part 2.2 Convergence of Monotone Sequences

2.2.1 Bounded monotone sequences

2.2.2 The monotone convergence theorem

2.2.3 Applications of monotone convergence

Part 2.3 Nested Intervals

2.3.1 Definition of nested intervals

2.3.2 Properties of nested intervals

2.3.3 Examples of nested intervals

Part 2.4 Nested Interval Theorem

2.4.1 Statement of the theorem

2.4.2 Proof outline

2.4.3 Applications of the theorem

Introduction

Sequences are not only central to Real Analysis but also provide a natural way of understanding how numbers behave when arranged in order. In the previous Series, we studied bounds and convergence, which gave us the tools to determine whether a sequence stabilises towards a particular value. Building on that foundation, this Series will focus on monotone sequences and the nested interval theorem, both of which are powerful concepts that guarantee convergence under certain conditions. These ideas are essential for proving deeper results in analysis and for understanding the completeness of the real number system.

The aim of this Series is to help you recognise monotone sequences, apply the monotone convergence theorem, and understand the nested interval theorem and its applications in analysis.

We shall commence this Series by defining monotone sequences and distinguishing between increasing and decreasing sequences with examples. Next, we will examine the convergence of monotone sequences, focusing on boundedness and the monotone convergence theorem. Subsequently, we will explore nested intervals, their properties, and illustrative examples. Finally, we will conclude with the nested interval theorem, presenting its statement, proof outline, and applications in analysis.

Series Learning Outcomes

By the end of **Series 2: Monotone Sequences and Nested Interval Theorem**, you should be able to:

SLO 2.1 define monotone sequences and distinguish between increasing and decreasing sequences.

SLO 2.2 explain the convergence of bounded monotone sequences and apply the monotone convergence theorem.

SLO 2.3 describe nested intervals and analyse their properties with examples.

SLO 2.4 state and outline the proof of the nested interval theorem, and apply it to problems in analysis.

2.1 Monotone Sequences

2.1.1 Definition of monotone sequences

A **monotone sequence** is a sequence in which the terms consistently increase or consistently decrease. In other words, the sequence does not change direction. Monotonicity is important because it provides a predictable structure, making it easier to study convergence.

Fill in the blank: A sequence that consistently increases or decreases without changing direction is called a __________________ sequence.

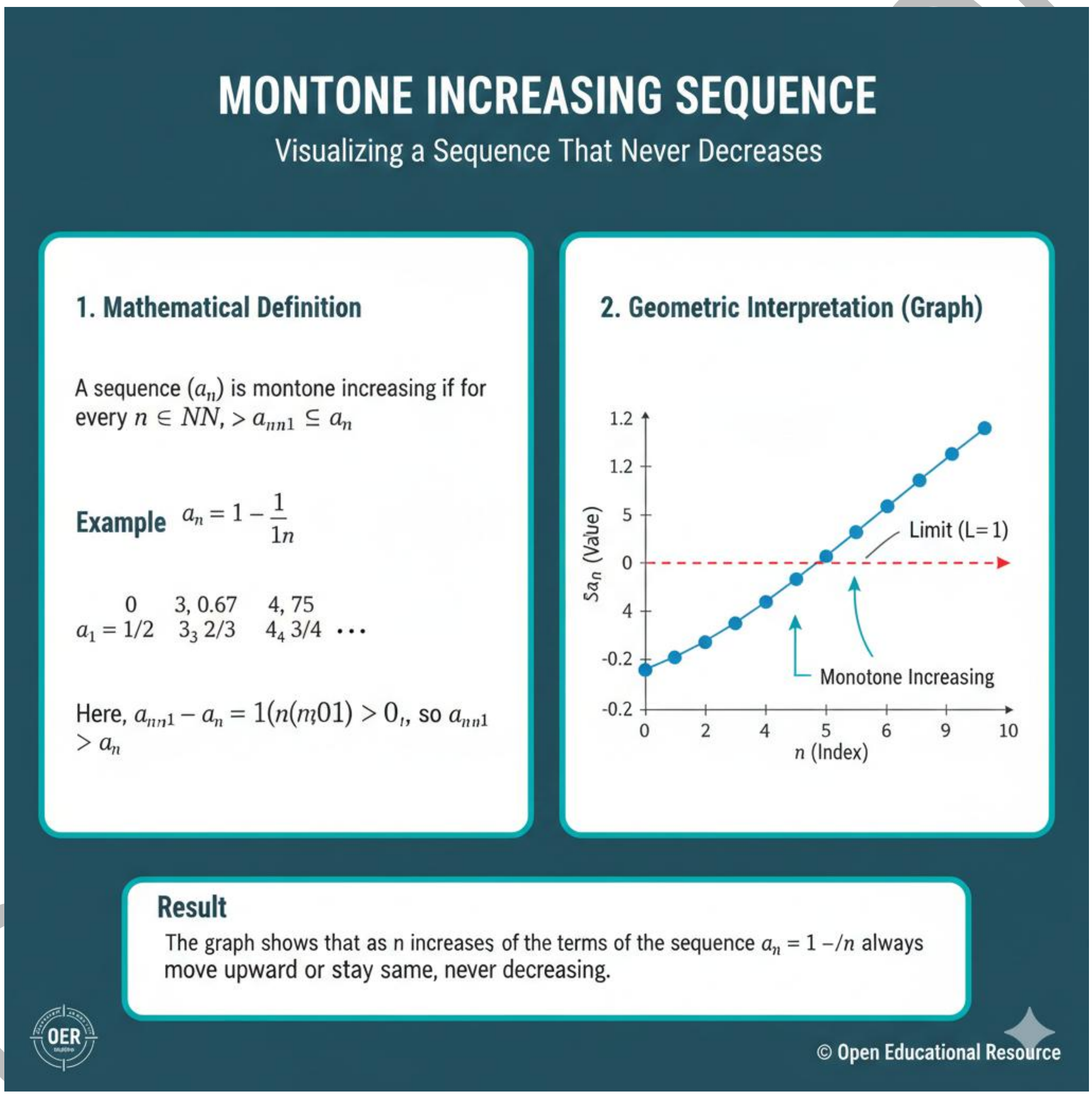
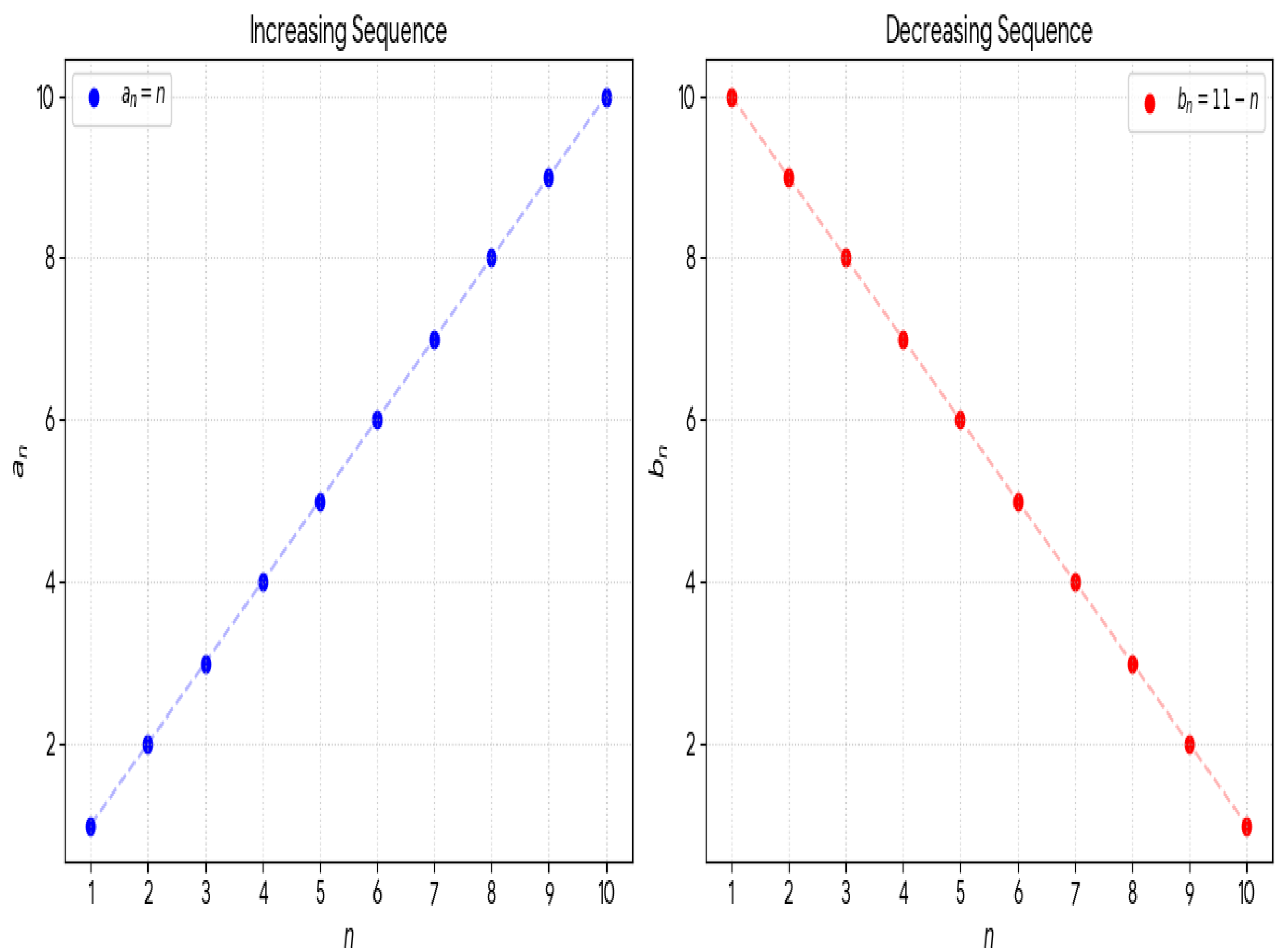


Illustration of a monotone increasing sequence on a graph

2.1.2 Increasing and decreasing sequences

An **increasing sequence** is one in which each term is greater than or equal to the previous term. For example, the sequence an=nis increasing. A **decreasing sequence** is one in which each term is less than or equal to the previous term. For example, the sequence an=1nis decreasing. These definitions allow us to classify sequences based on their behaviour.

Fill in the blank: A sequence in which each term is greater than or equal to the previous term is called an __________________ sequence.



Graph showing examples of increasing and decreasing sequences

2.1.3 Examples of monotone sequences

Examples help to clarify the concept of monotonicity. Consider the sequence =2n, which is monotone increasing because each term is larger than the last. Another example is an=1n, which is monotone decreasing because each term is smaller than the last. These examples show how monotone sequences can either grow without bound or shrink towards zero.

Fill in the blank: The sequence an=1nis an example of a ________________ decreasing sequence.

Examples of monotone sequences

Type	Sequence Formula	First Few Terms	Note
Monotone Increasing	$a_n = n^2$	$1, 4, 9, 16, \ldots$	Strictly increasing; diverges to infinity.
Monotone Increasing	$a_n = 2 - \frac{1}{n}$	$1, 1.5, 1.66, 1.75, \ldots$	Increases but is bounded above by 2.
Monotone Decreasing	$a_n = \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$	Strictly decreasing; converges to 0.
Monotone Decreasing	$a_n = e^{-n}$	$0.367, 0.135, 0.049, \ldots$	Decreases rapidly toward 0.
Monotone (Constant)	$a_n = 5$	$5, 5, 5, 5, \ldots$	Both non-increasing and non-decreasing.

Pilot Questions 2.1

A sequence that consistently increases or decreases without changing direction is called a: A. Divergent sequence B. Monotone sequence C. Bounded sequence D. Supremum sequence (Linked to SLO 2.1)

A sequence in which each term is greater than or equal to the previous term is called an: A. Decreasing sequence B. Increasing sequence C. Divergent sequence D. Infimum sequence (Linked to SLO 2.1)

The sequence an=nis: A. Increasing B. Decreasing C. Divergent D. Supremum (Linked to SLO 2.1)

The sequence an=1nis: A. Increasing B. Decreasing C. Divergent D. Supremum (Linked to SLO 2.1)

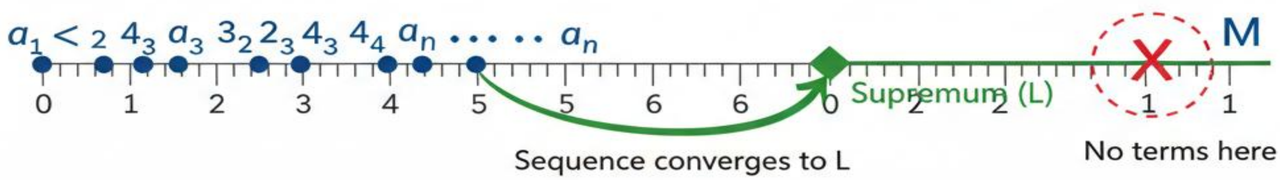
Which of the following statements is true? A. A monotone sequence always converges. B. A monotone sequence can be either increasing or decreasing. C. A monotone sequence must be bounded. D. A monotone sequence is always divergent. (Linked to SLO 2.1)

2.2 Convergence Of Monotone Sequences

2.2.1 Bounded monotone sequences

A **bounded sequence** is one that does not grow beyond certain limits. When a monotone sequence is bounded, it has a predictable behaviour. For example, a monotone increasing sequence that is bounded above will converge to its supremum, while a monotone decreasing sequence that is bounded below will converge to its infimum. This property makes bounded monotone sequences especially useful in analysis.

Fill in the blank: A monotone increasing sequence that is bounded above will converge to its __________________.



Graph showing a bounded monotone increasing sequence converging to its supremum

2.2.2 The monotone convergence theore

The **monotone convergence theorem** states that every bounded monotone sequence is convergent. This means that if a sequence is monotone increasing and bounded above, or monotone decreasing and bounded below, then it must converge. This theorem is a powerful tool because it guarantees convergence without needing to compute the limit directly.

Fill in the blank: The monotone convergence theorem states that every __________________ monotone sequence is convergent.

Statement of the monotone convergence theorem AI prompt:

The Theorem Statement

A sequence of real numbers (a_n) converges if and only if it is **monotone** and **bounded**.

Conditions for Convergence

To apply the theorem, two specific conditions must be met:

1. **Monotonicity**: The sequence must move in a single direction.
   - **Increasing:** $a_n \leq a_{n+1}$ for all n.
   - **Decreasing:** $a_n \geq a_{n+1}$ for all n.
2. **Boundedness**: The sequence must be contained within a specific range.
   - An **increasing** sequence must be **bounded above** ($a_n \leq M$).
   - A **decreasing** sequence must be **bounded below** ($a_n \geq m$).

Key Outcomes

- If (a_n) is increasing and bounded above, then:

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$$

- If (a_n) is decreasing and bounded below, then:

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$$

> **Why it matters:** In many proofs, it is difficult to calculate a limit directly. The MCT allows mathematicians to prove a limit *exists* simply by showing the sequence never changes direction and never crosses a certain "ceiling" or "floor."

2.2.3 Applications of monotone convergence

The monotone convergence theorem has many applications in analysis. For eample, it is used to prove the convergence of sequences defined recursively, such as those arising in iterative methods for solving equations. It also plays a role in integration theory, where monotone sequences of functions are used to establish convergence of integrals.

Fill in the blank: The monotone convergence theorem is often applied to prove the convergence of __________________ sequences defined recursively.

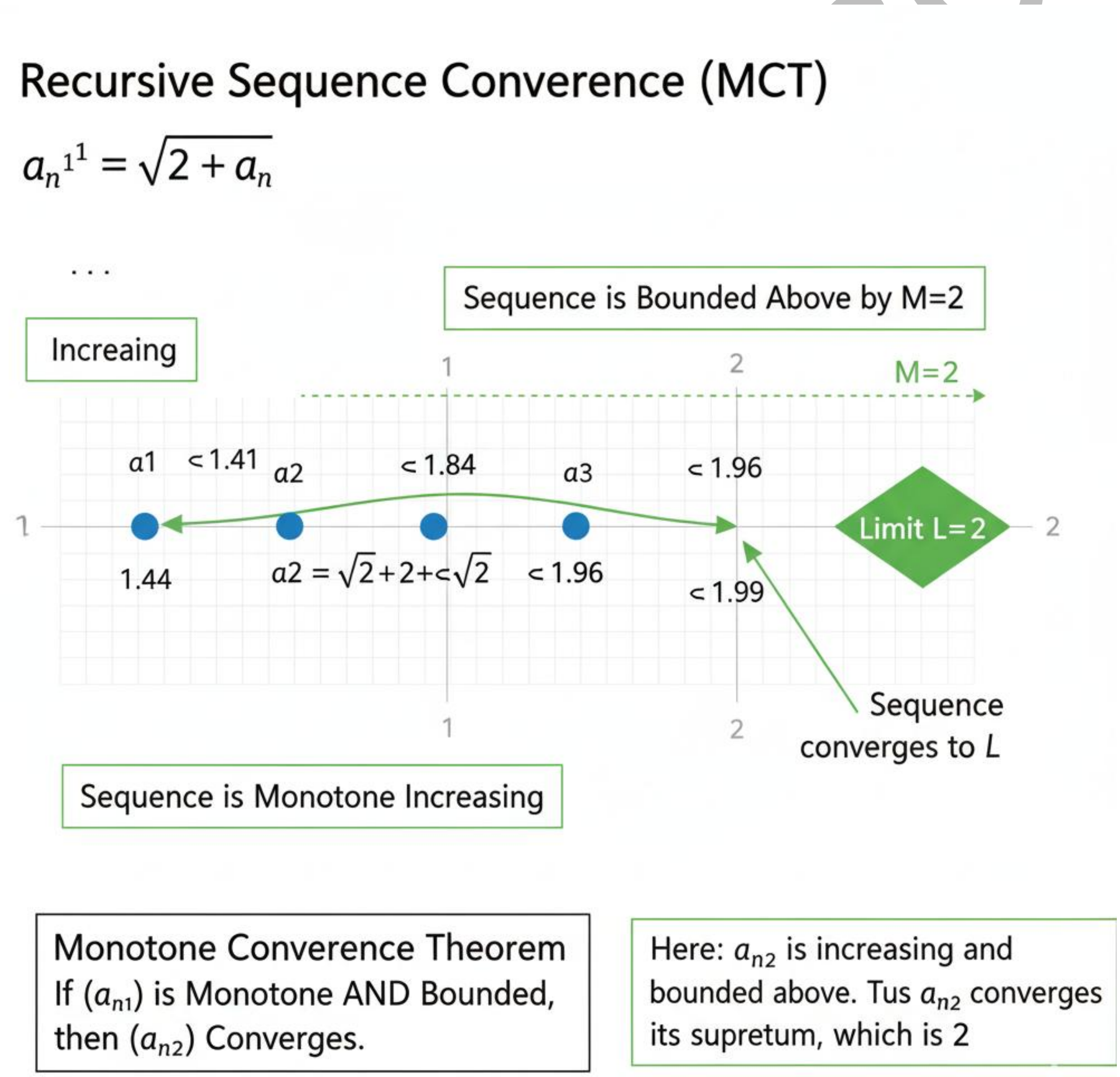


Illustration of recursive sequences converging under monotone convergence theorem

Pilot Questions 2.2

A monotone increasing sequence that is bounded above will converge to its: A. Infimum B. Supremum C. Limit inferior D. Divergent value (Linked to SLO 2.2)

The monotone convergence theorem states that every: A. Divergent sequence is convergent B. Bounded monotone sequence is convergent C. Oscillating sequence is convergent D. Supremum sequence is convergent (Linked to SLO 2.2)

A monotone decreasing sequence that is bounded below will converge to its: A. Supremum B. Infimum C. Maximum D. Divergent value (Linked to SLO 2.2)

Which of the following is a direct application of the monotone convergence theorem? A. Proving convergence of recursive sequences B. Defining divergence of sequences C. Establishing uniqueness of limits D. Identifying upper bounds (Linked to SLO 2.2)

The monotone convergence theorem is useful because it: A. Guarantees convergence without computing the limit directly B. Defines divergence of bounded sequences C. Explains oscillating behaviour of sequences D. Provides maximum and minimum values of sets (Linked to SLO 2.2)

2.3 Nested Intervals

2.3.1 Definition of nested intervals

A **nested interval** is a sequence of closed intervals on the real line where each interval is contained within the previous one. Formally, if we have intervals bn, then they are nested if an+1,bn+1]⊆[an,bn. This means the intervals get progressively smaller while remaining inside one another.

Fill in the blank: A sequence of closed intervals where each interval is contained within the previous one is called a __________________ interval.

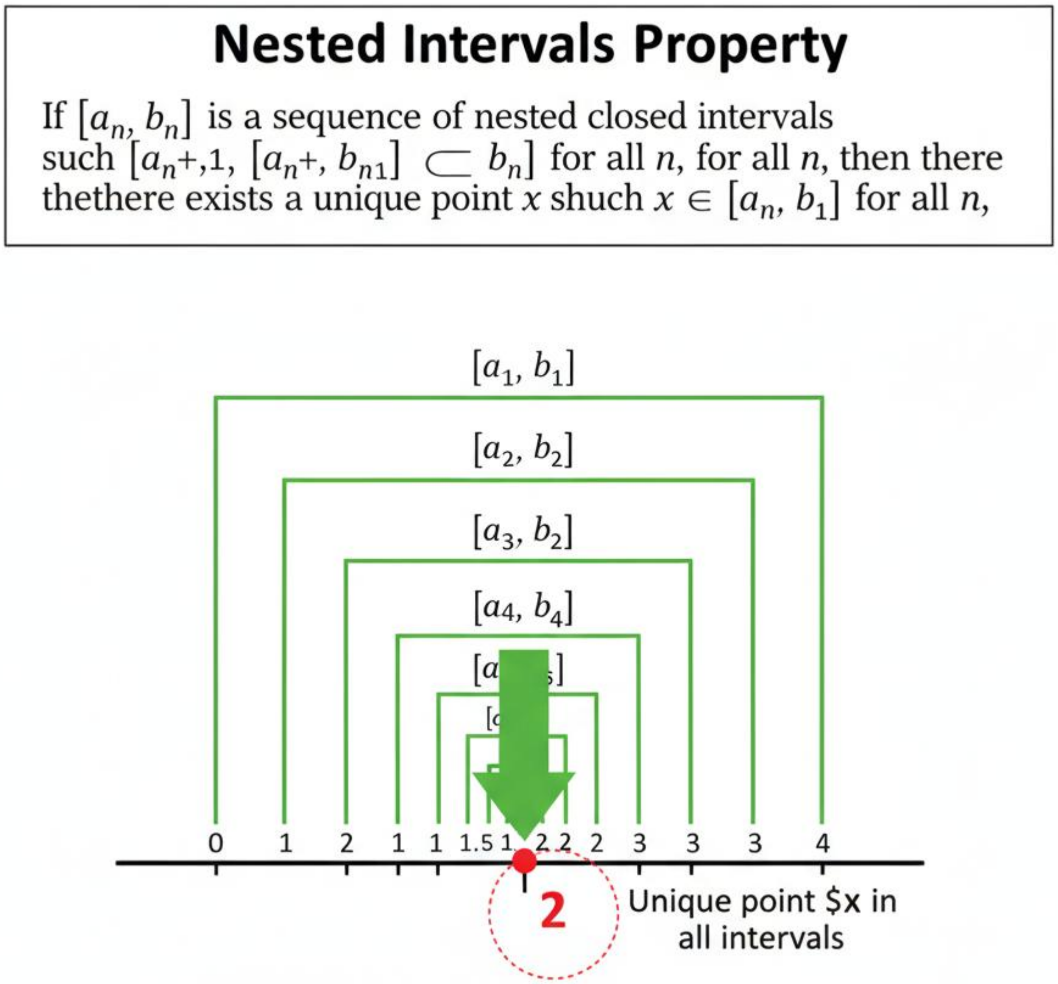


Illustration of nested intervals on the real line

2.3.2 Properties of nested intervals

Nested intervals have important properties. If the lengths of the intervals shrink to zero, then the intersection of all intervals contains exactly one point. If the lengths do not shrink to zero, the intersection may contain more than one point. These properties are crucial for proving results about convergence and completeness in real analysis.

Fill in the blank: If the lengths of nested intervals shrink to zero, their intersection contains exactly __________________ point.

Key properties of nested intervals

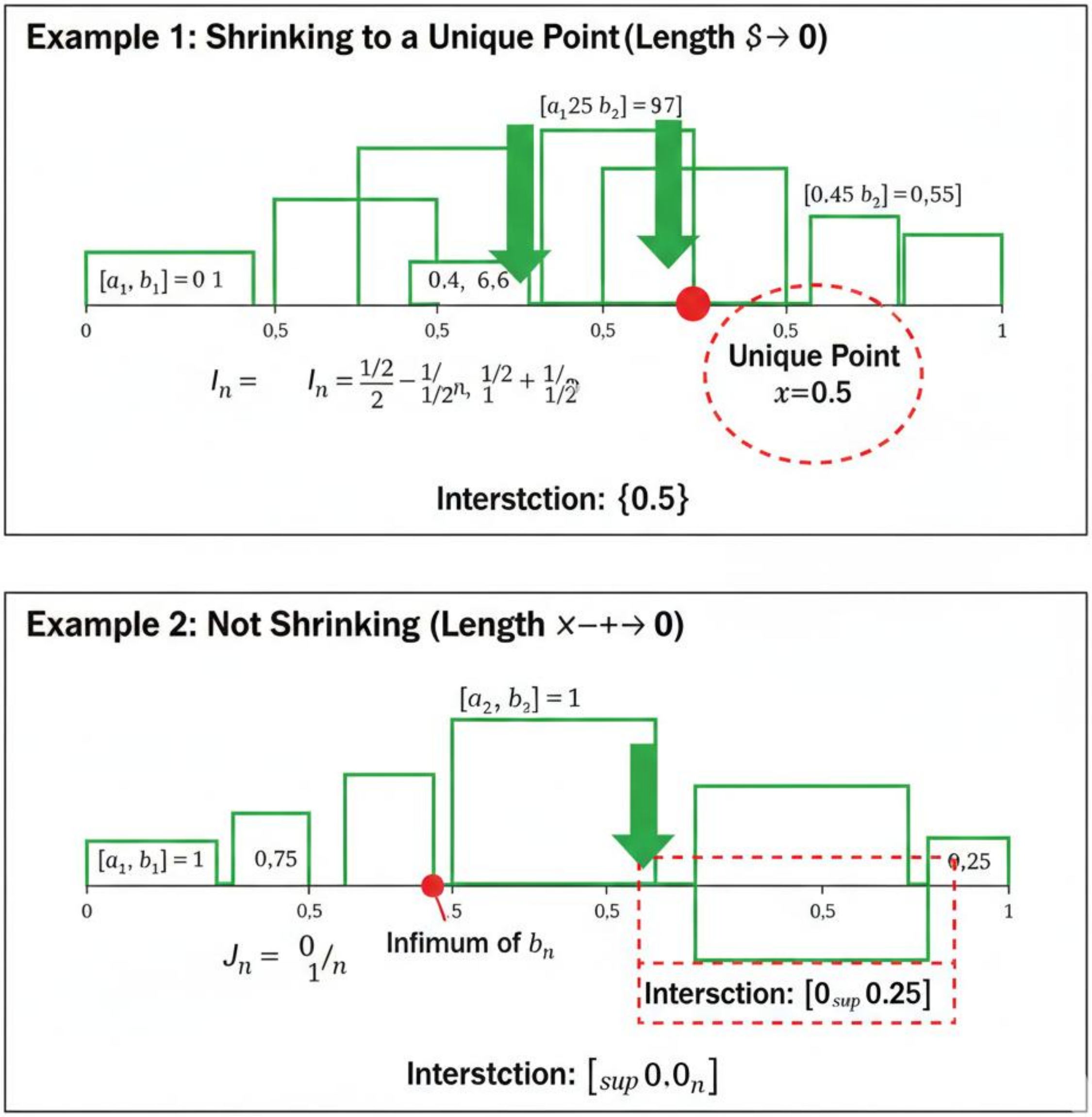
Core Properties

Property	Description	Mathematical Expression
Containment	Each interval is a subset of the one before it.	$I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$
Non-Emptiness	The intersection of all intervals in the sequence is never empty.	$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$
Closedness	The intervals must be **closed** $[a, b]$ for the property to hold.	$I_n = [a_n, b_n]$
Shrinking Length	As n increases, the length of the intervals approaches zero.	$\lim_{n\to\infty}(b_n - a_n) = 0$

2.3.3 Examples of nested intervals

Consider the sequence of intervals [0,1],[0,12],[0,13],.... Each interval is contained within the previous one, and their lengths shrink to zero. The intersection of all these intervals is the single point 0. Another example is [0,1],[0,34],[0,23],..., where the lengths do not shrink to zero, so the intersection contains more than one point.

Fill in the blank: The intersection of the intervals [0,1],[0,12],[0,13],...is the single point __________________.



Examples of nested intervals with shrinking and non-shrinking lengths

Pilot Questions 2.3

A sequence of closed intervals where each interval is contained within the previous one is called: A. Divergent intervals B. Nested intervals C. Bounded intervals D. Supremum intervals (Linked to SLO 2.3)

If the lengths of nested intervals shrink to zero, their intersection contains: A. No points B. Exactly one point C. Infinitely many points D. Supremum (Linked to SLO 2.3)

The intersection of the intervals [0,1],[0,12],[0,13],...is: A. 1 B. 0 C. Infinity D. 12(Linked to SLO 2.3)

If the lengths of nested intervals do not shrink to zero, their intersection may contain: A. More than one point B. Exactly one point C. No points D. Supremum only (Linked to SLO 2.3)

Which of the following statements is true? A. Nested intervals always shrink to zero length. B. Nested intervals can have intersections containing one or more points depending on their lengths. C. Nested intervals are unrelated to convergence. D. Nested intervals cannot exist on the real line. (Linked to SLO 2.3)

2.4 Nested Interval Theorem

2.4.1 Statement of the theorem

The **nested interval theorem** states that if we have a sequence of closed intervals bnsuch that each interval is contained within the previous one, and the lengths of the intervals shrink to zero, then the intersection of all these intervals contains exactly one point. This theorem is a direct application of the completeness property of the real numbers.

Fill in the blank: The nested interval theorem states that if the lengths of nested intervals shrink to zero, their intersection contains exactly ___________________ point.

Statement of the nested interval theorem

The Conditions

For the theorem to apply, the sequence of intervals I_n must satisfy the following three criteria:

1. **Nested Property**: Each interval must be contained within the previous one.

$$I_{n+1} \subseteq I_n \text{ for all } n \in \mathbb{N}$$

2. **Closed and Bounded**: Each interval must be a closed interval of the form $[a_n, b_n]$, where $a_n, b_n \in \mathbb{R}$ and $a_n \leq b_n$.
3. **Finite Endpoints**: The intervals must not be infinitely large (e.g., $[n, \infty)$ does not apply).

The Conclusion

If the conditions above are met, the theorem guarantees:

- **Existence**: The intersection of all intervals is non-empty. There exists at least one real number x such that:

$$x \in \bigcap_{n=1}^{\infty} I_n$$

- **Uniqueness (Optional Condition)**: If the lengths of the intervals approach zero as n goes to infinity—that is, $\lim_{n\to\infty}(b_n - a_n) = 0$—then the intersection contains **exactly one** unique real number.

2.4.2 Proof outline

The proof of the nested interval theorem relies on the fact that the sequence of left endpoints anis increasing and bounded above, while the sequence of right endpoints bnis decreasing and bounded below. By the monotone convergence theorem, both sequences converge, and since the lengths shrink to zero, they converge to the same point. That point is the unique element in the intersection of all intervals.

Fill in the blank: In the proof of the nested interval theorem, the sequence of left endpoints is __________________ and bounded above.

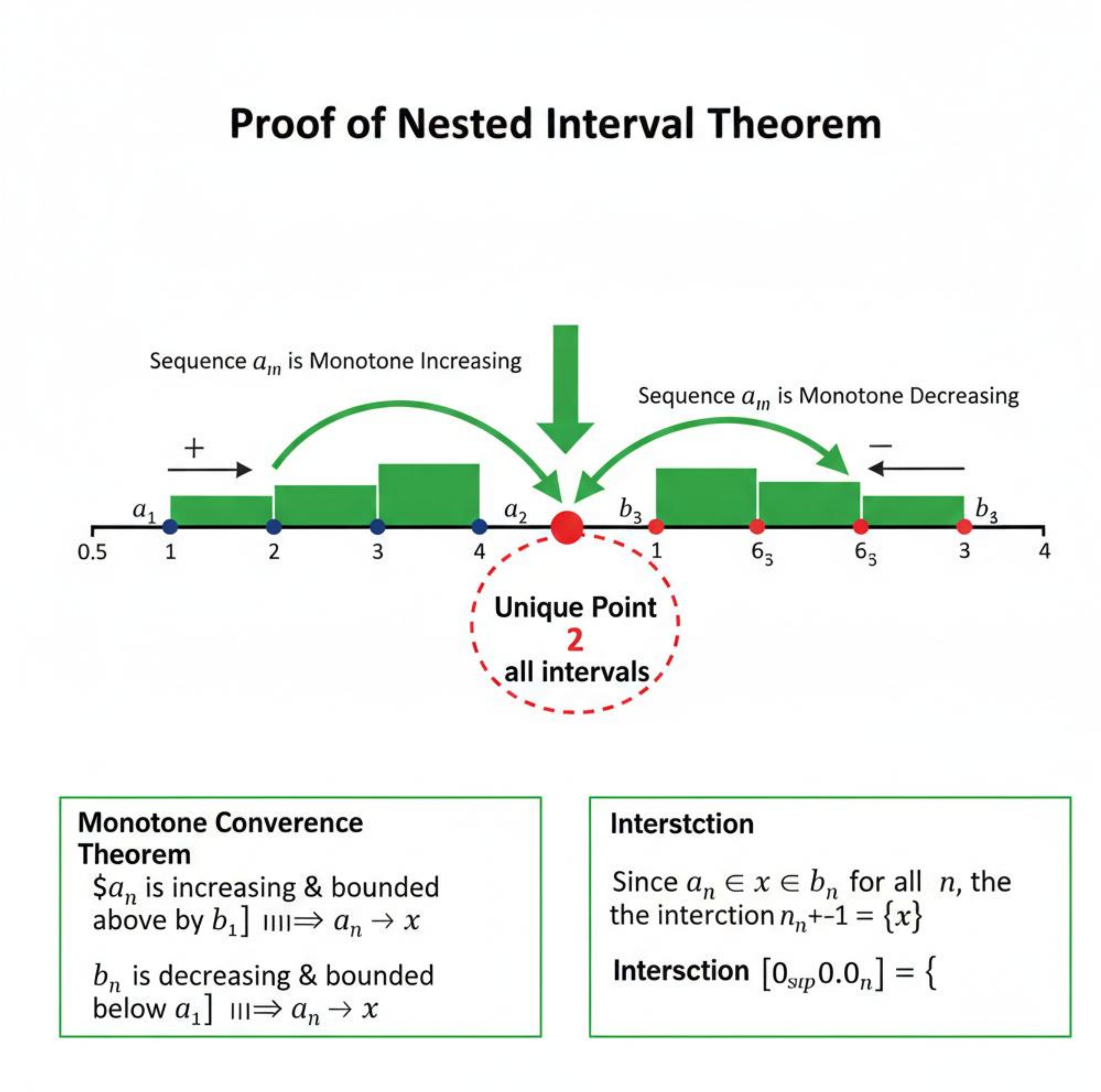
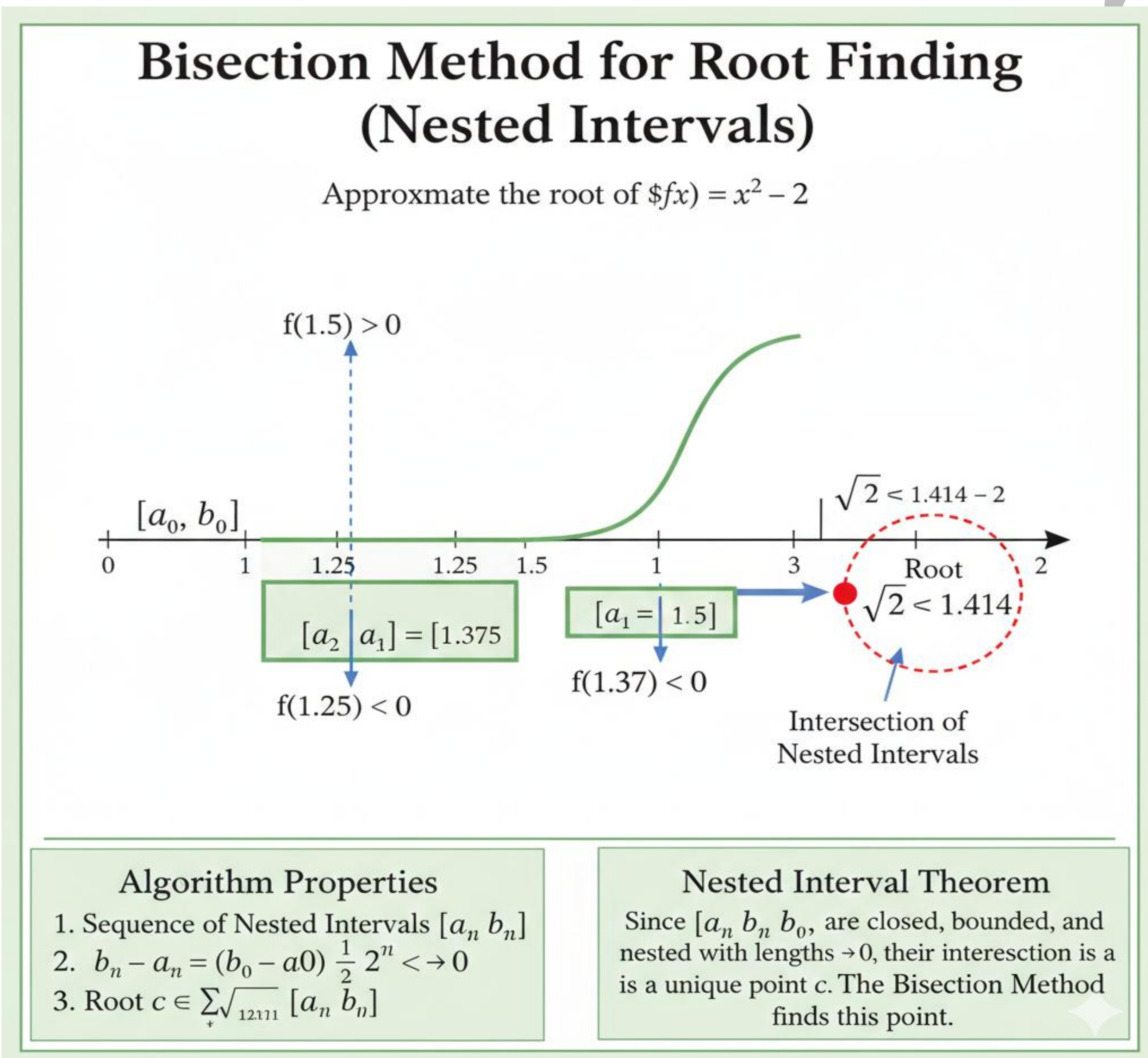


Illustration of left and right endpoints converging to the same point

2.4.3 Applications of the theorem

The nested interval theorem is widely used in analysis. It provides a constructive way of locating points on the real line, such as roots of equations. For example, the bisection method for finding roots of continuous functions is based on nested intervals. It also plays a role in demonstrating the completeness of the real numbers, ensuring that limits and intersections behave consistently.

Fill in the blank: The bisection method for finding roots of continuous functions is based on __________________ intervals.



Application of nested interval theorem in the bisection method

Pilot Questions 2.4

The nested interval theorem states that if the lengths of nested intervals shrink to zero, their intersection contains: A. No points B. Exactly one point C. Infinitely many points D. Supremum (Linked to SLO 2.4)

In the proof of the nested interval theorem, the sequence of left endpoints is: A. Decreasing and bounded below B. Increasing and bounded above C. Divergent D. Oscillating (Linked to SLO 2.4)

The sequence of right endpoints in the proof of the nested interval theorem is: A. Increasing and bounded above B. Decreasing and bounded below C. Divergent D. Oscillating (Linked to SLO 2.4)

The bisection method for finding roots of continuous functions is based on: A. Divergent sequences B. Nested intervals C. Supremum values D. Oscillating sequences (Linked to SLO 2.4)

Which of the following is an application of the nested interval theorem? A. Proving divergence of sequences B. Locating roots of equations using the bisection method C. Defining supremum and infimum D. Establishing uniqueness of limits (Linked to SLO 2.4)

Series Summary

This Series has introduced you to the concepts of **monotone sequences and the nested interval theorem**, which extend the ideas of bounds and convergence studied earlier. We began by defining monotone sequences and distinguishing between increasing and decreasing types with examples. We then examined the convergence of monotone sequences, focusing on boundedness and the monotone convergence theorem, which guarantees convergence under specific conditions. Next, we explored nested intervals, their properties, and illustrative examples. Finally, we studied the nested interval theorem, including its statement, proof outline, and applications such as the bisection method for locating roots.

By completing this Series, you should now be able to define monotone sequences, apply the monotone convergence theorem, describe nested intervals and their properties, and state and apply the nested interval theorem. These abilities directly reflect the Series Learning Outcomes (SLOs) and prepare you for deeper results in Real Analysis, particularly those involving completeness of the real numbers.

Glossary of Terms

Bounded sequence: A sequence whose terms do not exceed certain fixed limits.

Decreasing sequence: A sequence in which each term is less than or equal to the previous term.

Increasing sequence: A sequence in which each term is greater than or equal to the previous term.

Monotone sequence: A sequence that is either entirely increasing or entirely decreasing.

Monotone convergence theorem: The result stating that every bounded monotone sequence is convergent.

Nested intervals: A sequence of closed intervals where each interval is contained within the previous one.

Nested interval theorem: The result stating that if the lengths of nested intervals shrink to zero, their intersection contains exactly one point.

Supremum: The least upper bound of a set.

Infimum: The greatest lower bound of a set.

Concept Check Questions 2

Objective questions

A sequence that consistently increases or decreases without changing direction is called a: A. Divergent sequence B. Monotone sequence C. Bounded sequence D. Supremum sequence (Linked to SLO 2.1)

The monotone convergence theorem states that every: A. Divergent sequence is convergent B. Bounded monotone sequence is convergent C. Oscillating sequence is convergent D. Supremum sequence is convergent (Linked to SLO 2.2)

Short-answer questions

Define an increasing sequence and give an example. (Linked to SLO 2.1)

Explain why a bounded monotone sequence must converge. (Linked to SLO 2.2)

Long-answer questions

Describe the nested interval theorem, including its statement and proof outline. (Linked to SLO 2.4)

Basic response: State the theorem and explain the role of shrinking interval lengths.

Value-added response: Provide a structured proof outline using monotone convergence of endpoints.

Analyse how the nested interval theorem can be applied in the bisection method for finding roots of equations. (Linked to SLO 2.4)

Basic response: Explain how nested intervals are formed during bisection.

Value-added response: Extend the discussion to show how convergence guarantees the existence of a root.

Answers to Concept Check Questions 2

B. Monotone sequence

B. Bounded monotone sequence is convergent

An increasing sequence is one in which each term is greater than or equal to the previous term. Example: an=n.

A bounded monotone sequence must converge because its terms are restricted by bounds, and monotonicity ensures they approach a limit (supremum or infimum).

Basic response: The nested interval theorem states that if the lengths of nested intervals shrink to zero, their intersection contains exactly one point.

Value-added response: The proof uses the fact that left endpoints form an increasing bounded sequence and right endpoints form a decreasing bounded sequence. Both converge to the same point, which is the unique intersection.

Basic response: In the bisection method, intervals are repeatedly halved, forming nested intervals that shrink to zero length.

Value-added response: The nested interval theorem guarantees that the intersection of these intervals is a single point, which is the root of the equation.

Answers to Pilot Questions 2

Part 2.1

Monotone sequence

Increasing sequence

Increasing

Decreasing

A monotone sequence can be either increasing or decreasing

Part 2.2

Supremum

Bounded monotone sequence is convergent

Infimum

Proving convergence of recursive sequences

Guarantees convergence without computing the limit directly

Part 2.3

Nested intervals

Exactly one point

0

More than one point

Nested intervals can have intersections containing one or more points depending on their lengths

Part 2.4

Exactly one point

Increasing and bounded above

Decreasing and bounded below

Nested intervals

Locating roots of equations using the bisection method

Series 3 Outline

Part 3.1 Cauchy Sequences

3.1.1 Definition of Cauchy sequences

3.1.2 Examples of Cauchy sequences

3.1.3 Importance of Cauchy sequences in analysis

Part 3.2 Completeness of Real Numbers

3.2.1 Relationship between Cauchy sequences and convergence

3.2.2 Completeness property of R

3.2.3 Contrast with incomplete spaces

Part 3.3 Convergence of Series

3.3.1 Definition of series and partial sums

3.3.2 Convergent and divergent series

3.3.3 Examples of convergent series

Part 3.4 Tests for Convergence of Series

3.4.1 Comparison test

3.4.2 Ratio test

3.4.3 Root test

Introduction

In Real Analysis, one of the most powerful ideas is that of Cauchy sequences, which provide a way of characterising convergence without explicitly knowing the limit. This concept is fundamental because it leads directly to the notion of completeness of the real numbers, distinguishing Rfrom other spaces such as Q. Alongside this, the study of series—infinite sums of sequences—extends our understanding of convergence and introduces techniques for handling infinite processes rigorously.

The aim of this Series is to enable you to define and recognise Cauchy sequences, explain the completeness of the real numbers, analyse the convergence of series, and apply standard tests for convergence.

We shall commence this Series by defining Cauchy sequences, providing examples, and explaining their importance in analysis. Next, we will examine the completeness of the real numbers, showing how Cauchy sequences and convergence are linked. Subsequently, we will study the convergence of series, including definitions, partial sums, and examples of convergent and divergent series. Finally, we will conclude with tests for convergence of series, such as the comparison test, ratio test, and root test, which provide practical tools for determining convergence.

Series Learning Outcomes

By the end of **Series 3: Cauchy Sequences and Convergence of Series**, you should be able to:

SLO 3.1 define Cauchy sequences, provide examples, and explain their importance in analysis.

SLO 3.2 explain the relationship between Cauchy sequences and convergence, and describe the completeness property of the real numbers.

SLO 3.3 define series and partial sums, distinguish between convergent and divergent series, and illustrate with examples.

SLO 3.4 apply standard tests for convergence of series, including the comparison test, ratio test, and root test.

3.1 Cauchy Sequences

3.1.1 Definition of Cauchy sequences

A **Cauchy sequence** is a sequence in which the terms get arbitrarily close to each other as the sequence progresses. Formally, a sequence anis Cauchy if for every $\varepsilon>0$, there exists an integer Nsuch that for all $m,n\geq N$, $|am-an|<\varepsilon$. This definition focuses on the closeness of terms rather than their approach to a specific limit.

Fill in the blank: A sequence in which the terms get arbitrarily close to each other as the sequence progresses is called a __________________ sequence.

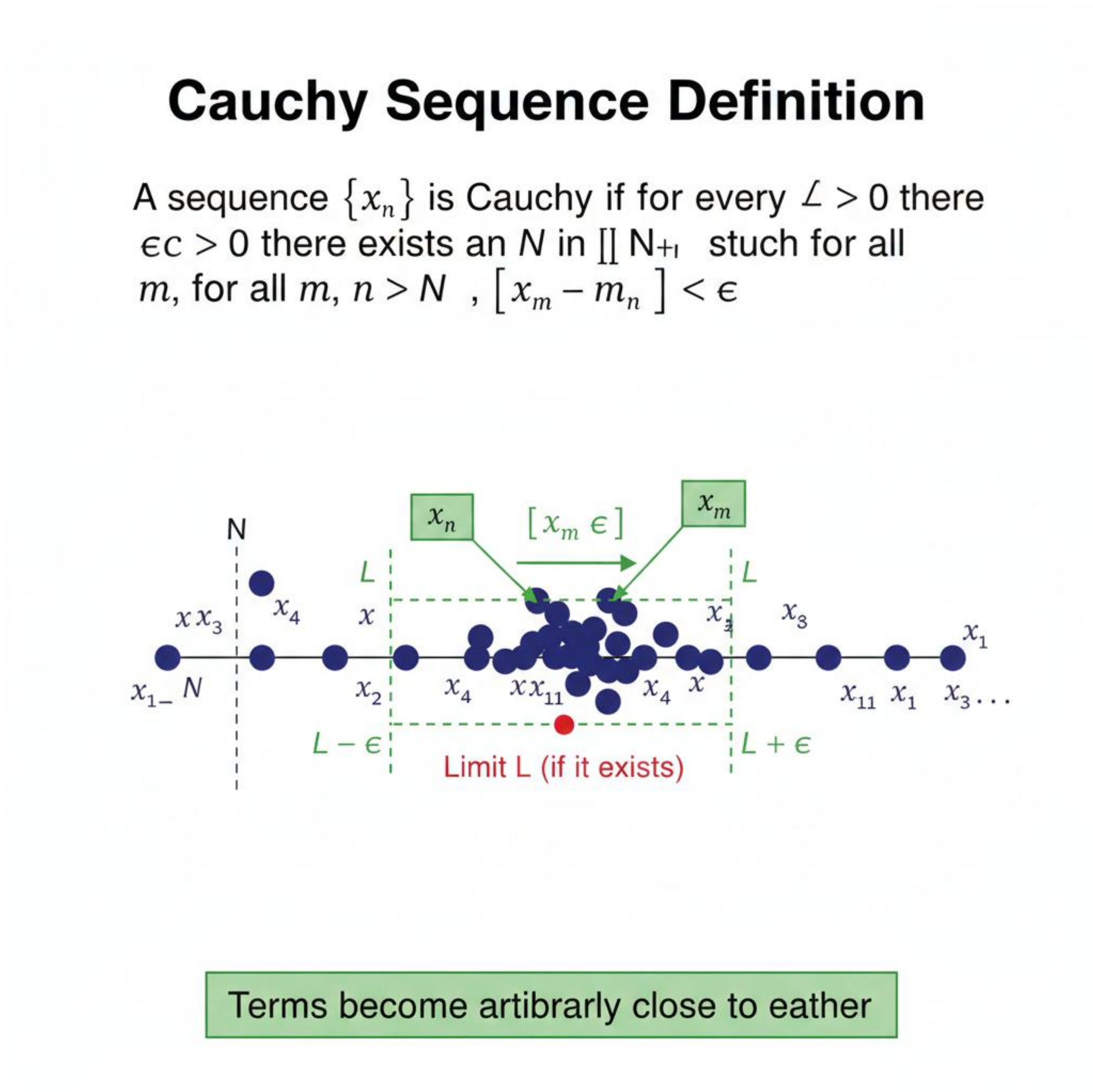


Illustration of a Cauchy sequence with terms clustering closer together

3.1.2 Examples of Cauchy sequences

The sequence an=1nis Cauchy because the difference between terms becomes arbitrarily small as nincreases.

The sequence of decimal approximations of (e.g., 3, 3.1, 3.14, 3.141, …) is Cauchy because the approximations get closer together.

Conversely, the sequence an=nis not Cauchy because the terms grow farther apart.

Fill in the blank: The sequence an=nis not a Cauchy sequence because its terms grow __________________ apart.

Examples of Cauchy and non-Cauchy sequences

Examples of Cauchy and Non-Cauchy Sequences

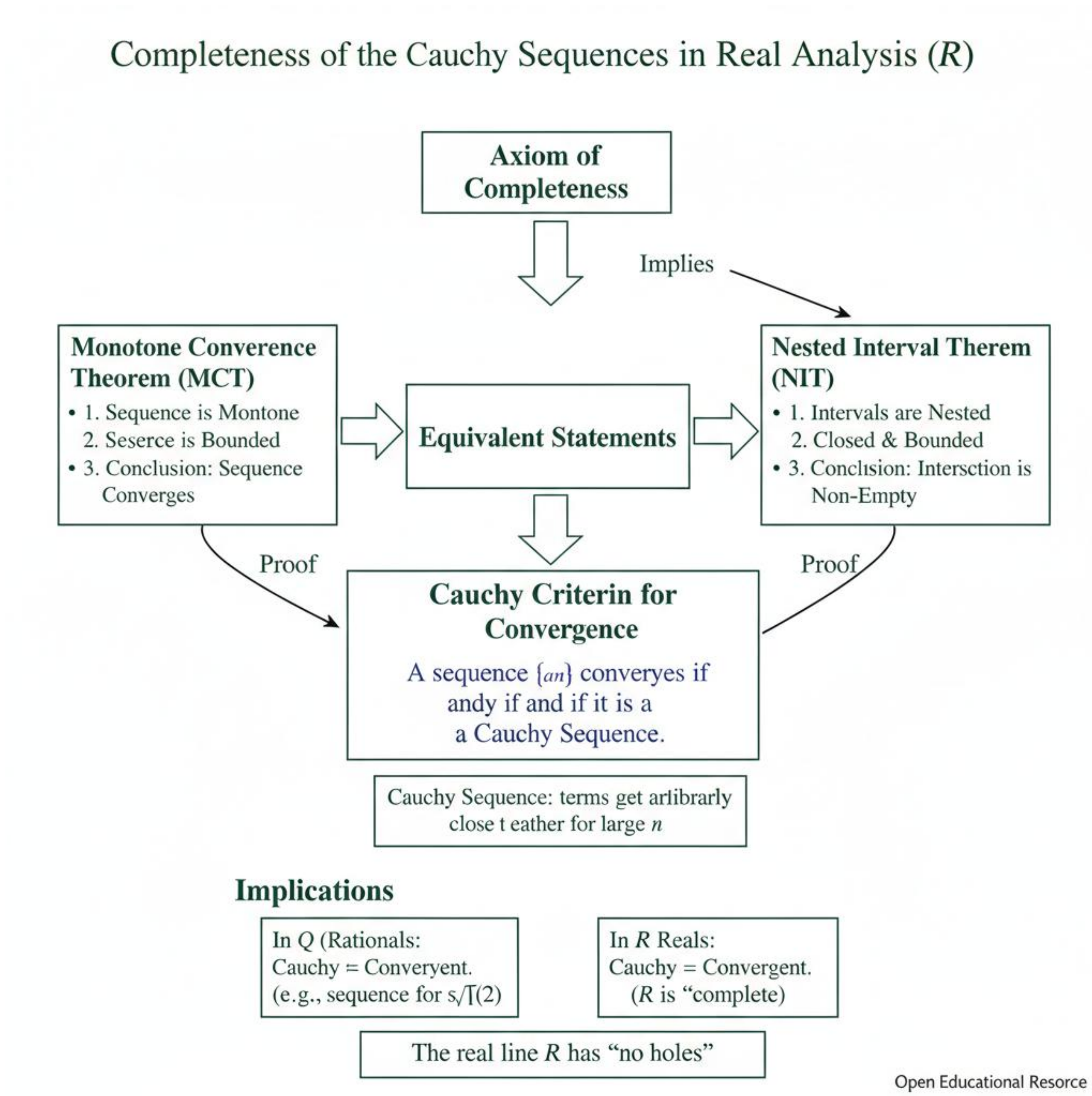
In real analysis, a sequence (a_n) is **Cauchy** if the terms of the sequence become arbitrarily close to each other as n increases. In the space of real numbers, a sequence is Cauchy if and only if it converges.

Sequence Formula	First Few Terms	Classification	Reasoning
$a_n = \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$	**Cauchy**	Converges to 0; distance between terms $
$a_n = \frac{n+1}{n}$	$2, 1.5, 1.33, 1.25, \dots$	**Cauchy**	Converges to 1.
$a_n = (-1)^n$	$-1, 1, -1, 1, \dots$	**Non-Cauchy**	Diverges (oscillates); distance between terms is always 0 or 2.
$a_n = n^2$	$1, 4, 9, 16, \dots$	**Non-Cauchy**	Diverges to ∞; distance between terms grows larger.
$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$	$1, 1.25, 1.36, 1.42, \dots$	**Cauchy**	This is a convergent series (the p-series with $p = 2$).

3.1.3 Importance of Cauchy sequences in analysis

Cauchy sequences are important because they provide a way to define convergence without explicitly knowing the limit. In fact, in the real numbers, every Cauchy sequence converges, which is a consequence of completeness. This property is fundamental in analysis and distinguishes complete spaces like Rfrom incomplete ones like Q.

Fill in the blank: In the real numbers, every Cauchy sequence __________________.



Importance of Cauchy sequences in defining convergence and completeness

Pilot Questions 3.1

A sequence in which the terms get arbitrarily close to each other as the sequence progresses is called a: A. Divergent sequence B. Cauchy sequence C. Monotone sequence D. Supremum sequence (Linked to SLO 3.1)

- The sequence an=1nis: A. Divergent B. Cauchy C. Increasing D. Supremum (Linked to SLO 3.1)
- The sequence an=nis not Cauchy because its terms grow: A. Closer together B. Farther apart C. Bounded D. Supremum (Linked to SLO 3.1)
- The sequence of decimal approximations of (3, 3.1, 3.14, …) is: A. Divergent B. Cauchy C. Decreasing D. Supremum (Linked to SLO 3.1)
- In the real numbers, every Cauchy sequence: A. Diverges B. Converges C. Oscillates D. Shrinks to zero length (Linked to SLO 3.1)

3.1 Cauchy Sequences

3.1.1 Definition of Cauchy sequences

A **Cauchy sequence** is a sequence in which the terms get arbitrarily close to each other as the sequence progresses. Formally, a sequence anis Cauchy if for every ε>0, there exists an integer Nsuch that for all m,n≥N, |am-an|<ε. This definition focuses on the closeness of terms rather than their approach to a specific limit.

Fill in the blank: A sequence in which the terms get arbitrarily close to each other as the sequence progresses is called a __________________ sequence.

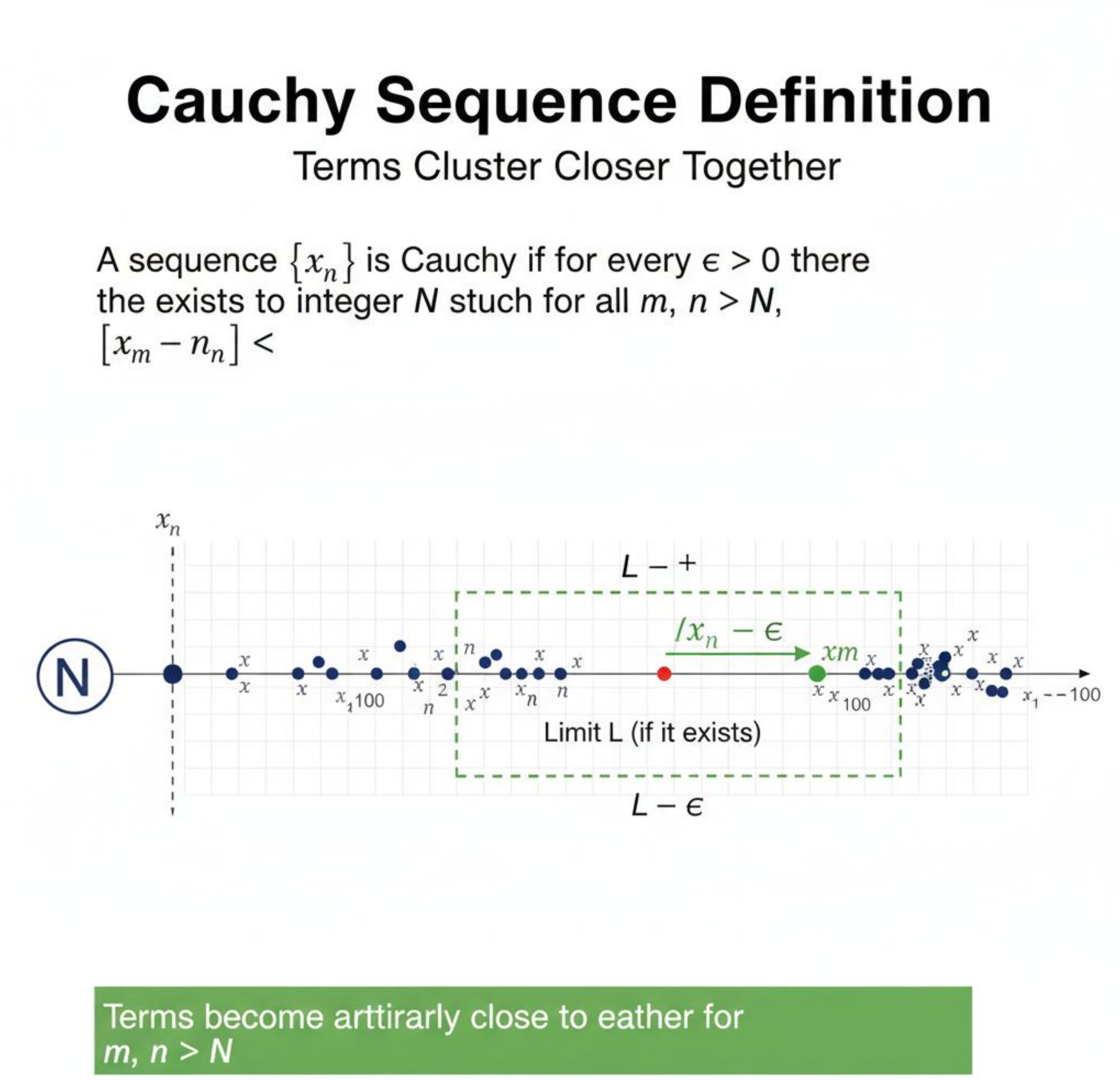


Illustration of a Cauchy sequence with terms clustering closer together

3.1.2 Examples of Cauchy sequences

The sequence an=1nis Cauchy because the difference between terms becomes arbitrarily small as nincreases.

The sequence of decimal approximations of (e.g., 3, 3.1, 3.14, 3.141, …) is Cauchy because the approximations get closer together.

Conversely, the sequence an=nis not Cauchy because the terms grow farther apart.

Fill in the blank: The sequence an=nis not a Cauchy sequence because its terms grow __________________ apart.

Examples of Cauchy and non-Cauchy sequences

Examples of Cauchy and Non-Cauchy Sequences

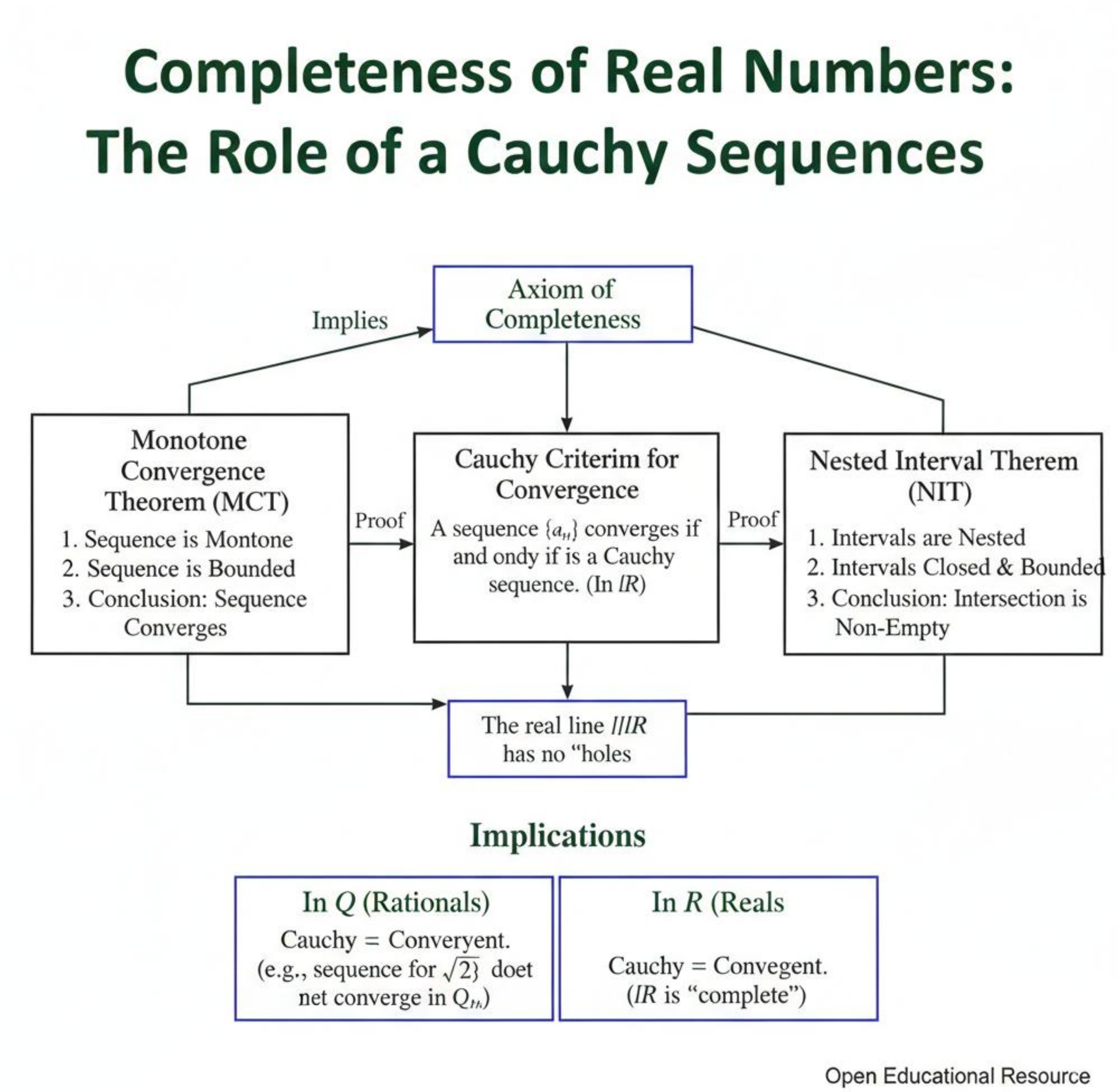
In real analysis, a sequence (a_n) is **Cauchy** if its terms become arbitrarily close to each other as the index increases. In the set of real numbers $\mathbb{R}$, being a Cauchy sequence is equivalent to being a convergent sequence.

Sequence Formula	First Few Terms	Classification	Reasoning
$a_n = \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \dots$	**Cauchy**	Converges to 0. For any ϵ, terms eventually stay within ϵ of each other.
$a_n = \frac{n+1}{n}$	$2, 1.5, 1.33, \dots$	**Cauchy**	Converges to 1.
$a_n = (-1)^n$	$-1, 1, -1, 1, \dots$	**Non-Cauchy**	Divergent (oscillates). The distance between a_n and a_{n+1} is always 2.
$a_n = n$	$1, 2, 3, 4, \dots$	**Non-Cauchy**	Divergent to ∞. Terms get further apart, not closer.
$a_n = \sum_{k=1}^{n} \frac{1}{k}$	$1, 1.5, 1.83, \dots$	**Non-Cauchy**	The Harmonic Series diverges. Although $a_{n+1} - a_n \to 0$, the terms don't "cluster."

3.1.3 Importance of Cauchy sequences in analysis

Cauchy sequences are important because they provide a way to define convergence without explicitly knowing the limit. In fact, in the real numbers, every Cauchy sequence converges, which is a consequence of completeness. This property is fundamental in analysis and distinguishes complete spaces like Rfrom incomplete ones like Q.

Fill in the blank: In the real numbers, every Cauchy sequence ___________________.



Importance of Cauchy sequences in defining convergence and completeness

Pilot Questions 3.1

A sequence in which the terms get arbitrarily close to each other as the sequence progresses is called a: A. Divergent sequence B. Cauchy sequence C. Monotone sequence D. Supremum sequence (Linked to SLO 3.1)

The sequence an=1nis: A. Divergent B. Cauchy C. Increasing D. Supremum (Linked to SLO 3.1)

The sequence an=nis not Cauchy because its terms grow: A. Closer together B. Farther apart C. Bounded D. Supremum (Linked to SLO 3.1)

The sequence of decimal approximations of (3, 3.1, 3.14, …) is: A. Divergent B. Cauchy C. Decreasing D. Supremum (Linked to SLO 3.1)

In the real numbers, every Cauchy sequence: A. Diverges B. Converges C. Oscillates D. Shrinks to zero length (Linked to SLO 3.1)

3.3 Convergence of Series

3.3.1 Definition of series and partial sums

A **series** is the sum of the terms of a sequence. If anis a sequence, then the series is written as n=1an. To study convergence, we look at the **partial sums**:

Sk=n=1kan

If the sequence of partial sums Skconverges to a finite limit, then the series is said to converge. Otherwise, it diverges.

Fill in the blank: A series converges if the sequence of __________________ converges to a finite limit.

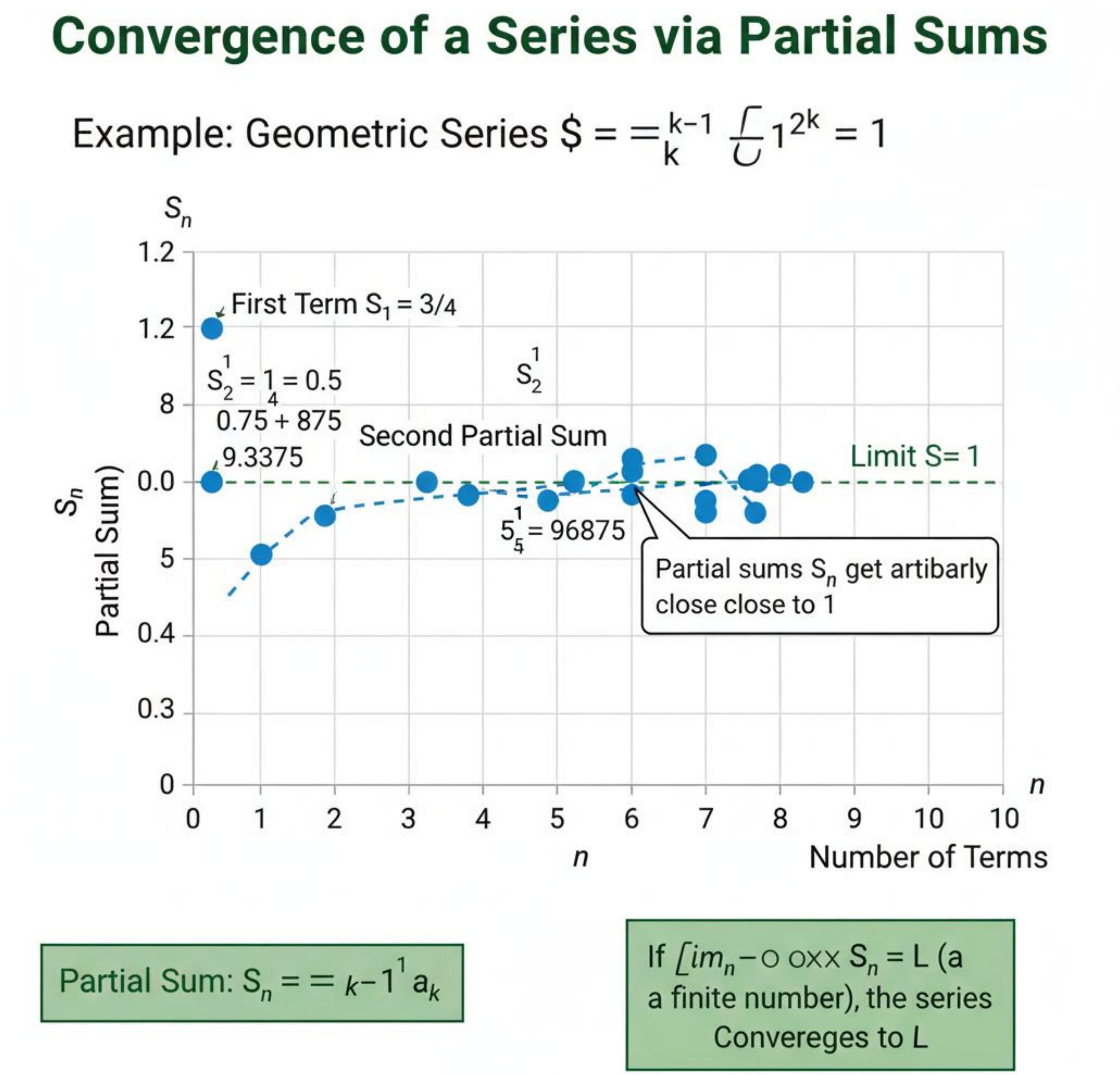


Illustration of partial sums approaching a finite limit

3.3.2 Convergent and divergent series

Convergent series: The geometric series n=012nconverges to 2 because its partial sums approach a finite limit.

Divergent series: The harmonic series n=11ndiverges because its partial sums grow without bound.

Fill in the blank: The harmonic series n=11nis _________________ because its partial sums grow without bound.

Examples of convergent and divergent series

Series Type	Formula	Convergence Status	Explanation
Geometric Series	$\sum_{n=0}^{\infty}(\frac{1}{2})^n$	**Convergent**	Common ratio $
Harmonic Series	$\sum_{n=1}^{\infty}\frac{1}{n}$	**Divergent**	Although terms $\to 0$, the sum grows infinitely (very slowly).
Alternating Series	$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$	**Convergent**	Terms decrease in magnitude and $\to 0$. Sums to $\ln(2)$.
P-Series ($p > 1$)	$\sum_{n=1}^{\infty}\frac{1}{n^2}$	**Convergent**	Known as the Basel Problem; sums to $\frac{\pi^2}{6}$.
P-Series ($p \leq 1$)	$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$	**Divergent**	Since $p = 0.5$, the terms do not decrease fast enough.
Telescoping Series	$\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$	**Convergent**	Middle terms cancel out; sums to 1.

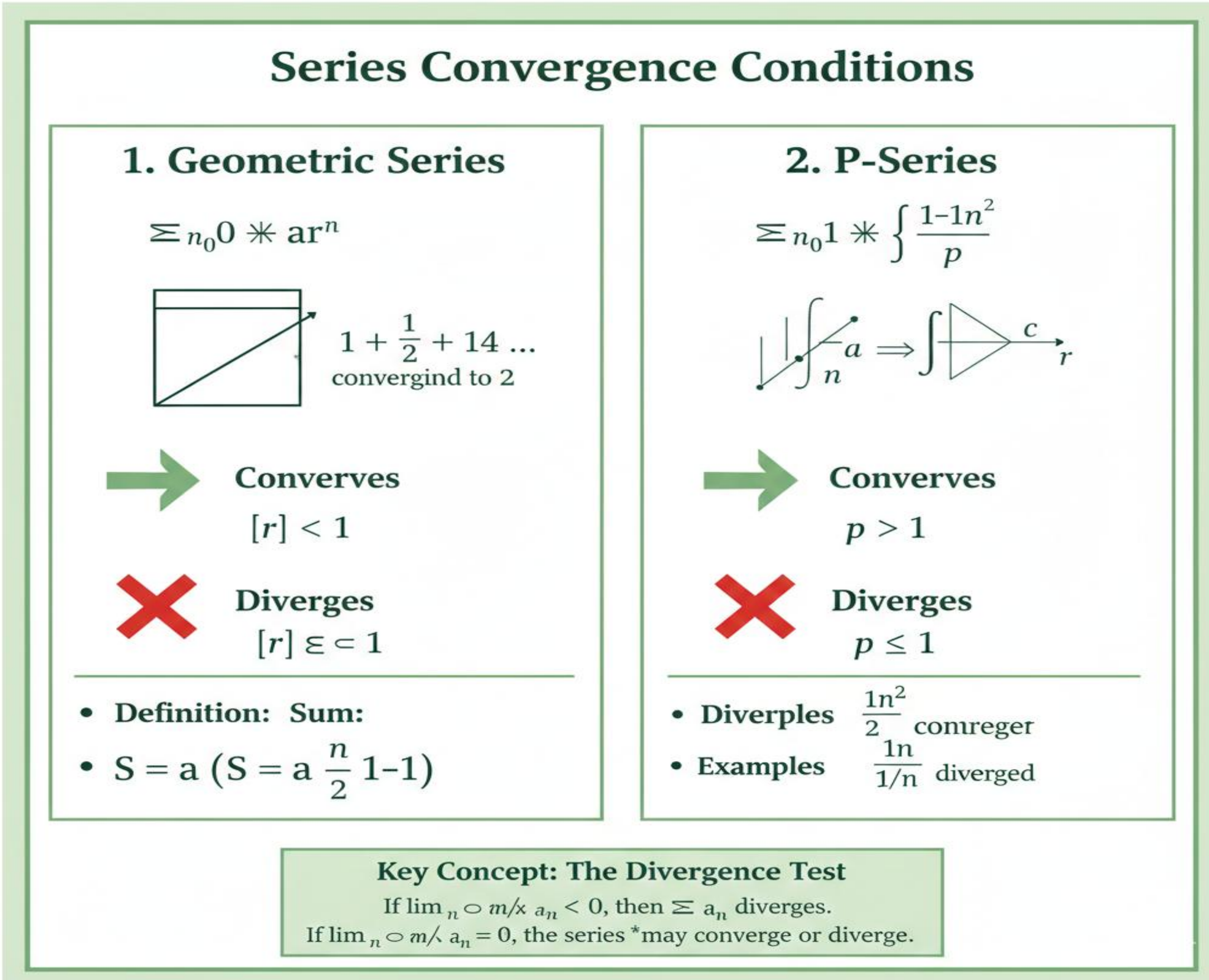
3.3.3 Examples of convergent series

Geometric series: n=0rnconverges if |r|<1, with sum 11-r.

p-series: n=11npconverges if p>1and diverges if p≤1. These examples are fundamental in analysis and often serve as benchmarks for testing convergence.

Fill in the blank: The geometric series n=0rnconverges if |r| < _________________.

Examples of geometric and p-series convergence



Pilot Questions 3.3

A series converges if the sequence of: A. Terms B. Partial sums C. Supremum values D. Divergent sequences (Linked to SLO 3.3)

The geometric series n=012nconverges to: A. 1 B. 2 C. Infinity D. 0 (Linked to SLO 3.3)

The harmonic series n=11nis: A. Convergent B. Divergent C. Supremum D. Oscillating (Linked to SLO 3.3)

The geometric series n=0rnconverges if: A. |r|>1B. |r|<1C. r=1D. r=0(Linked to SLO 3.3)

The p-series n=11npconverges if: A. p>1B. p<1C. p=1D. p=0(Linked to SLO 3.3)

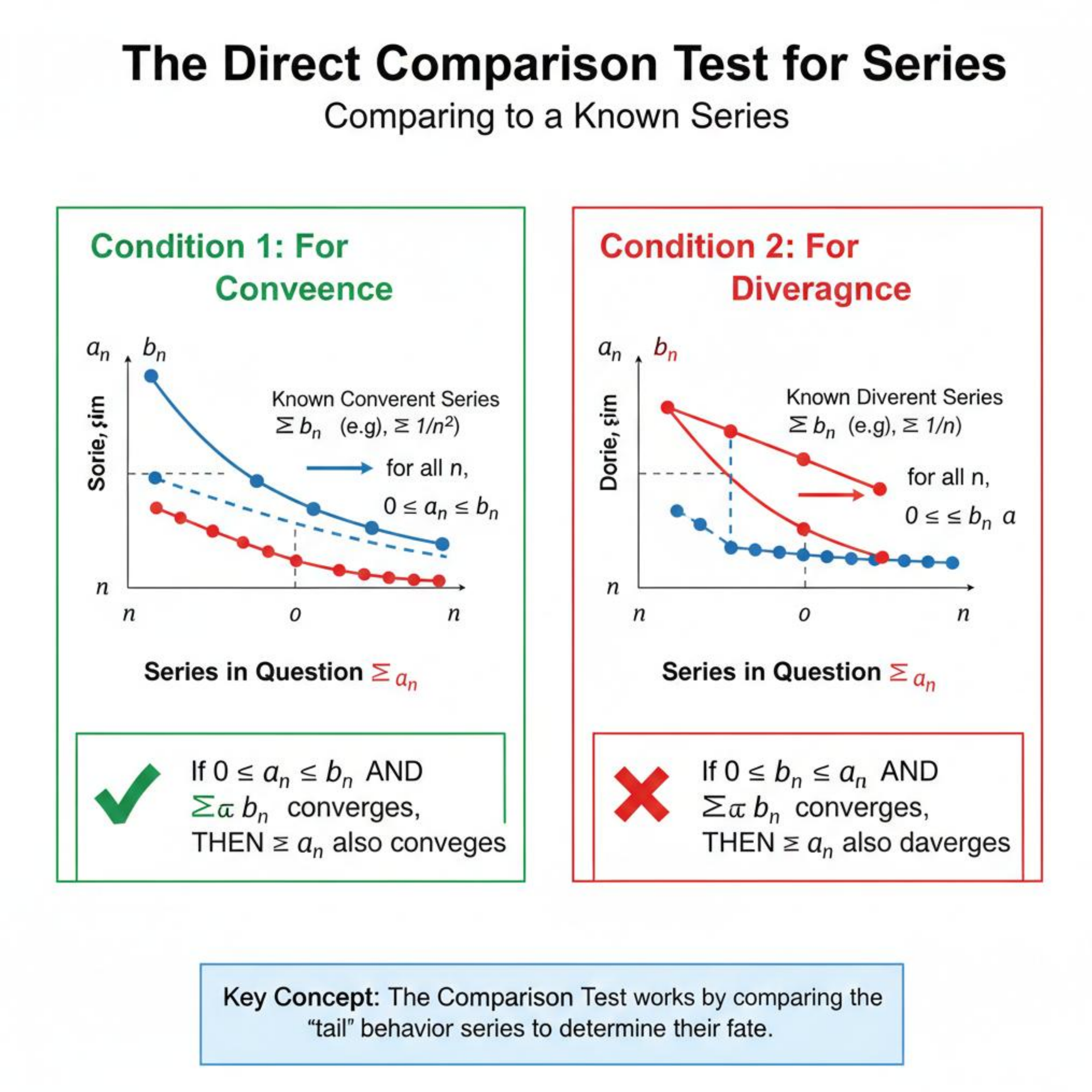
3.4 Tests for Convergence of Series

3.4.1 Comparison test

The **comparison test** is used to determine the convergence of a series by comparing it with another series whose convergence is already known. If 0≤anbnfor all n, and ∑bnconverges, then ∑analso converges. Conversely, if ∑andiverges and anbn≥0, then ∑bnalso diverges.

Fill in the blank: The comparison test determines convergence by comparing a series with another series whose ___________________ is already known.

Illustration of the comparison test with two series



3.4.2 Ratio test

The **ratio test** involves examining the ratio of successive terms. For a series $\sum a_n$, compute

$$L=\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$$

If $L<1$, the series converges.

If $L>1$, the series diverges.

If $L=1$, the test is inconclusive.

Fill in the blank: In the ratio test, if L<1, the series ________________.

Statement of the ratio test with conditions for convergence and divergence

The Method

To apply the test, calculate the limit of the absolute ratio of consecutive terms:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

The Conditions and Conclusions

Result of Limit (L)	Conclusion	Implications
$L < 1$	**Converges Absolutely**	The series $\sum a_n$ converges. The terms decrease fast enough (faster than a geometric series).
$L > 1$ (or $L = \infty$)	**Diverges**	The series $\sum a_n$ diverges. The terms do not approach zero; they grow or stay large.
$L = 1$	**Inconclusive**	The test fails. The series may converge or diverge (e.g., it fails for all p-series).

3.4.3 Root test

The **root test** examines the nth root of the terms of a series. For a series ∑an, compute

$$L=\lim n \to n|an|$$

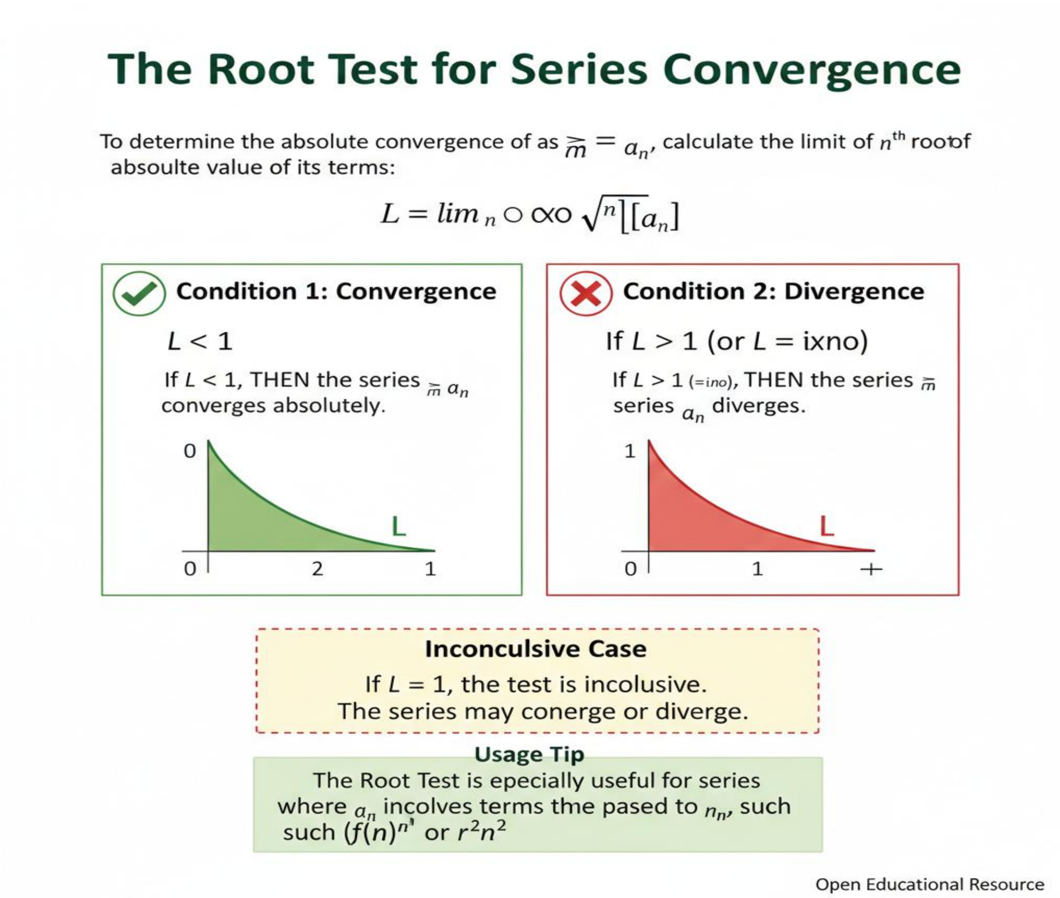
If L<1, the series converges.

If L>1, the series diverges.

If L=1, the test is inconclusive.

Fill in the blank: In the root test, if L>1, the series _________________.

Illustration of the root test applied to a series



Pilot Questions 3.4

The comparison test determines convergence by comparing a series with another series whose: A. Divergence is unknown B. Convergence is already known C. Supremum is infinite D. Oscillation is bounded (Linked to SLO 3.4)

In the ratio test, if L<1, the series: A. Diverges B. Converges C. Oscillates D. Is inconclusive (Linked to SLO 3.4)

In the ratio test, if L=1, the test is: A. Conclusive B. Inconclusive C. Divergent D. Supremum (Linked to SLO 3.4)

In the root test, if L>1, the series: A. Converges B. Diverges C. Oscillates D. Is inconclusive (Linked to SLO 3.4)

Which of the following tests involves examining the nth root of the terms of a series? A. Comparison test B. Ratio test C. Root test D. Supremum test (Linked to SLO 3.4)

Series 3 Summary

This Series has introduced you to the concepts of **Cauchy sequences and convergence of series**, which are central to Real Analysis. We began by defining Cauchy sequences, providing examples, and explaining their importance in analysis. Next, we examined the completeness of the real numbers, showing how every Cauchy sequence converges in R, while incomplete spaces like Qfail this property. We then studied the convergence of series, including definitions, partial sums, and examples of convergent and divergent series. Finally, we explored standard tests for convergence of series, such as the comparison test, ratio test, and root test, which provide practical tools for determining convergence.

By completing this Series, you should now be able to define Cauchy sequences, explain completeness, analyse convergence of series, and apply convergence tests. These abilities directly reflect the Series Learning Outcomes (SLOs) and prepare you for advanced topics in Real Analysis.

Glossary of Terms

Cauchy sequence: A sequence whose terms get arbitrarily close to each other as the sequence progresses.

Completeness: The property of a space where every Cauchy sequence converges to a point within the space.

Partial sum: The sum of the first kterms of a series.

Series: The sum of the terms of a sequence, often written as ∑an.

Convergent series: A series whose partial sums approach a finite limit.

Divergent series: A series whose partial sums do not approach a finite limit.

Comparison test: A test for convergence by comparing with another known series.

Ratio test: A test for convergence using the ratio of successive terms.

Root test: A test for convergence using the nth root of terms.

Concept Check Questions 3

Objective questions

A sequence in which the terms get arbitrarily close to each other is called a: A. Divergent sequence B. Cauchy sequence C. Monotone sequence D. Supremum sequence (Linked to SLO 3.1)

The completeness property of the real numbers states that every: A. Divergent sequence converges B. Cauchy sequence converges to a real number C. Rational sequence converges to a rational number D. Supremum sequence converges (Linked to SLO 3.2)

Short-answer questions

Define a partial sum and explain its role in determining convergence of a series. (Linked to SLO 3.3)

State the ratio test and explain its conditions for convergence. (Linked to SLO 3.4)

Long-answer questions

Discuss the importance of Cauchy sequences in analysis. (Linked to SLO 3.1)

Basic response: Explain how Cauchy sequences describe closeness of terms.

Value-added response: Extend to their role in defining completeness and convergence.

Analyse the convergence of the geometric series n=0rn. (Linked to SLO 3.3)

Basic response: State the condition |r|<1and the sum formula.

Value-added response: Provide examples with specific values of rand explain divergence when |r|≥1.

Answers to Concept Check Questions 3

B. Cauchy sequence

B. Cauchy sequence converges to a real number

A partial sum is the sum of the first kterms of a series. The sequence of partial sums determines whether the series converges to a finite limit.

The ratio test states that for a series ∑an, if L=lim⁡n→|an+1||an|, then the series converges if L<1, diverges if L>1, and is inconclusive if L=1.

Basic response: Cauchy sequences describe how terms get closer together as the sequence progresses.

Value-added response: They are fundamental in analysis because they characterise convergence and define completeness of the real numbers.

Basic response: The geometric series converges if |r|<1, with sum 11-r.

Value-added response: For example, if r=12, the series converges to 2. If r=2, the series diverges because terms grow without bound.

Answers to Pilot Questions 3

Part 3.1

Cauchy sequence

Cauchy

Farther apart

Cauchy

Converges

Part 3.2

Convergent

Cauchy sequence converges to a real number

Convergent in Rbut not in Q

Converge within Q

Every Cauchy sequence in Rconverges

Part 3.3

Partial sums

2

Divergent

|r|<1

$p>1$

Part 3.4

Convergence is already known

Converges

Inconclusive

Diverges

Root test

Series 4: Limits Superior and Inferior, and Subsequences

Series Outline

Part 4.1 Subsequences

4.1.1 Definition of subsequences

4.1.2 Examples of subsequences

4.1.3 Importance of subsequences in analysis

Part 4.2 Limit Superior and Limit Inferior

4.2.1 Definition of limit superior (lim sup)

4.2.2 Definition of limit inferior (lim inf)

4.2.3 Examples illustrating lim sup and lim inf

Part 4.3 Properties of Limit Superior and Inferior

4.3.1 Relationship with subsequences

4.3.2 Bounds and convergence behaviour

4.3.3 Applications in analysis

Part 4.4 Applications of Subsequences and Limits Superior/Inferior

4.4.1 Role in oscillating sequences

4.4.2 Use in convergence proofs

4.4.3 Applications in advanced analysis

Introduction

In Real Analysis, sequences often exhibit complex behaviours that cannot be fully captured by simple convergence. To study these behaviours, mathematicians use subsequences and the concepts of limit superior ($\limsup$) and limit inferior ($\liminf$). Subsequences allow us to focus on selected terms of a sequence, while $\limsup$ and $\liminf$ provide a way to describe the limiting behaviour of sequences that oscillate or fail to converge in the usual sense.

The aim of this Series is to help you understand subsequences, define and compute limit superior and limit inferior, explore their properties, and apply these concepts to analyse sequences that do not converge straightforwardly.

We shall commence this Series by defining subsequences, illustrating with examples, and explaining their importance in analysis. Next, we will introduce limit superior and limit inferior, providing definitions and examples. Subsequently, we will study the properties of $\limsup$ and $\liminf$, including their relationship with subsequences and their role in bounding sequence behaviour. Finally, we will conclude with applications of subsequences and limits superior/inferior, particularly in oscillating sequences, convergence proofs, and advanced analysis.

Series Learning Outcomes

By the end of Series 4: Limits Superior and Inferior, and Subsequences, you should be able to:

SLO 4.1 define subsequences, provide examples, and explain their importance in analysis.

SLO 4.2 define limit superior ($\limsup$) and limit inferior ($\liminf$), and compute them for given sequences.

SLO 4.3 explain the properties of $\limsup$ and $\liminf$, including their relationship with subsequences and bounds.

SLO 4.4 apply subsequences and limits superior/inferior to analyse oscillating sequences, prove convergence results, and explore applications in advanced analysis.

4.1 Subsequences

4.1.1 Definition of subsequences

A subsequence is formed by selecting certain terms of a sequence while preserving their original order. Formally, if anis a sequence, then a subsequence is ank, where n1<n2<n3<⋯. This means we "thin out" the sequence but keep the order intact.

Fill in the blank: A subsequence is formed by selecting certain terms of a sequence while preserving their __________________.

Illustration of a sequence and one of its subsequences

4.1.2 Examples of subsequences

From the sequence an=n, the subsequence a2n=2nconsists of even terms.

From the sequence an=(-1)n, the subsequence a2n=1consists of only the positive terms.

From the sequence an=1n, the subsequence an2=1n2converges faster to 0.

Fill in the blank: From the sequence an=(-1)n, the subsequence a2nconsists of only the __________________ terms.

: Examples of subsequences drawn from different sequences

Parent Sequence (a_n)	Type	Example Subsequence (a_{n_k})	Selection Rule (n_k)
Arithmetic $2, 4, 6, 8, 10, \dots$	$a_n = 2n$	$4, 8, 12, 16, \dots$	$n_k = 2k$ (Even terms)
Alternating $1, -1, 1, -1, \dots$	$a_n = (-1)^{n+1}$	$-1, -1, -1, -1, \dots$	$n_k = 2k$ (Even terms)
Reciprocal $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$	$a_n = \frac{1}{n}$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$	$n_k = 2^k$ (Powers of 2)
Fibonacci-like $1, 2, 3, 5, 8, \dots$	f_n	$1, 3, 8, 21, \dots$	$n_k = 2k - 1$ (Odd indices)

4.1.3 Importance of subsequences in analysis

Subsequences are important because they help us study the behaviour of sequences, especially when the original sequence does not converge. For example, an oscillating sequence may have subsequences that converge to different limits. Subsequences also play a key role in defining limit superior and limit inferior, and in proving convergence results.

Fill in the blank: Subsequences are useful in studying sequences that ____________________ to converge.

Importance of subsequences in analysing oscillating and non-convergent sequences

Pilot Questions 4.1

A subsequence is formed by selecting certain terms of a sequence while preserving their: A. Supremum B. Order C. Divergence D. Oscillation (Linked to SLO 4.1)

From the sequence $a_n=n$, the subsequence a_{2n} consists of: A. Odd terms B. Even terms C. Supremum values D. Oscillating terms (Linked to SLO 4.1)

From the sequence $a_n=(-1)^n$, the subsequence a_{2n} consists of only the: A. Negative terms B. Positive terms C. Divergent terms D. Supremum terms (Linked to SLO 4.1)

From the sequence $a_n=\frac{1}{n}$, the subsequence a_{n^2} converges: A. Slower to 0 B. Faster to 0 C. Diverges D. Oscillates (Linked to SLO 4.1)

Subsequences are useful in studying sequences that: A. Always converge B. Fail to converge C. Supremum only D. Oscillate without bounds (Linked to SLO 4.1)

4.2 Limit Superior and Limit Inferior

4.2.1 Definition of limit superior ($\limsup$)

The limit superior of a sequence a_n, denoted a_n , is the largest limit point of the sequence. Equivalently, it is the limit of the sequence of suprema of the tails:

$$\limsup a_n=\lim_{n\to\infty}\sup\{a_k:k\geq n\}.$$

It captures the "upper bound" behaviour of the sequence as n grows.

Fill in the blank: The limit superior of a sequence is the largest ____________________ point of the sequence.

Illustration of a sequence oscillating with its limit superior marked

4.2.2 Definition of limit inferior ($\liminf$)

The limit inferior of a sequence a_n, denoted a_n , is the smallest limit point of the sequence. Equivalently, it is the limit of the sequence of infima of the tails:

$$\liminf a_n=\lim_{n\to\infty}\inf\{a_k:k\geq n\}.$$

It captures the "lower bound" behaviour of the sequence as n grows.

Fill in the blank: The limit inferior of a sequence is the smallest ___________________ point of the sequence.

Definitions of limit superior and limit inferior

Formal Definitions

For a bounded sequence (a_n), we define two new sequences based on the "tails" of the original sequence:

1. $b_k = \sup\{a_n : n \geq k\}$ (the supremum of the remaining terms).
2. $c_k = \inf\{a_n : n \geq k\}$ (the infimum of the remaining terms).

Concept	Notation	Formula	Intuition
Limit Superior	$\limsup_{n\to\infty} a_n$	$\lim_{k\to\infty}(\sup_{n\geq k} a_n)$	The largest accumulation point; the highest value the sequence approaches infinitely often.
Limit Inferior	$\liminf_{n\to\infty} a_n$	$\lim_{k\to\infty}(\inf_{n\geq k} a_n)$	The smallest accumulation point; the lowest value the sequence approaches infinitely often.

4.2.3 Examples illustrating lim sup and lim inf

For the sequence an=(-1)n, the subsequence of even terms converges to 1, and the subsequence of odd terms converges to -1. Thus, lim sup an=1and lim inf an=-1.

For the sequence an=1n, both an and an equal 0, since the sequence converges to 0.

For the sequence an=sin(n), the values oscillate between -1 and 1, so lim sup an=1and lim inf an=-1.

Fill in the blank: For the sequence an=(-1)n, the limit superior is 1 and the limit inferior is ___________________.

Examples of sequences with different lim sup and lim inf values

Pilot Questions 4.2

The limit superior of a sequence is the: A. Smallest limit point B. Largest limit point C. Supremum of the sequence itself D. Infimum of the sequence itself (Linked to SLO 4.2)

The limit inferior of a sequence is the: A. Largest limit point B. Smallest limit point C. Supremum of the sequence itself D. Divergent value (Linked to SLO 4.2)

For the sequence an=(-1)n, the limit superior is: A. -1 B. 0 C. 1 D. Infinity (Linked to SLO 4.2)

For the sequence an=1n, both an and an equal: A. 1 B. 0 C. Infinity D. -1 (Linked to SLO 4.2)

For the sequence an=sin(n), the limit superior and limit inferior are: A. 1 and -1 B. 0 and 0 C. Infinity and -Infinity D. Supremum and Infimum (Linked to SLO 4.2)

4.3 Properties of Limit Superior and Inferior

4.3.1 Relationship with subsequences

The limit superior and limit inferior are closely tied to subsequences.

The limit superior ($\limsup$) is the largest subsequential limit of a sequence.

The limit inferior ($\liminf$) is the smallest subsequential limit of a sequence. This means that every subsequence has limits bounded between $\liminf$ and $\limsup$.

Fill in the blank: The limit superior of a sequence is the largest ____________________ limit of the sequence.

Illustration showing subsequences converging to different limits bounded by $\limsup$ and $\liminf$

4.3.2 Bounds and convergence behaviour

If a sequence converges, then $\limsup = \liminf =$ limit of the sequence.

- If a sequence oscillates, then lim sup and lim inf describe the bounds of oscillation.
- Always, lim inf ≤ lim sup.

Fill in the blank: If a sequence converges, then lim sup and lim inf are __________________ to the limit of the sequence.

Properties of bounds and convergence behaviour of lim sup and lim inf

Core Mathematical Properties

Property	Mathematical Expression	Significance
Ordering	$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$	The "eventual floor" is never above the "eventual ceiling."
Convergence	$\liminf a_n = \limsup a_n = L$	This is a necessary and sufficient condition for $\lim_{n\to\infty} a_n = L$.
Subsequential Limits	$S = \{\text{limits of all subsequences}\}$	$\limsup a_n = \max(S)$ and $\liminf a_n = \min(S)$.
Boundedness	$m \leq a_n \leq M \implies m \leq \liminf a_n \leq \limsup a_n \leq M$	If a sequence is bounded, its $\limsup$ and $\liminf$ are finite real numbers.

4.3.3 Applications in analysis

Limit superior and limit inferior are powerful tools in analysis:

- They help describe the behaviour of sequences that do not converge.
- They are used in defining convergence in probability and measure theory.
- They provide bounds for subsequential limits, ensuring rigorous treatment of oscillating sequences.

Fill in the blank: Limit superior and limit inferior are useful in describing the behaviour of sequences that do not __________________.

Applications of limsup and liminf in oscillating sequences and advanced analysis

Pilot Questions 4.3

The limit superior of a sequence is the: A. Smallest subsequential limit B. Largest subsequential limit C. Supremum of the sequence itself D. Infimum of the sequence itself (Linked to SLO 4.3)

The limit inferior of a sequence is the: A. Largest subsequential limit B. Smallest subsequential limit C. Supremum of the sequence itself D. Divergent value (Linked to SLO 4.3)

If a sequence converges, then $\limsup$ and $\liminf$ are: A. Equal to the limit of the sequence B. Different from the limit of the sequence C. Infinity and -Infinity D. Supremum and Infimum (Linked to SLO 4.3)

For an oscillating sequence, $\limsup$ and $\liminf$ describe: A. The bounds of oscillation B. The exact limit C. Supremum only D. Divergence without bounds (Linked to SLO 4.3)

Limit superior and limit inferior are useful in describing the behaviour of sequences that: A. Always converge B. Fail to converge C. Supremum only D. Oscillate without bounds (Linked to SLO 4.3)

4.4 Applications of Subsequences and Limits Superior/Inferior

4.4.1 Role in oscillating sequences

Subsequences and limits superior/inferior are especially useful in analysing oscillating sequences. For example, the sequence $a_n=(-1)^n$ does not converge, but its subsequences converge to different values (1 and -1). Here, $\limsup a_n=1$ and $\liminf a_n=-1$, which describe the bounds of oscillation.

Fill in the blank: For the sequence $a_n=(-1)^n$, subsequences converge to different values, so the sequence does not __________________.

Oscillating sequence with subsequences converging to different limits ,tf

4.4.2 Use in convergence proofs

Subsequences are often used in proofs of convergence. For example, if every subsequence of a sequence converges to the same limit, then the entire sequence converges to that limit. Similarly, $\limsup$ and $\liminf$ provide bounds that can be used to show convergence when they coincide.

Fill in the blank: If every subsequence of a sequence converges to the same limit, then the entire sequence __________________ to that limit.

Role of subsequences and limits in convergence proofs

Summary of Proof Utilities

Tool	Primary Use in Proofs	Related Theorem
Subsequences	Finding "candidates" for a limit in bounded sets.	Bolzano-Weierstrass
$\limsup / \liminf$	Proving a limit exists without knowing its value.	Monotone Convergence Theorem
Cauchy Sequences	Proving convergence without referencing the limit L.	Completeness Axiom

4.4.3 Applications in advanced analysis

In advanced analysis, subsequences and limits superior/inferior are used in:

Measure theory: defining convergence almost everywhere.

Probability theory: describing convergence in distribution.

Functional analysis: studying bounded sequences in Banach and Hilbert spaces.

Fill in the blank: In probability theory, limit superior and limit inferior are used to describe convergence in __________________.

: Applications of subsequences and limits superior/inferior in advanced analysis

Pilot Questions 4.4

- For the sequence an=(-1)n, subsequences converge to different values, so the sequence: A. Converges B. Does not converge C. Supremum only D. Oscillates without bounds (Linked to SLO 4.4)
- If every subsequence of a sequence converges to the same limit, then the entire sequence: A. Diverges B. Converges to that limit C. Oscillates D. Supremum only (Linked to SLO 4.4)
- $\limsup$ and $\liminf$ can be used to show convergence when they: A. Are equal B. Are different C. Are infinite D. Oscillate (Linked to SLO 4.4)
- In probability theory, limit superior and limit inferior are used to describe convergence in: A. Distribution B. Supremum C. Oscillation D. Divergence (Linked to SLO 4.4)
- Which of the following is an application of subsequences and limits superior/inferior? A. Studying convergence in measure theory B. Defining divergence of bounded sequences C. Explaining supremum of sets D. Identifying monotone sequences (Linked to SLO 4.4)

Series 4 Summary

This Series has introduced you to subsequences, limit superior ($\limsup$), and limit inferior ($\liminf$), which extend the study of sequences beyond simple convergence. We began by defining subsequences, illustrating with examples, and explaining their importance in analysis. Next, we introduced $\limsup$ and $\liminf$, providing definitions and examples. We then studied their properties, including relationships with subsequences, bounds, and convergence behaviour. Finally, we explored applications in oscillating sequences, convergence proofs, and advanced areas such as measure theory, probability, and functional analysis.

By completing this Series, you should now be able to define subsequences, compute $\limsup$ and $\liminf$, explain their properties, and apply them in analysis. These abilities directly reflect the Series Learning Outcomes (SLOs) and prepare you for deeper results in Real Analysis.

Glossary of Terms

- Subsequence: A sequence formed by selecting certain terms of another sequence while preserving their order.

- Limit superior ($\limsup$): The largest subsequential limit of a sequence, or the limit of the suprema of its tails.
- Limit inferior ($\liminf$): The smallest subsequential limit of a sequence, or the limit of the infima of its tails.
- Oscillating sequence: A sequence that does not converge but fluctuates between values.
- Subsequential limit: A limit of a subsequence of a given sequence.

Concept Check Questions 4

Objective questions

1. A subsequence is formed by selecting certain terms of a sequence while preserving their: A. Supremum B. Order C. Divergence D. Oscillation (Linked to SLO 4.1)
2. The limit superior of a sequence is the: A. Smallest limit point B. Largest limit point C. Supremum of the sequence itself D. Infimum of the sequence itself (Linked to SLO 4.2)

Short-answer questions

Define limit inferior and give an example. (Linked to SLO 4.2)

Explain the relationship between subsequences and $\limsup$/$\liminf$. (Linked to SLO 4.3)

Long-answer questions

Discuss the properties of $\limsup$ and $\liminf$. (Linked to SLO 4.3)

- Basic response: State definitions and inequalities.
- Value-added response: Explain their role in convergence and oscillation.

Analyse how subsequences and $\limsup$/$\liminf$ are applied in oscillating sequences and advanced analysis. (Linked to SLO 4.4)

- Basic response: Describe their role in bounding oscillations.
- Value-added response: Extend to applications in measure theory and probability.

Answers to Concept Check Questions 4

1. B. Order

B. Largest limit point

Limit inferior is the smallest subsequential limit of a sequence. Example: For $a_n=(-1)^n$, $\liminf a_n=-1$.

Subsequences converge to limits bounded between $\liminf$ and $\limsup$. These values describe the range of possible subsequential limits.

Basic response: $\limsup$ is the largest subsequential limit, $\liminf$ is the smallest, and always $\liminf \le \limsup$.

Value-added response: If a sequence converges, then $\limsup = \liminf$. For oscillating sequences, they describe bounds of oscillation.

Basic response: In oscillating sequences, subsequences converge to different limits, and $\limsup$/$\liminf$ capture the bounds.

Value-added response: In advanced analysis, they are used in measure theory (convergence almost everywhere) and probability (convergence in distribution).

Answers to Pilot Questions 4

Part 4.1

Order

Even terms

Positive terms

Faster to 0

Fail to converge

Part 4.2

Largest limit point

Smallest limit point

1

0

1 and -1

Part 4.3

Largest subsequential limit

Smallest subsequential limit

Equal to the limit of the sequence

Bounds of oscillation

Fail to converge

Part 4.4

Does not converge

Converges to that limit

Are equal

Distribution

Studying convergence in measure theory

Series 5: Infinite Series and Power Series

Series Outline

Part 5.1 Infinite Series

5.1.1 Definition of infinite series

5.1.2 Convergence and divergence of infinite series

5.1.3 Examples of infinite series

Part 5.2 Power Series

5.2.1 Definition of power series

5.2.2 Radius and interval of convergence

5.2.3 Examples of power series

Part 5.3 Properties of Power Series

Introduction

Infinite series and power series are central tools in Real Analysis and beyond. An infinite series is the sum of infinitely many terms of a sequence, and its convergence or divergence reveals deep properties of functions and numbers. A power series is a special type of infinite series where terms involve powers of a variable, often used to represent functions and approximate them with remarkable accuracy.

The aim of this Series is to help you understand infinite series and power series, explore their convergence behaviour, study their properties, and apply them in mathematics, physics, and engineering.

We shall commence this Series by defining infinite series, explaining convergence and divergence, and providing examples. Next, we will introduce power series, including their definition, radius of convergence, and interval of convergence. Subsequently, we will study the properties of power series, such as differentiation, integration, and representation of functions. Finally, we will conclude with applications of infinite and power series, including approximations of functions, solving differential equations, and applications in physics and engineering.

Series Learning Outcomes

By the end of Series 5: Infinite Series and Power Series, you should be able to:

SLO 5.1 define infinite series, explain convergence and divergence, and provide examples.

SLO 5.2 define power series, determine their radius and interval of convergence, and illustrate with examples.

SLO 5.3 explain the properties of power series, including differentiation, integration, and representation of functions.

SLO 5.4 apply infinite and power series to approximate functions, solve differential equations, and explore applications in physics and engineering.

5.1 Infinite Series

5.1.1 Definition of infinite series

An infinite series is the sum of infinitely many terms of a sequence. If anis a sequence, the infinite series is written as:

$$n=1an$$

To study whether this sum makes sense, we examine the partial sums:

$$Sk=n=1kan$$

If the sequence of partial sums Skconverges to a finite limit, the series converges; otherwise, it diverges.

Fill in the blank: An infinite series converges if the sequence of ___________________ converges to a finite limit.

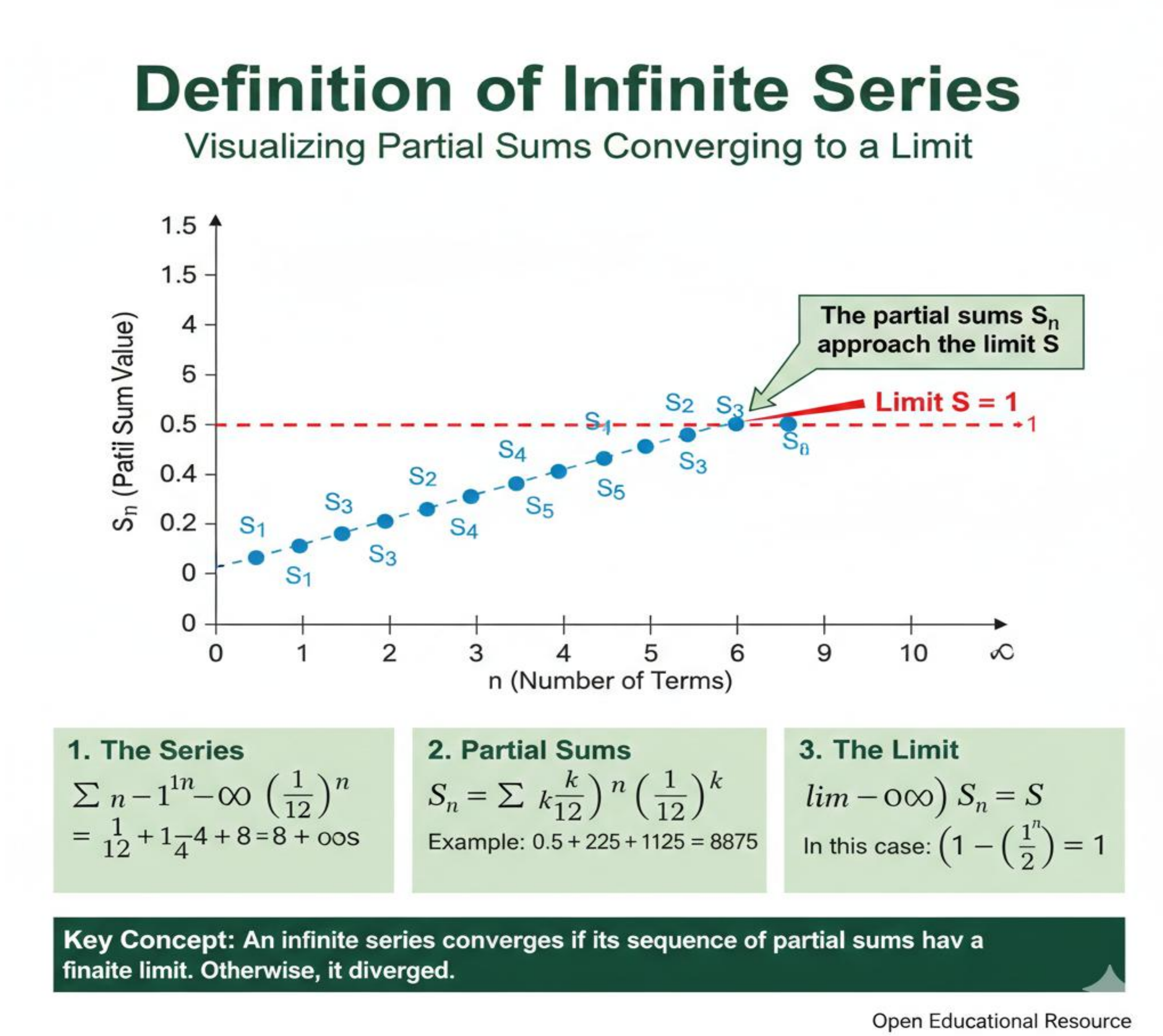


Illustration of partial sums approaching a finite limit in an infinite series

5.1.2 Convergence and divergence of infinite series

Convergent series: The geometric series n=012nconverges to 2.

Divergent series: The harmonic series n=11ndiverges because its partial sums grow without bound.

Condition: For convergence, the terms anmust approach zero, but this alone is not sufficient (e.g., harmonic series).

Fill in the blank: The harmonic series n=11nis __________________ because its partial sums grow without bound.

Examples of convergent and divergent infinite series

Convergent Series

These series approach a specific finite value as more terms are added.

Series Type	Example	Behavior
Geometric ($ | r | < 1$)		
p-series ($p > 1$)	$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$	Converges (specifically to $\frac{\pi^2}{6}$).
Alternating	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$	Converges to $\ln(2)$ via the Alternating Series Test.
Telescoping	$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots$	Terms cancel out, converging to 1.

Divergent Series

These series do not settle at a finite sum; they often grow toward infinity or oscillate.

Series Type	Example	Reason for Divergence
Harmonic	$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$	Grows infinitely slow, but never stops (p-series with $p = 1$).
Geometric ($ | r | \ge 1$)		
Divergence Test	$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$	The limit of the terms is 1, not 0.
Oscillating	$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 \dots$	Partial sums jump between -1 and 0.

5.1.3 Examples of infinite series

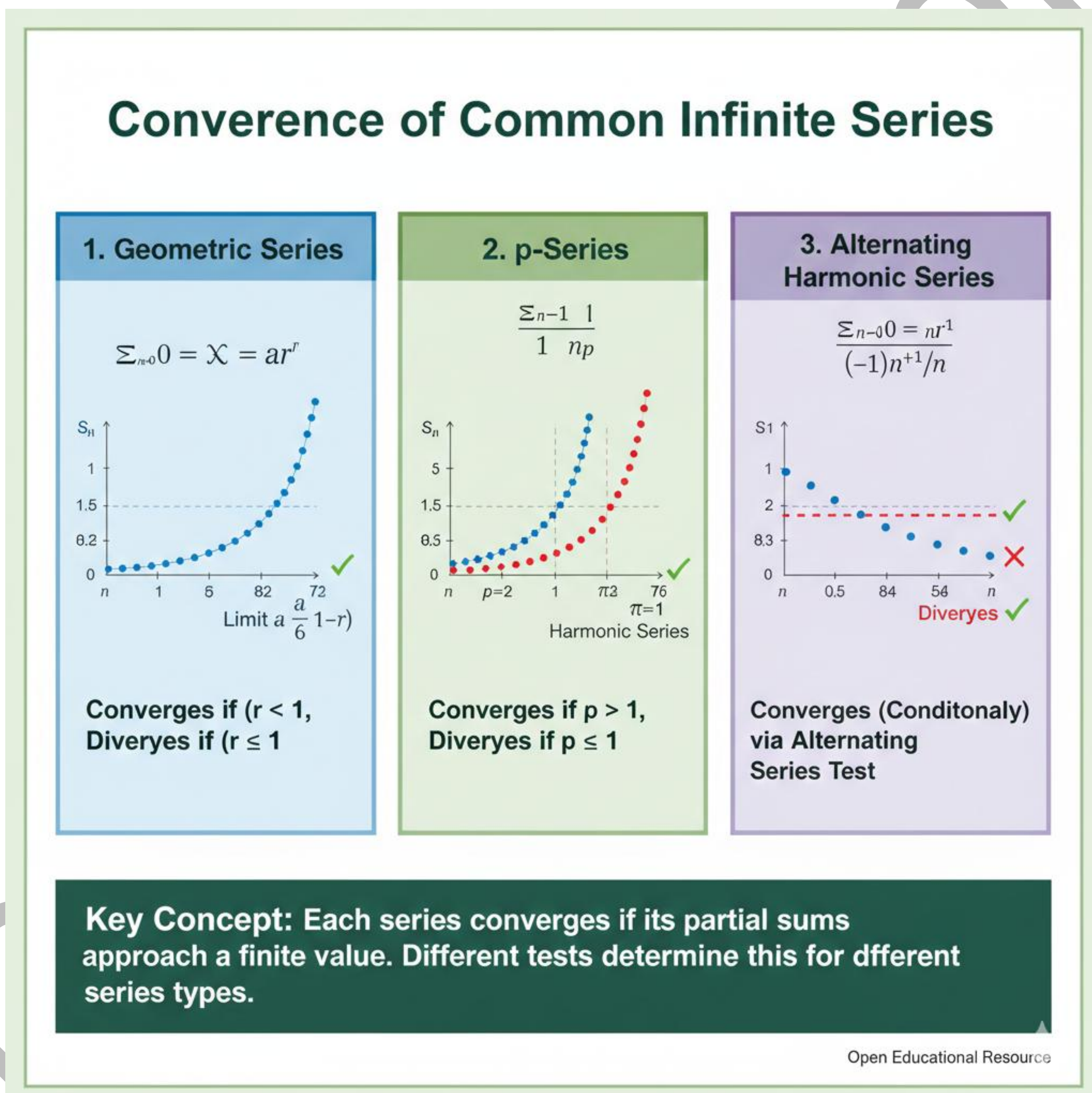
Geometric series: n=0rnconverges if |r|<1, with sum 11-r.

p-series: n=11npconverges if p>1and diverges if p≤1.

Alternating harmonic series: n=1(1)n+1nconverges to ln(2).

Fill in the blank: The geometric series n=0rnconverges if |r| < ____________________.

Examples of geometric, p-series, and alternating harmonic series



Pilot Questions 5.1

An infinite series converges if the sequence of: A. Terms B. Partial sums C. Supremum values D. Divergent sequences (Linked to SLO 5.1)

The geometric series n=012nconverges to: A. 1 B. 2 C. Infinity D. 0 (Linked to SLO 5.1)

The harmonic series n=11nis: A. Convergent B. Divergent C. Supremum D. Oscillating (Linked to SLO 5.1)

The geometric series n=0rnconverges if: A. |r|>1B. |r|<1C. r=1D. r=0(Linked to SLO 5.1)

The alternating harmonic series n=1(1)n+1nconverges to: A. 0 B. 1 C. ln(2)D. Infinity (Linked to SLO 5.1)

5.2 Power Series

5.2.1 Definition of power series

A power series is an infinite series of the form:

n=0an(x-c)n

where anare coefficients, cis the center of the series, and xis the variable. Power series are used to represent functions as infinite polynomials.

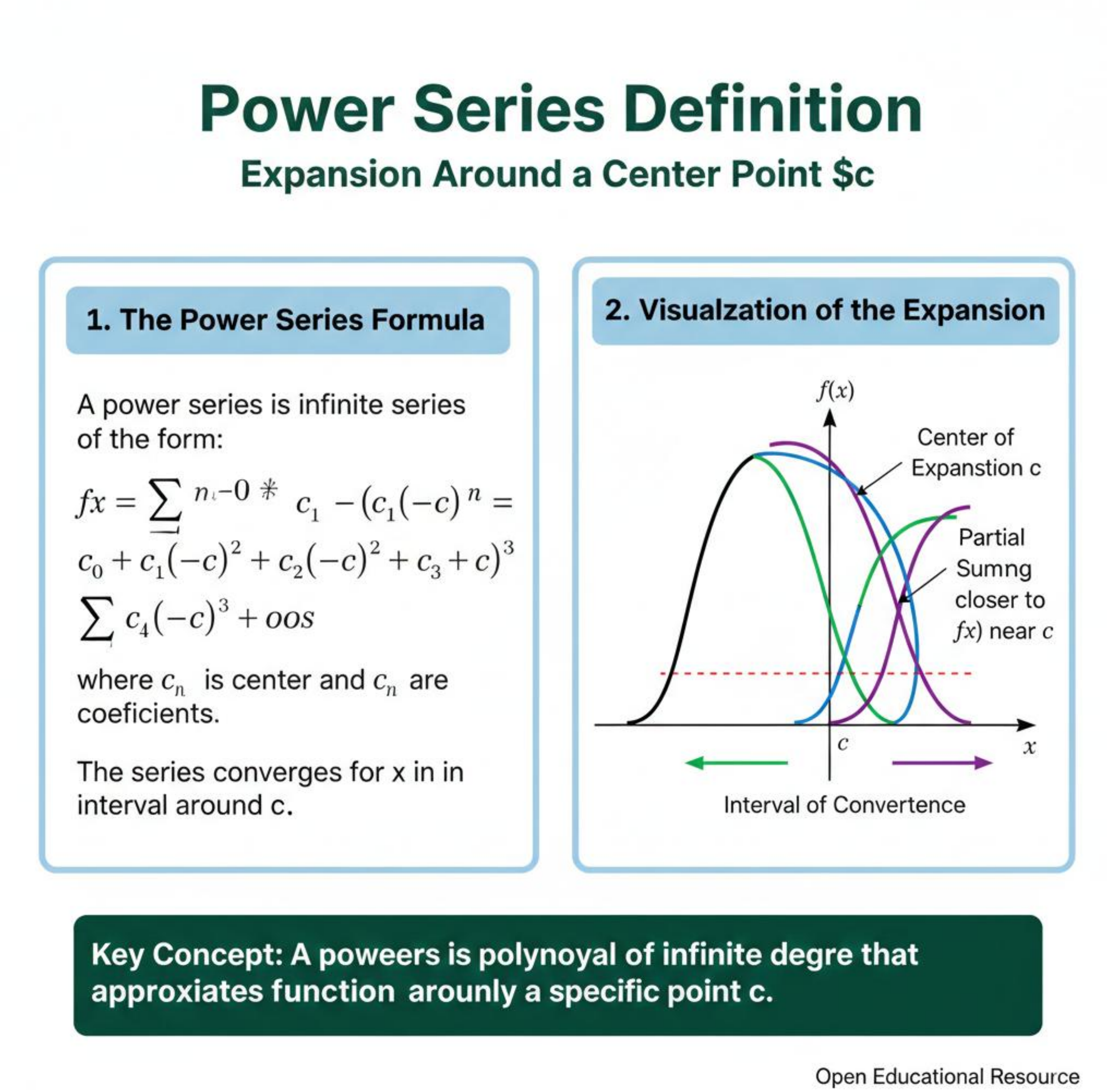


Illustration of a power series expansion around a center cAI prompt:

5.2.2 Radius and interval of convergence

The radius of convergence Rof a power series is the distance from the center cwithin which the series converges.

If |x-c|<R, the series converges.

If |x-c|>R, the series diverges.

At the boundary |x-c|=R, convergence must be checked separately.

Fill in the blank: A power series converges if |x - c| < ___________________.

Radius and interval of convergence of a power series

1. The Three Possibilities for Convergence

According to the **Abel-Refinement Theorem**, every power series follows one of these three patterns:

- **Case 1:** The series converges **only at** $x = a$. In this case, $R = 0$.
- **Case 2:** The series converges for **all real numbers** x. In this case, $R = \infty$.
- **Case 3:** There exists a real number $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

---

2. Finding the Radius (R)

We typically use the **Ratio Test** or the **Root Test** (derived from the Cauchy-Hadamard Theorem) to find R:

Method	Formula for L	Radius of Convergence
Ratio Test	$L = \lim_{n \to \infty} \left	\frac{c_{n+1}}{c_n} \right
Root Test	$L = \lim_{n \to \infty} \sqrt[n]{	c_n

Note: If $L = 0$, then $R = \infty$. If $L = \infty$, then $R = 0$.

---

3. Determining the Interval of Convergence (I)

The **Interval of Convergence** is the set of all x-values for which the series converges.

1. **Solve the inequality:** $|x - a| < R$, which expands to $a - R < x < a + R$.
2. **Test the Endpoints:** The Ratio/Root tests are inconclusive when $|x - a| = R$. You must manually plug $x = a - R$ and $x = a + R$ into the original series and use other tests (like the p-series or Alternating Series test) to check for convergence.

> **Key Distinction:** The **Radius** is a distance (R), while the **Interval** is a set of points (e.g., $(a - R, a + R]$, $[a - R, a + R]$, etc.).

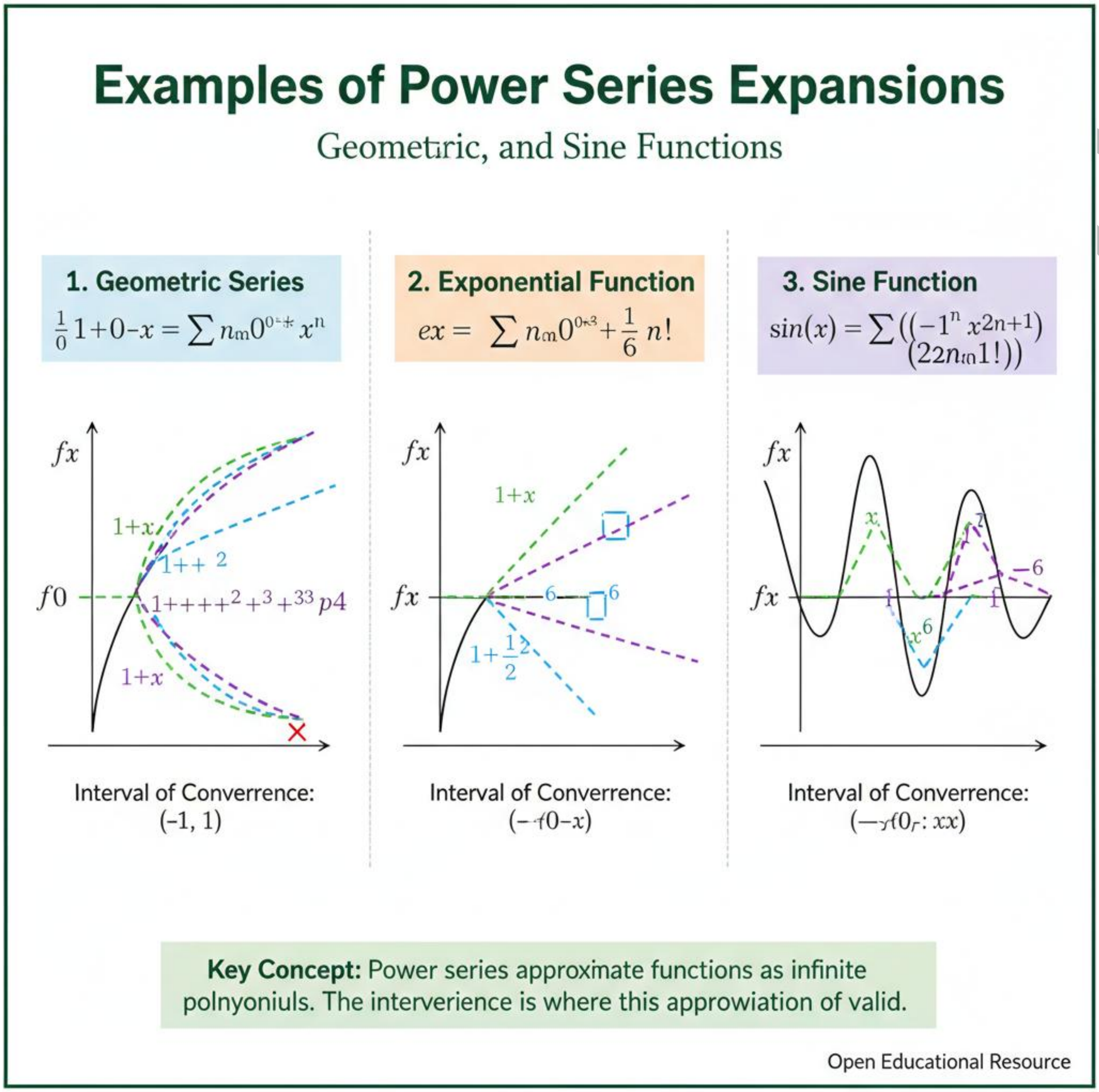
5.2.3 Examples of power series

Geometric series: n=0xnconverges for |x|<1.

Exponential function: ex=n=0xnn!, converges for all real x.

Sine function: sin(x)=n=0(1)n(2n+1)!x2n+1, converges for all real x.

Fill in the blank: The exponential function excan be represented as a power series that converges for ___________________ real x.



Examples of power series expansions for geometric, exponential, and sine functions

Pilot Questions 5.2

A power series is an infinite series of the form: A. ∑anB. ∑an(x-c)nC. ∑1nD. ∑(-1)n(Linked to SLO 5.2)

The radius of convergence of a power series is the distance from the: A. Supremum B. Center C. Divergence point D. Oscillation point (Linked to SLO 5.2)

A power series converges if: A. |x-c|<RB. |x-c|>RC. |x-c|=Ralways D. x=0only (Linked to SLO 5.2)

The geometric series n=0xnconverges for: A. |x|>1B. |x|<1C. x=1D. x=0(Linked to SLO 5.2)

The exponential function excan be represented as a power series that converges for: A. No real xB. Some real xC. All real xD. Only positive x(Linked to SLO 5.2)

5.3 Properties of Power Series

5.3.1 Differentiation and integration of power series

One of the most powerful features of power series is that they can be **differentiated** and **integrated** term by term within their interval of convergence.

If n=0an(x-c)nconverges for |x-c|<R, then:

Differentiation:

ddxn=0an(x-c)n=n=1nan(x-c)n-1

Integration:

∫n=0an(x-c)ndx=n=0ann+1(x-c)n+1+C

Fill in the blank: Power series can be differentiated and integrated ___________________ term by term within their interval of convergence.

5.3.2 Representation of functions by power series

Many important functions can be represented exactly by power series:

ex=n=0xnn!

sin(x)=n=0(1)n(2n+1)!x2n+1

cos(x)=n=0(1)n(2n)!x2n

These series converge for all real x, making them extremely useful in analysis and applications.

Fill in the blank: The sine function can be represented as a power series involving only ___________________ powers of x.

5.3.3 Applications in analysis

Power series are used to:

Approximate functions locally (Taylor series expansions).

Solve differential equations by expressing solutions as series.

Provide analytic continuation of functions beyond their original domain.

Fill in the blank: Power series are often used to approximate functions locally through __________________ series expansions.

Pilot Questions 5.3

Power series can be differentiated and integrated: A. Only once B. Term by term within their interval of convergence C. Only outside their interval of convergence D. Not at all (Linked to SLO 5.3)

The exponential function excan be represented as: A. ∑xnn!B. ∑xnC. ∑(-1)nxnD. ∑1n(Linked to SLO 5.3)

The sine function's power series involves only: A. Even powers of xB. Odd powers of xC. Supremum values D. Divergent terms (Linked to SLO 5.3)

The cosine function's power series involves only: A. Odd powers of xB. Even powers of xC. Supremum values D. Divergent terms (Linked to SLO 5.3)

Power series are often used to approximate functions locally through: A. Fourier series expansions B. Taylor series expansions C. Supremum expansions D. Divergent expansions (Linked to SLO 5.3)

5.4 Applications of Infinite and Power Series

5.4.1 Approximations of functions

Infinite and power series allow us to approximate complicated functions with polynomials.

Taylor series expansions provide local approximations near a point.

Example: sin(x)≈x-x33!+x55!for small x.

These approximations are widely used in numerical analysis and engineering.

Fill in the blank: Taylor series expansions provide local __________________ of functions near a point.

5.4.2 Use in solving differential equations

Power series are a powerful method for solving differential equations, especially when closed-form solutions are difficult.

Example: Solutions to y''+y=0can be expressed using sine and cosine power series.

Series methods are also used in physics to solve equations in quantum mechanics and wave theory.

Fill in the blank: Power series methods are useful in solving __________________ equations when closed-form solutions are difficult.

5.4.3 Applications in physics and engineering

Infinite and power series are applied in:

Physics: Approximating wave functions, expansions in quantum mechanics, and perturbation theory.

Engineering: Signal processing, control theory, and approximations in electrical circuits.

Applied mathematics: Numerical methods and approximations of integrals.

Fill in the blank: In engineering, power series are often used in __________________ processing and control theory.

Pilot Questions 5.4

Taylor series expansions provide local: A. Divergence B. Approximation C. Supremum D. Oscillation (Linked to SLO 5.4)

Power series methods are useful in solving: A. Algebraic equations B. Differential equations C. Supremum problems D. Divergent series (Linked to SLO 5.4)

The sine function can be approximated near 0 by: A. x-x33!+x55!B. 1 + x +) B. \(1 + x + x^2 C. 1nD. (1)n(Linked to SLO 5.4)

In physics, power series are used in: A. Quantum mechanics B. Supremum theory C. Divergence analysis D. Oscillation without bounds (Linked to SLO 5.4)

In engineering, power series are often used in: A. Signal processing B. Supremum analysis C. Divergent expansions D. Oscillation theory (Linked to SLO 5.4)

Series 5 Summary

This Series has introduced you to **infinite series and power series**, which are fundamental tools in Real Analysis and widely applied in mathematics, physics, and engineering. We began by defining infinite series, explaining convergence and divergence, and illustrating with examples such as geometric and harmonic series. Next, we studied power series, including their definition, radius and interval of convergence, and examples like exponential, sine, and cosine expansions. We then explored properties of power series, such as differentiation, integration, and representation of functions. Finally, we examined applications of infinite and power series in approximating functions, solving differential equations, and practical uses in physics and engineering.

By completing this Series, you should now be able to define infinite and power series, determine convergence behaviour, explain their properties, and apply them in analysis and applied contexts. These abilities directly reflect the Series Learning Outcomes (SLOs) and prepare you for deeper explorations in Real and Complex Analysis.

Glossary of Terms

Infinite series: The sum of infinitely many terms of a sequence.

Partial sum: The sum of the first kterms of a series, used to test convergence.

Convergent series: A series whose partial sums approach a finite limit.

Divergent series: A series whose partial sums do not approach a finite limit.

Power series: An infinite series of the form ∑an(x-c)n, often used to represent functions.

Radius of convergence: The distance from the center cwithin which a power series converges.

Taylor series: A power series expansion of a function around a point.

Concept Check Questions 5

Objective questions

1. An infinite series converges if the sequence of: A. Terms B. Partial sums C. Supremum values D. Divergent sequences (Linked to SLO 5.1)
2. A power series converges if: A. $|x-c|<R$B. $|x-c|>R$C. $|x-c|=R$always D. $x=0$only (Linked to SLO 5.2)

Short-answer questions

Define a power series and give an example. (Linked to SLO 5.2)

Explain how power series can be differentiated and integrated. (Linked to SLO 5.3)

Long-answer questions

Discuss the properties of power series. (Linked to SLO 5.3)

- **Basic response**: State differentiation and integration rules.
- **Value-added response**: Explain representation of functions and convergence behaviour.

Analyse the applications of infinite and power series in approximating functions and solving differential equations. (Linked to SLO 5.4)

- **Basic response**: Describe Taylor series approximations and series solutions.
- **Value-added response**: Extend to applications in physics and engineering.

Answers to Concept Check Questions 5

1. B. Partial sums
2. A. $|x-c|<R$
3. A power series is an infinite series of the form $\sum an(x-c)n$. Example: $ex=\sum xnn!$.
4. Power series can be differentiated and integrated term by term within their interval of convergence.

Basic response: Power series can be differentiated and integrated term by term.

Value-added response: They represent functions like $e^x, \sin(x), \cos(x)$, and converge within a radius of convergence.

Basic response: Taylor series approximate functions locally; series methods solve differential equations.

Value-added response: Applications extend to quantum mechanics, wave theory, and engineering signal processing.

Answers to Pilot Questions 5

Part 5.1

Partial sums

2

Divergent

$|r|<1$

$\ln(2)$

Part 5.2

$\sum a_n(x-c)^n$

Center

$|x-c|<R$

$|x|<1$

All real x

Part 5.3

Term by term within their interval of convergence

$\sum \frac{x^n}{n!}$

Odd powers of x

Even powers of x

Taylor series expansions

Part 5.4

Approximation

Differential equations

x-x33!+x55!

Quantum mechanics

Signal processing