La-Net

MTH103

ELEMENTARY MATHEMATICS III

1 Vector Algebra and Calculus

Series outline

This Series will be organised into four main Parts, each with relevant subparts to ensure coverage of the entire scope of vector algebra and calculus.

Proposed Parts and Subparts

1.1 Geometric Representation of Vectors

- 1.1.1 Vectors in one, two, and three dimensions
- 1.1.2 Components of vectors
- 1.1.3 Direction cosines

1.2 Vector Operations

- 1.2.1 Addition and subtraction of vectors
- 1.2.2 Scalar multiplication of vectors
- 1.2.3 Linear independence of vectors

1.3 Scalar and Vector Products

- 1.3.1 Scalar (dot) product of two vectors
- 1.3.2 Vector (cross) product of two vectors
- 1.3.3 Applications of scalar and vector products

1.4 Differentiation and Integration of Vectors

- 1.4.1 Differentiation of vectors with respect to a scalar variable
- 1.4.2 Integration of vectors with respect to a scalar variable
- 1.4.3 Applications in motion and dynamics

Introduction

Vectors are fundamental tools in mathematics and physics, as they allow us to represent quantities that have both magnitude and direction. From describing the motion of a car along a road to analysing forces acting on a structure, vectors provide a language that connects geometry with dynamics. In this Series, we shall build a strong foundation in vector algebra and calculus, which will serve as the basis for later topics in coordinate geometry and particle motion.

The aim of this Series is to equip you with the ability to represent vectors geometrically, perform operations such as addition, scalar multiplication, and products, and apply differentiation and integration of vectors to problems in motion and dynamics.
We shall commence this Series by examining the geometric representation of vectors in one, two, and three dimensions, including their components and direction cosines. Next, we will move on to vector operations such as addition, subtraction, and scalar multiplication, as well as the concept of linear independence. Following that, we will explore scalar and vector products, considering both their definitions and applications. Finally, we will wrap up by studying differentiation and integration of vectors with respect to a scalar variable, highlighting how these techniques are applied in analysing motion.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 define vectors in one, two, and three dimensions, and explain their geometric representation, components, and direction cosines,

SLO 2.2 demonstrate how to perform vector operations including addition, subtraction, and scalar multiplication, and analyse the concept of linear independence,

SLO 2.3 calculate scalar (dot) and vector (cross) products of two vectors, and apply these products to solve mathematical and physical problems,

SLO 2.4 carry out differentiation of vectors with respect to a scalar variable, and interpret its significance in motion analysis,

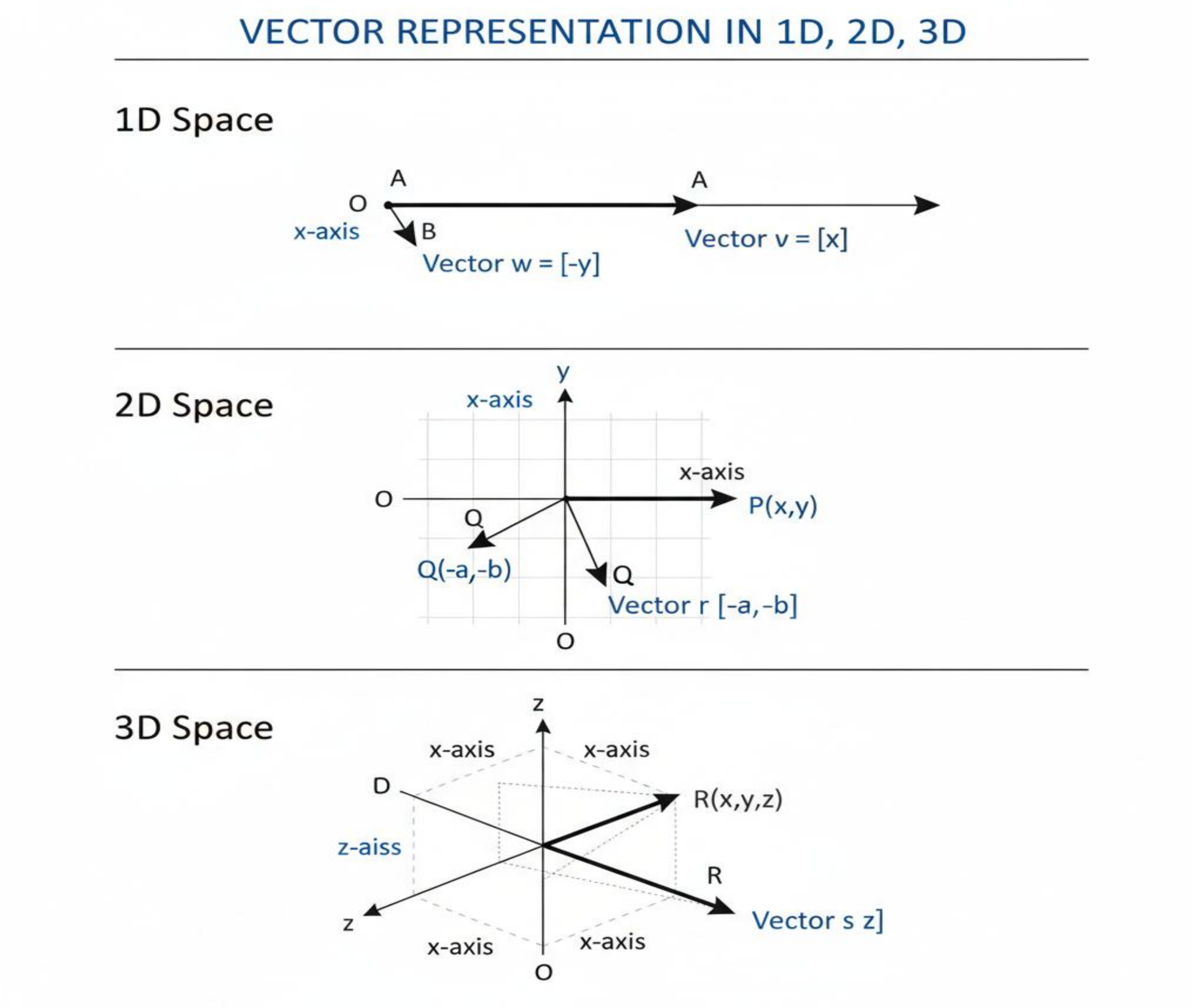
SLO 2.5 perform integration of vectors with respect to a scalar variable, and

apply the results to problems in dynamics.

1.1 Geometric Representation Of Vectors

1.1.1 Vectors in one, two, and three dimensions

A vector is a quantity that has both magnitude and direction. In one dimension, a vector can be represented as a line segment along a straight line, such as displacement along the x-axis. In two dimensions, vectors are represented in a plane using ordered pairs, for example y. In three dimensions, vectors extend into space and are represented as ordered triples, for example z.

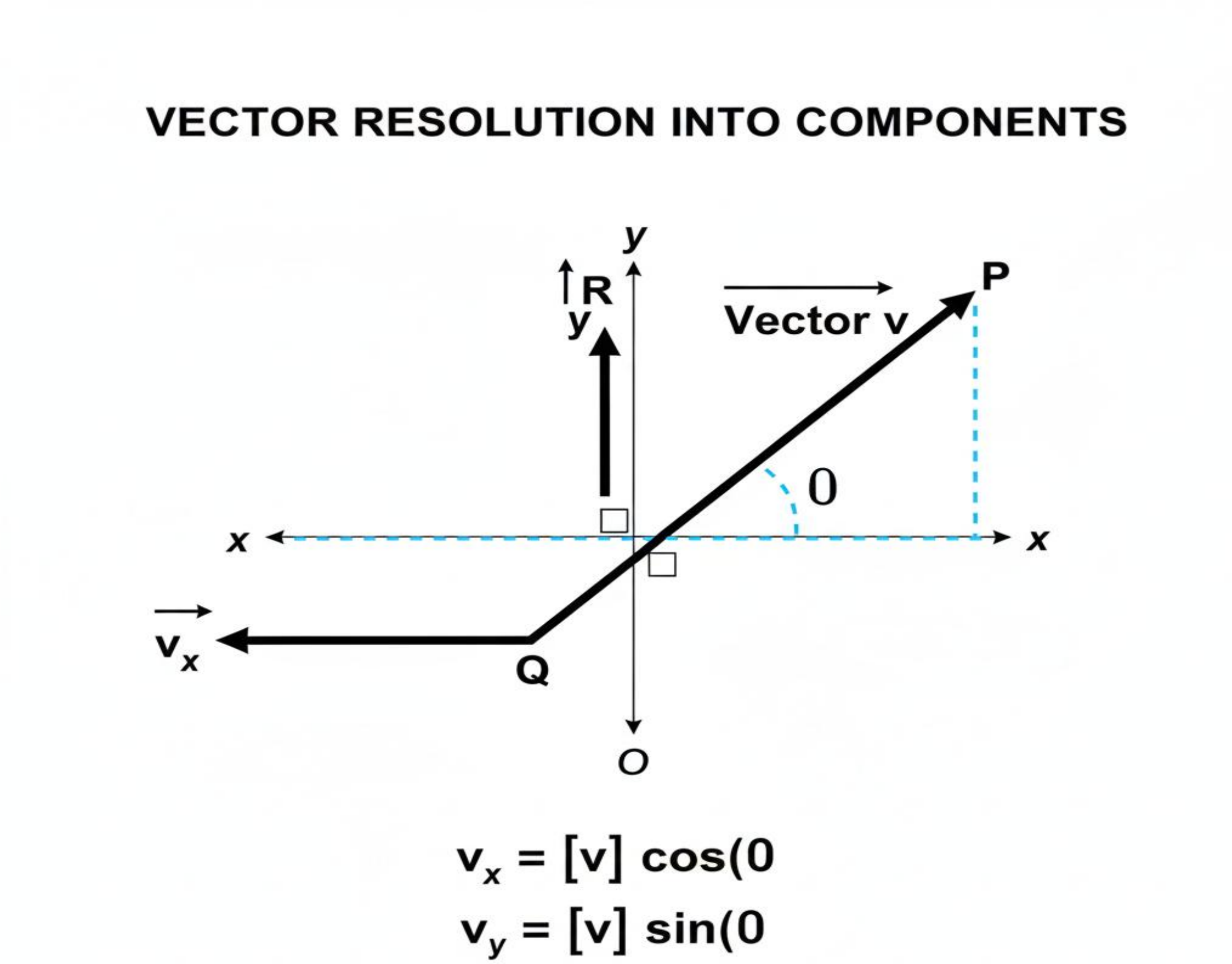


Representation of vectors in one, two, and three dimensions

Fill in the blank: A vector is defined as a quantity that has both ______ and direction.

1.1.2 Components of vectors

The components of a vector are the projections of the vector along the coordinate axes. For example, a vector in two dimensions can be expressed as v=vxi+vyj, where vxand vyare the components along the x and y axes respectively. In three dimensions, a vector is expressed as v=vxi+vyj+vzk.

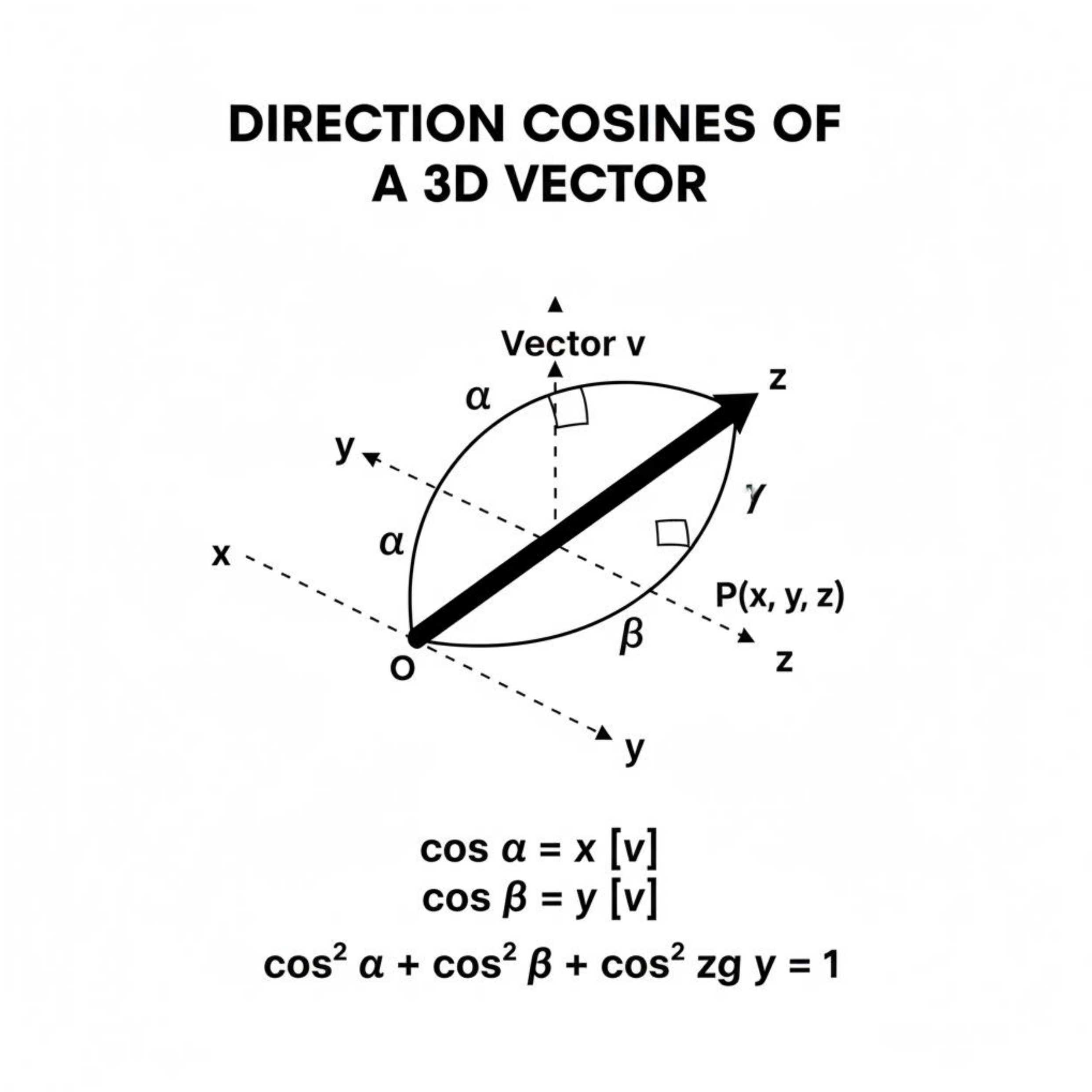


Resolution of a vector into components AI prompt suggestion: “

Fill in the blank: The x-component of a vector represents its projection along the ______ axis.

1.1.3 Direction cosines

Direction cosines are the cosines of the angles that a vector makes with the coordinate axes. For a vector $v=(x,y,z)$, the direction cosines are given by $\cos\alpha=x|v|$, $\cos\beta=y|v|$, and $\cos\gamma=z|v|$, where $|v|$is the magnitude of the vector. These values help in determining the orientation of the vector in space.



Direction cosines of a vector in three dimensions

Fill in the blank: The cosine of the angle between a vector and the x-axis is called the ______ cosine.

Pilot questions 1.1

Which property distinguishes a vector from a scalar?

How is a vector represented in two dimensions?

What are the components of a vector in three dimensions?

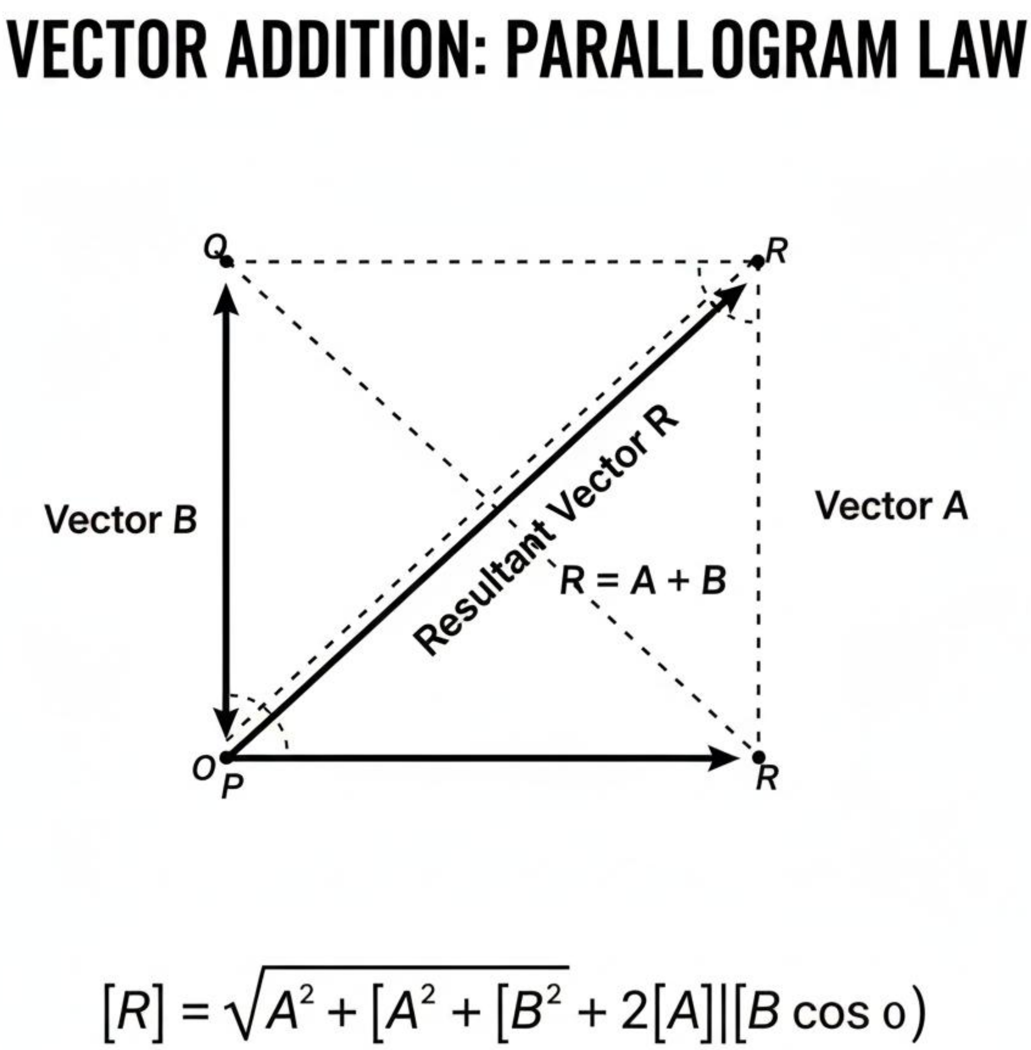
Define direction cosines in relation to a vector.

If a vector makes angles of 60°, 45°, and 60° with the x, y, and z axes respectively, what are its direction cosines?

1.2 Vector Operations

1.2.1 Addition and subtraction of vectors

The **addition of vectors** involves combining two or more vectors to produce a resultant vector. This is done by adding their corresponding components. For example, if u=(ux,uy)and v=(vx,vy), then u+v=(ux+vx,uy+vy). Similarly, **subtraction of vectors** is performed by subtracting the components of one vector from another.

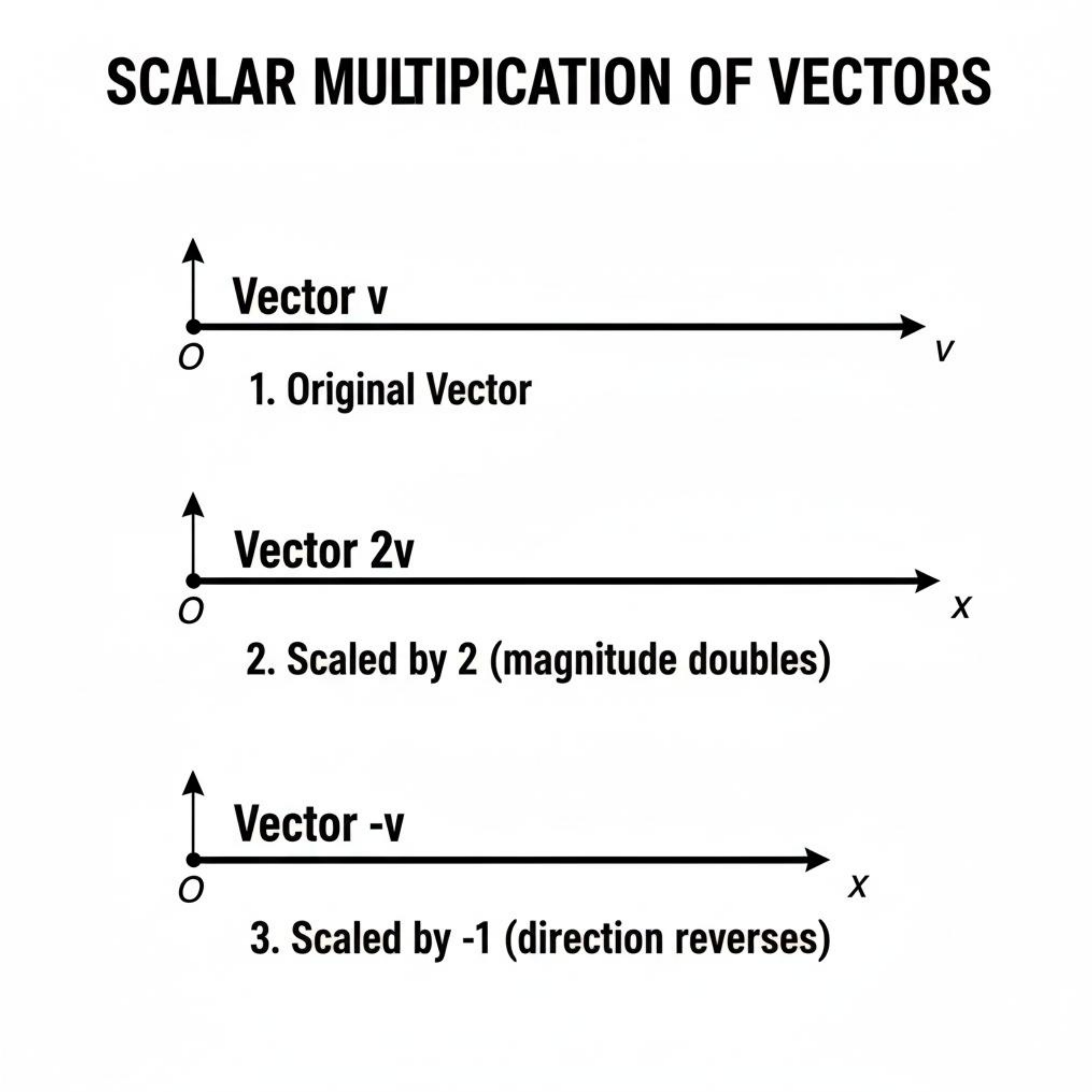


Vector addition using the parallelogram law AI prompt suggestion:

Fill in the blank: The resultant of two vectors is obtained by adding their corresponding ______.

1.2.2 Scalar multiplication of vectors

Scalar multiplication means multiplying a vector by a real number (scalar). This changes the magnitude of the vector but not its direction, unless the scalar is negative, in which case the direction is reversed. For example, if v=(x,y), then kv=(kx,ky).

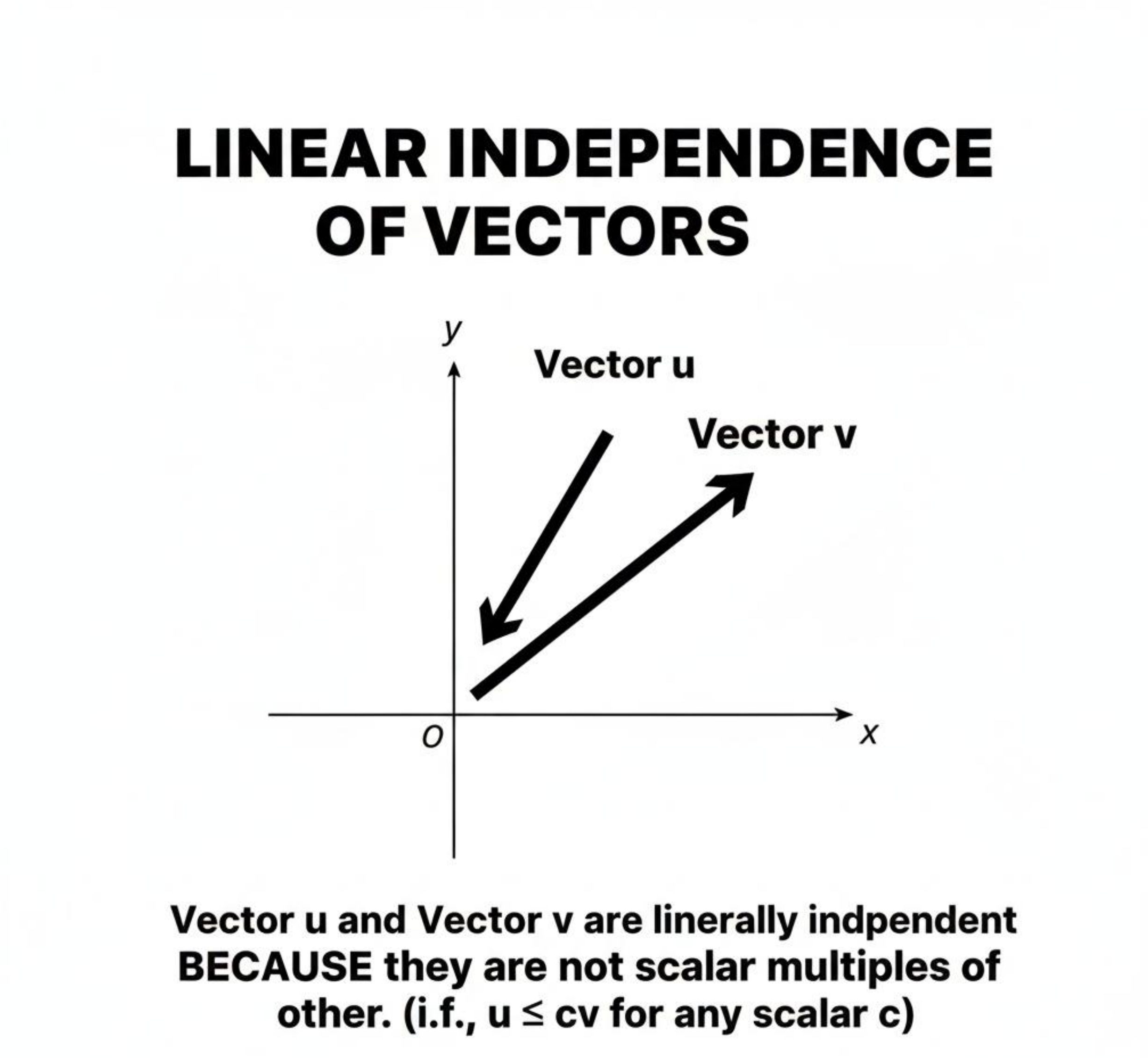


Scalar multiplication of a vector AI prompt suggestion: “

Fill in the blank: Multiplying a vector by a negative scalar reverses its ______.

1.2.3 Linear independence of vectors

Linear independence refers to a set of vectors where no vector can be expressed as a linear combination of the others. For example, in two dimensions, the vectors iand jare linearly independent because neither can be written as a multiple of the other. If vectors are linearly dependent, one vector can be expressed as a combination of the others, reducing the dimensionality of the space they span.



Linearly independent vectors in two dimensions

Fill in the blank: Two vectors are linearly independent if neither is a ______ of the other.

Pilot questions 1.2

The resultant of two vectors is obtained by: A. Multiplying their magnitudes B. Adding their corresponding components C. Subtracting their magnitudes D. Dividing their components

What hap\

pens to the direction of a vector when multiplied by a negative scalar?

Express the scalar multiplication of $v=(3,-2)$by $k=4$.

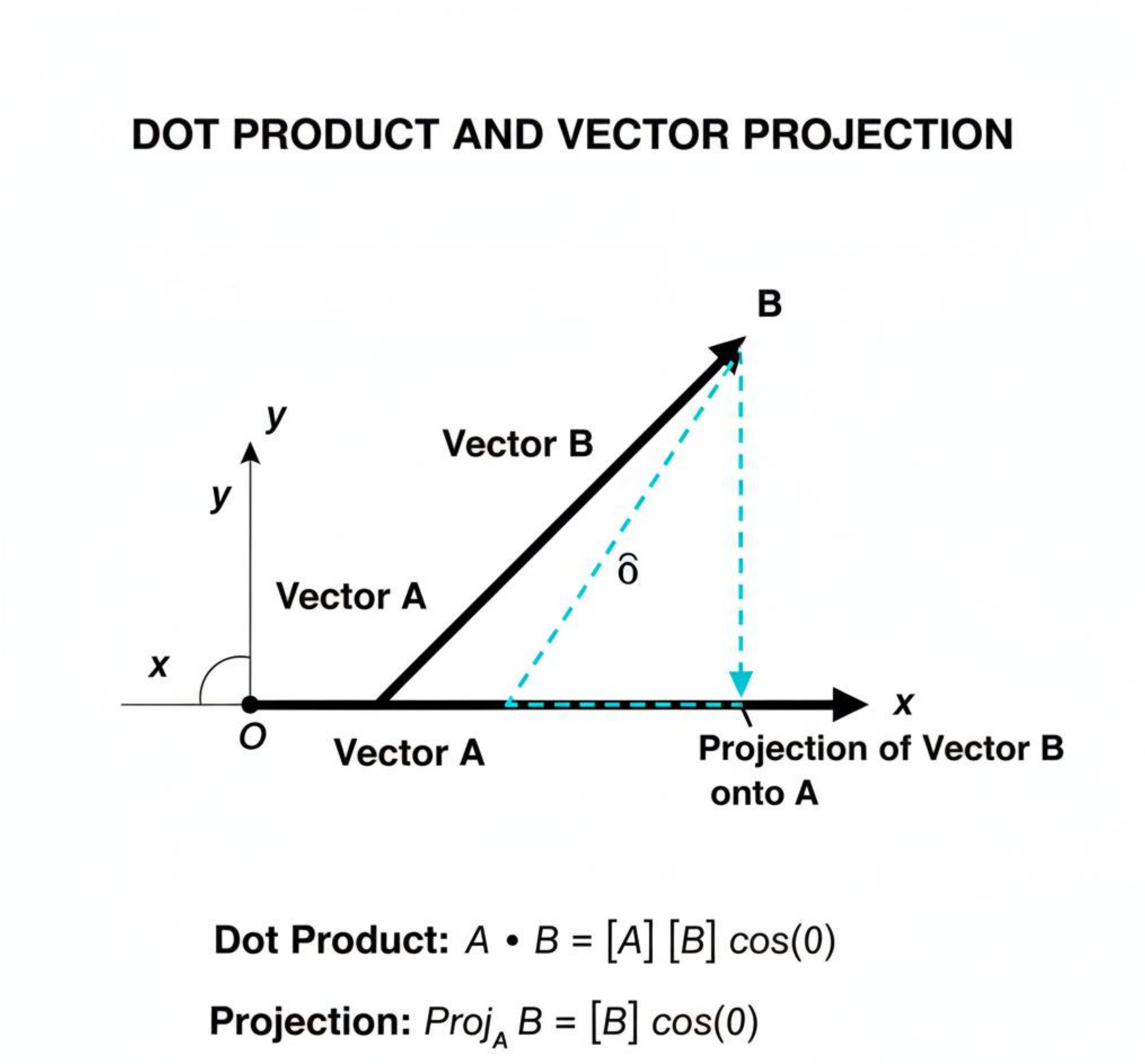
Define linear independence of vectors.

Give an example of two linearly dependent vectors in two dimensions.

1.3 Scalar And Vector Products

1.3.1 Scalar (dot) product of two vectors

The **scalar product**, also called the **dot product**, is an operation that takes two vectors and produces a scalar. It is defined as $uv=|u||v|\cos\theta$, where is the angle between the vectors. In component form, if $u=(ux,uy,uz)$and $v=(vx,vy,vz)$, then $uv=uxvx+uyvy+uzvz$. The scalar product is useful for determining angles between vectors and checking orthogonality.



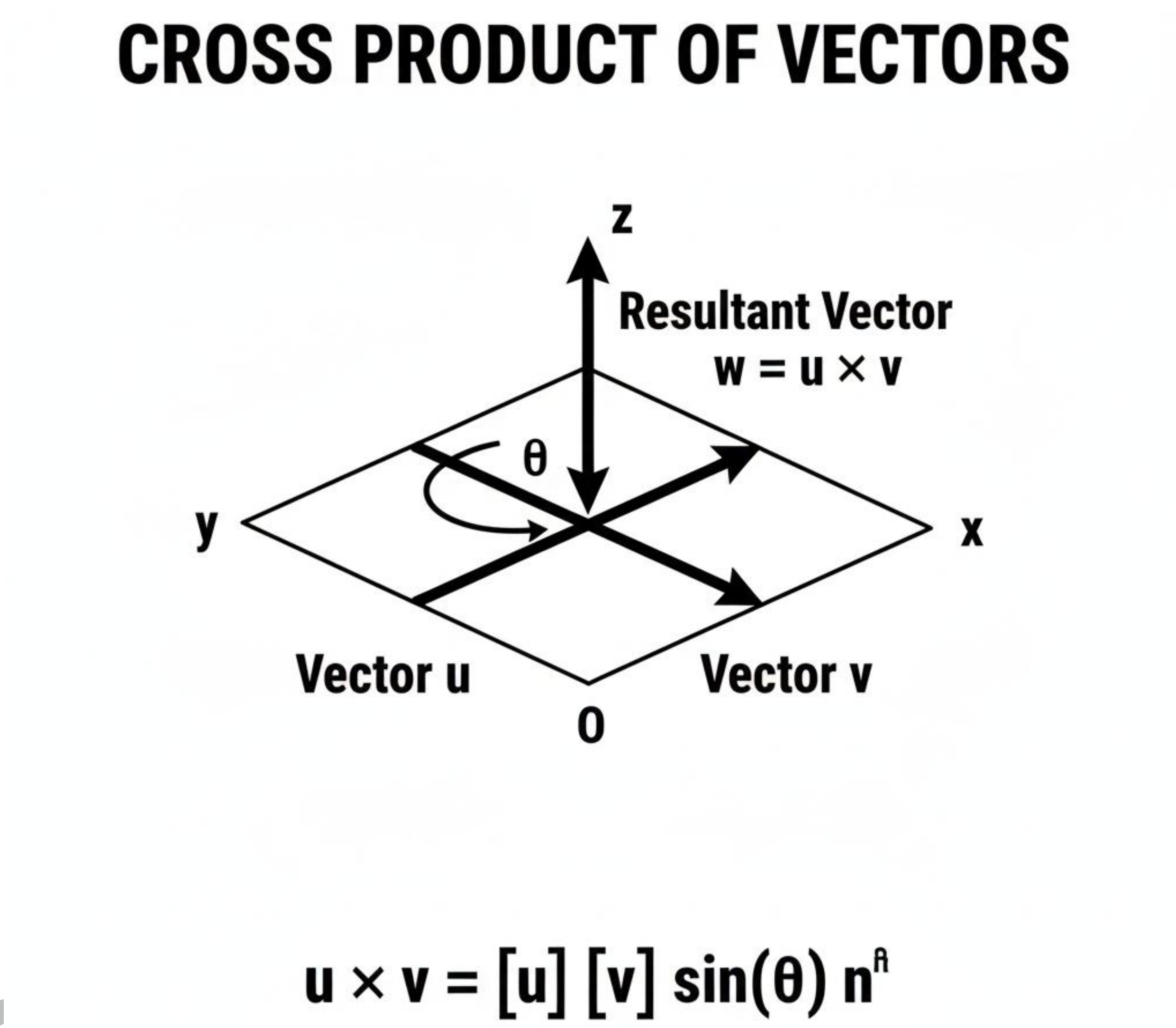
Scalar product of two vectors

Fill in the blank: The scalar product of two perpendicular vectors is always ______.

1.3.2 Vector (cross) product of two vectors

The **vector product**, also called the **cross product**, is an operation that takes two vectors and produces another vector. It is defined as $uv=|u||v|\sin\theta\ n$, where is the angle between the vectors and nis a unit vector perpendicular to both. In component form, if $u=(ux,uy,uz)$ and $v=(vx,vy,vz)$, then uv=(uyvz-uzvy)i-(uxvz-uzvx)j+(uxvy-uyvx)k.

The cross product is useful for finding areas of parallelograms and determining perpendicular directions.



Vector product of two vectors

Fill in the blank: The vector product of two parallel vectors is always ______.

1.3.3 Applications of scalar and vector products

The scalar product is widely used in physics and engineering. For example, it can be applied to calculate **work**, defined as the product of force and displacement in the direction of the force. The vector product is used to calculate **torque**, defined as the cross product of force and position vector. These applications show how vector operations connect abstract mathematics with real-world physical problems.

Applications of scalar and vector products Short description: A box listing examples of applications of dot and cross products

Dot Product (Scalar Product)	Cross Product (Vector Product)
Work Done: Calculated as $W = \vec{F} \cdot \vec{d}$. It measures how much force is applied in the direction of displacement.	**Torque:** Calculated as $\vec{\tau} = \vec{r} \times \vec{F}$. It measures the tendency of a force to cause rotation about a pivot.
Angle Between Vectors: Used to find the cosine of the angle θ between two vectors: \cos \theta = \frac{\vec{A} \cdot \vec{B}}{	$\vec{A}$
Power: In physics, power is the dot product of force and velocity: $P = \vec{F} \cdot \vec{v}$.	**Magnetic Force:** The force on a moving charge in a magnetic field: $\vec{F} = q(\vec{v} \times \vec{B})$.
Projection: Determining the component of one vector that lies along the direction of another.	**Angular Momentum:** Calculated as $\vec{L} = \vec{r} \times \vec{p}$, describing the rotational motion of an object.

Fill in the blank: The dot product of force and displacement gives the physical quantity called ______.

Pilot questions 1.3

The scalar product of two vectors results in: A. A vector B. A scalar C. A matrix D. A tensor

Write the scalar product of u=(2,3,1)and v=(1,-1,4).

What is the direction of the vector product of two vectors?

State one physical application of the scalar product.

State one physical application of the vector product.

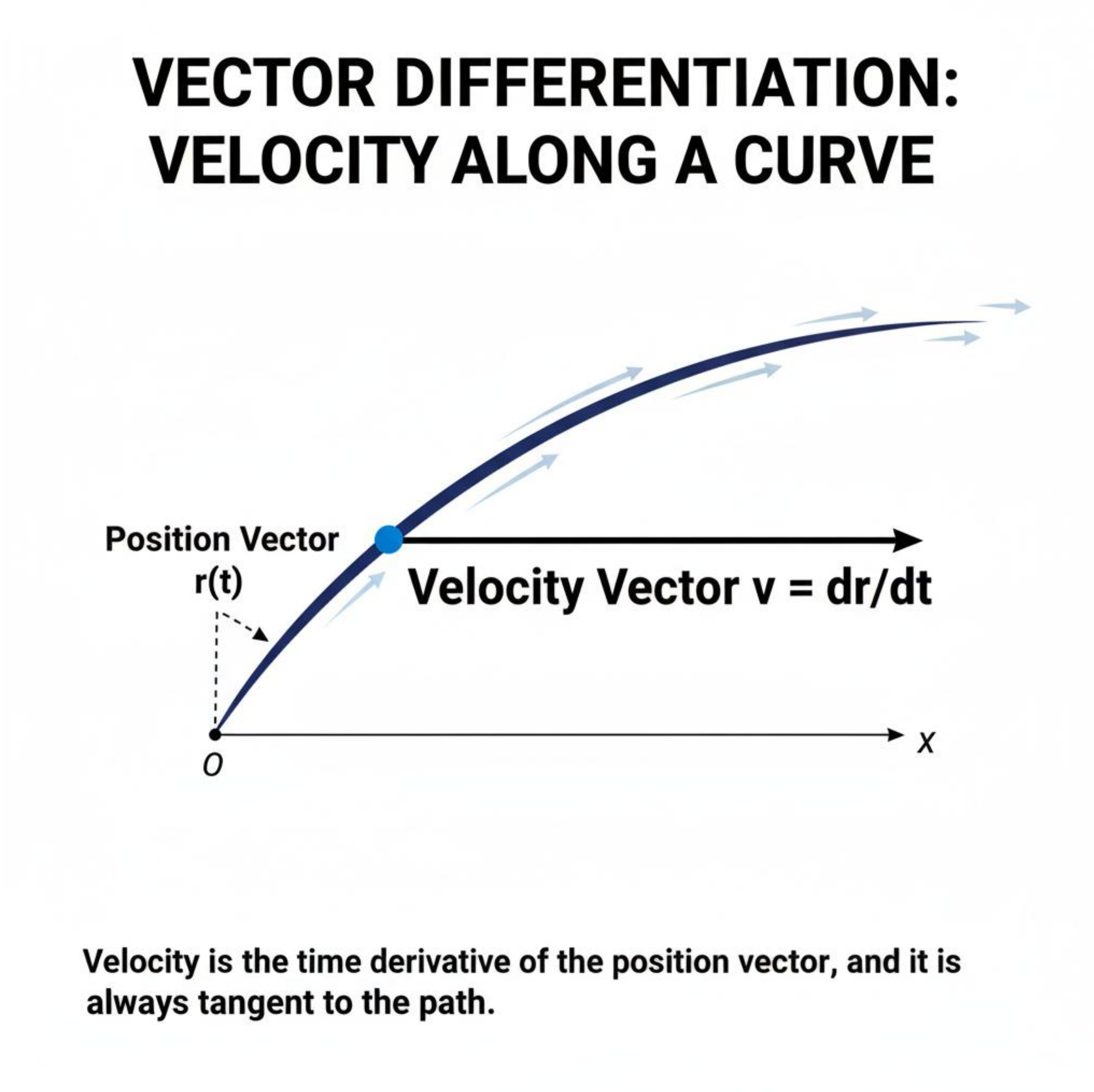
1.4 Differentiation And Integration Of Vectors

1.4.1 Differentiation of vectors with respect to a scalar variable

Differentiation of vectors involves finding the rate of change of a vector function with respect to a scalar variable, usually time. If r(t)=x(t)i+y(t)j+z(t)k, then

drdt=dxdti+dydtj+dzdtk.

This derivative represents the **velocity vector** of a particle moving along a path. Differentiating again gives the **acceleration vector**, which shows how velocity changes with time.



Differentiation of a vector function representing velocity AI prompt suggestion: "

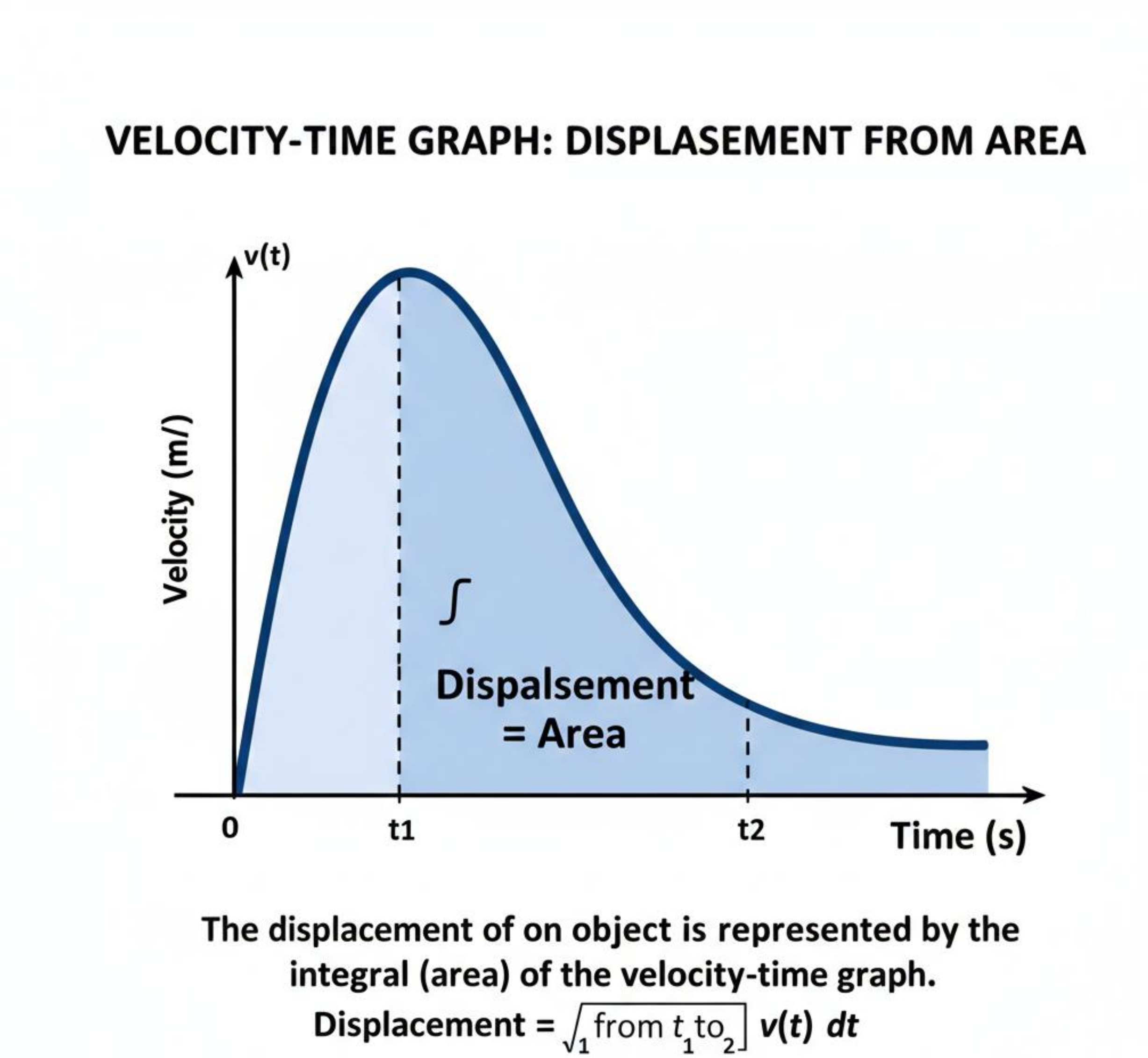
Fill in the blank: The derivative of a position vector with respect to time gives the ______ vector.

1.4.2 Integration of vectors with respect to a scalar variable

Integration of vectors involves summing up vector quantities over a scalar variable. If v(t)is a velocity vector, then the position vector is obtained by integrating:

$$r(t)=\int v(t)\,dt.$$

Similarly, integrating acceleration gives velocity. Integration is therefore used to reconstruct motion from rates of change.



Integration of velocity vector to obtain displacement AI prompt suggestion: "

Fill in the blank: Integrating the velocity vector with respect to time gives the ______ vector.

1.4.3 Applications in motion and dynamics

Differentiation and integration of vectors are essential in analysing motion. Differentiation allows us to determine instantaneous velocity and acceleration, while integration helps us calculate displacement and position from velocity or acceleration data. These techniques are widely applied in physics, engineering, and mechanics, such as predicting the trajectory of projectiles or analysing oscillations in pendulums.

A box listing examples of applications of vector differentiation and integration in dynamics.

Vector Differentiation (Rates of Change)	Vector Integration (Accumulation)
Velocity ($\vec{v}$): The first derivative of the position vector $\vec{r}(t)$ with respect to time. It indicates the instantaneous speed and direction of motion. $\vec{v}(t) = \frac{d\vec{r}}{dt}$	**Displacement $\Delta\vec{r}$):** The integral of the velocity vector over a time interval. It represents the net change in position. $\Delta\vec{r} = \int_{t_1}^{t_2} \vec{v}(t) \, dt$
Acceleration ($\vec{a}$): The derivative of the velocity vector. It describes how the velocity changes, including speeding up, slowing down, or changing direction. $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$	**Trajectory/Position:** Finding the path of an object by integrating acceleration twice (given initial conditions), commonly used in projectile motion. $\vec{r}(t) = \iint \vec{a}(t) \, dt^2$

Vector Differentiation (Rates of Change)	Vector Integration (Accumulation)
Jerk: The rate of change of acceleration ($d\vec{a}/dt$), often used in engineering to ensure smooth transitions in motion (like elevators or rollercoasters).	**Impulse:** The integral of a force vector over time, which relates to the change in an object's linear momentum. $\vec{J} = \int \vec{F} \, dt$

Fill in the blank: Differentiation gives instantaneous velocity, while integration gives total ______.

Pilot questions 1.4

- The derivative of a position vector with respect to time represents: A. Acceleration B. Velocity C. Displacement D. Force
- If $r(t)=t2i+3tj+\sin tk$, find $drdt$.
- What does integrating acceleration with respect to time give?
- State one physical application of vector differentiation.
- State one physical application of vector integration.

Series Summary

This Series has provided you with a structured foundation in vector algebra and calculus. We began with the geometric representation of vectors in one, two, and three dimensions, including their components and direction cosines. We then examined vector operations such as addition, subtraction, scalar multiplication, and linear independence. Following that, we studied scalar and vector products, highlighting their definitions and applications in mathematics and physics. Finally, we explored differentiation and integration of vectors with respect to a scalar variable, showing how these techniques are applied in motion and dynamics.

By completing this Series, you should now be able to define and represent vectors geometrically, demonstrate vector operations, calculate scalar and vector products, and apply differentiation and integration of vectors to problems in motion. These abilities directly align with the Series Learning Outcomes, preparing you for the next Series on coordinate geometry.

Glossary of Terms

Acceleration vector: The derivative of velocity with respect to time, representing the rate of change of velocity.

Components of a vector: The projections of a vector along the coordinate axes, expressed as coefficients of unit vectors.

Direction cosines: The cosines of the angles between a vector and the coordinate axes, used to describe its orientation in space.

Linear independence: A property of vectors where no vector in a set can be expressed as a linear combination of the others.

Scalar product (dot product): A multiplication of two vectors that results in a scalar, defined as the product of their magnitudes and the cosine of the angle between them.

Scalar variable: A single-valued quantity, often time, with respect to which vectors can be differentiated or integrated.

Vector: A quantity that has both magnitude and direction, represented geometrically by an arrow.

Vector product (cross product): A multiplication of two vectors that results in another vector, defined as the product of their magnitudes and the sine of the angle between them, directed perpendicular to both.

Velocity vector: The derivative of position with respect to time, representing the rate of change of displacement.

Concept Check Questions 1

Objective questions

Which property distinguishes a vector from a scalar? (Linked to SLO 2.1)

The scalar product of two perpendicular vectors is: A. Zero B. One C. Equal to their magnitudes D. Undefined (Linked to SLO 2.3)

Short-answer questions

Write the components of a vector $v=(3,4,5)$in terms of unit vectors. (Linked to SLO 2.1)

Explain what is meant by linear independence of vectors. (Linked to SLO 2.2)

Long-answer questions

Derive the formula for the direction cosines of a vector in three dimensions and explain their significance. (Linked to SLO 2.1)

Basic response: Show the derivation using the definition of cosines and vector magnitude.

Value-added response: Extend the explanation to applications in physics, such as determining the orientation of forces in space.

Differentiate the vector function $r(t)=t2i+3tj+\sin tk$with respect to t, and interpret the result in terms of motion. (Linked to SLO 2.4)

Basic response: Provide the derivative and state it represents velocity.

Value-added response: Discuss how the derivative can be used to analyse acceleration and trajectory in dynamics.

Answers to Concept Check Questions 1

A vector has both magnitude and direction, while a scalar has only magnitude.

A. Zero

$v=3i+4j+5k$

Vectors are linearly independent if none of them can be expressed as a linear combination of the others.

Basic response: $\cos\alpha=x|v|,\cos\beta=y|v|,\cos\gamma=z|v|$. They describe the orientation of the vector. **Value-added response**: In physics, direction cosines are used to resolve forces along axes and analyse equilibrium in three dimensions.

Basic response: $drdt=2ti+3j+\cos tk$. This represents velocity. **Value-added response**: Differentiating again gives acceleration, which helps in predicting motion under varying forces.

Answers to Pilot Questions 1

Magnitude and direction

Ordered pair y

vx,vy,vz

Cosines of the angles between the vector and the coordinate axes

$\cos 60, \cos 45, \cos 60$

2 Coordinate Geometry In Two Dimensions

Series outline

This Series will be organised into four main Parts, each with relevant sub-Parts to cover straight lines, circles, conic sections, and tangents/normals.

Proposed Parts and Sub-Parts

2.1 Straight lines

2.1.1 Equation of a straight line

2.1.2 Slope and intercept forms

2.1.3 Distance between points and lines

2.2 Circles

2.2.1 Equation of a circle

2.2.2 Centre and radius

2.2.3 Properties of circles

2.3 Conic sections: parabola, ellipse, and hyperbola

2.3.1 Standard equations of conic sections

2.3.2 Geometric properties of each conic

2.3.3 Applications of conic sections

2.4 Tangents and normals

2.4.1 Tangents to straight lines and circles

2.4.2 Tangents to conic sections

2.4.3 Normals and their applications

Introduction

Geometry in two dimensions provides a powerful framework for understanding shapes, curves, and their relationships within a plane. From the straight lines that define paths and boundaries to the circles and conic sections that describe more complex curves, coordinate geometry allows us to represent these figures algebraically and analyse their properties with precision. This Series builds directly on your knowledge of vectors, offering tools to study geometric figures and their applications in mathematics and dynamics.

The aim of this Series is to equip you with the ability to derive and interpret equations of straight lines, circles, and conic sections, as well as to apply the concepts of tangents and normals in solving geometric problems.

We shall commence this Series by examining straight lines, focusing on their equations, slopes, and distances. Next, we will move on to circles, exploring their equations, centres, and radii. Following that, we will study conic sections, including the parabola, ellipse, and hyperbola, along with their geometric properties and applications. Finally, we will conclude with tangents and normals, highlighting how they are constructed and applied to both simple and complex curves.

Series Learning Outcomes

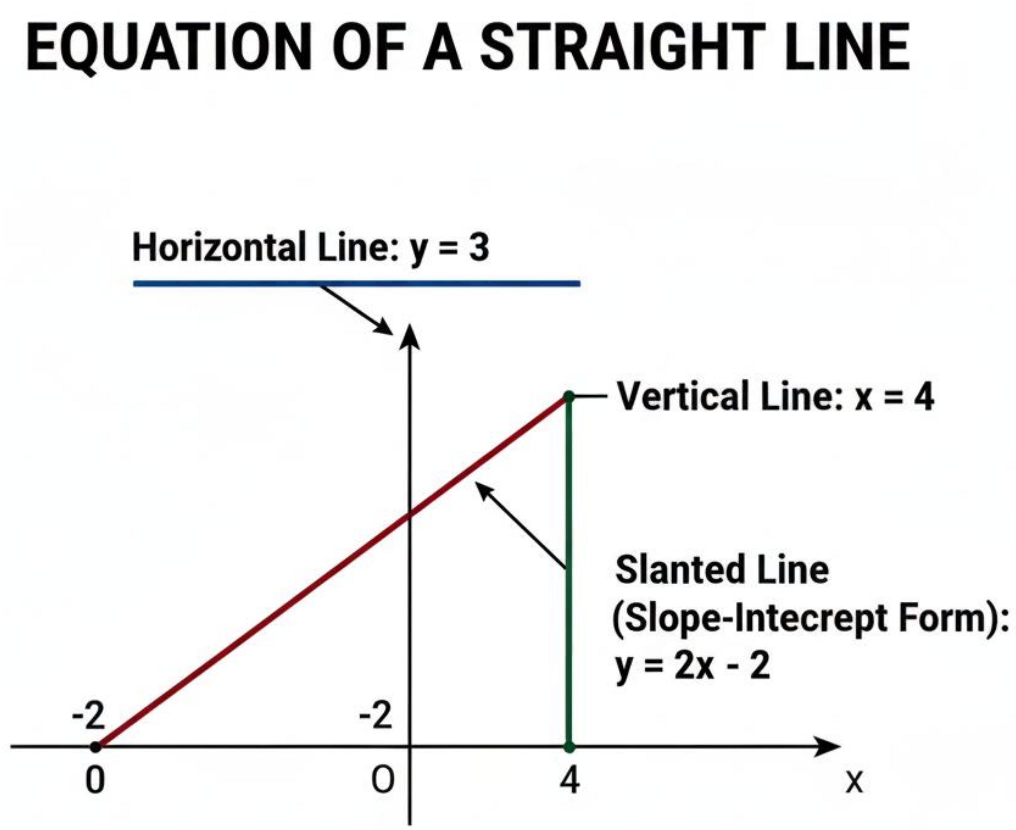
Upon completing this Series, you should be able to:

- **SLO 3.1** derive and explain the equations of straight lines in two dimensions, including slope and intercept forms, and calculate distances between points and lines,
- **SLO 3.2** formulate and interpret the equation of a circle, identify its centre and radius, and analyse its properties,
- **SLO 3.3** define and describe the standard equations of conic sections (parabola, ellipse, hyperbola), and evaluate their geometric properties and applications,
- **SLO 3.4** construct and apply equations of tangents and normals to straight lines, circles, and conic sections, and demonstrate their use in solving geometric problems.

2.1 Straight Lines

2.1.1 Equation of a straight line

A **straight line** in two-dimensional coordinate geometry is the simplest curve, defined by a linear relationship between xand y. The general equation of a straight line is ax+by+c=0, where a, b, and care constants. This form is flexible, as it can represent vertical, horizontal, and slanted lines depending on the values of aand b.

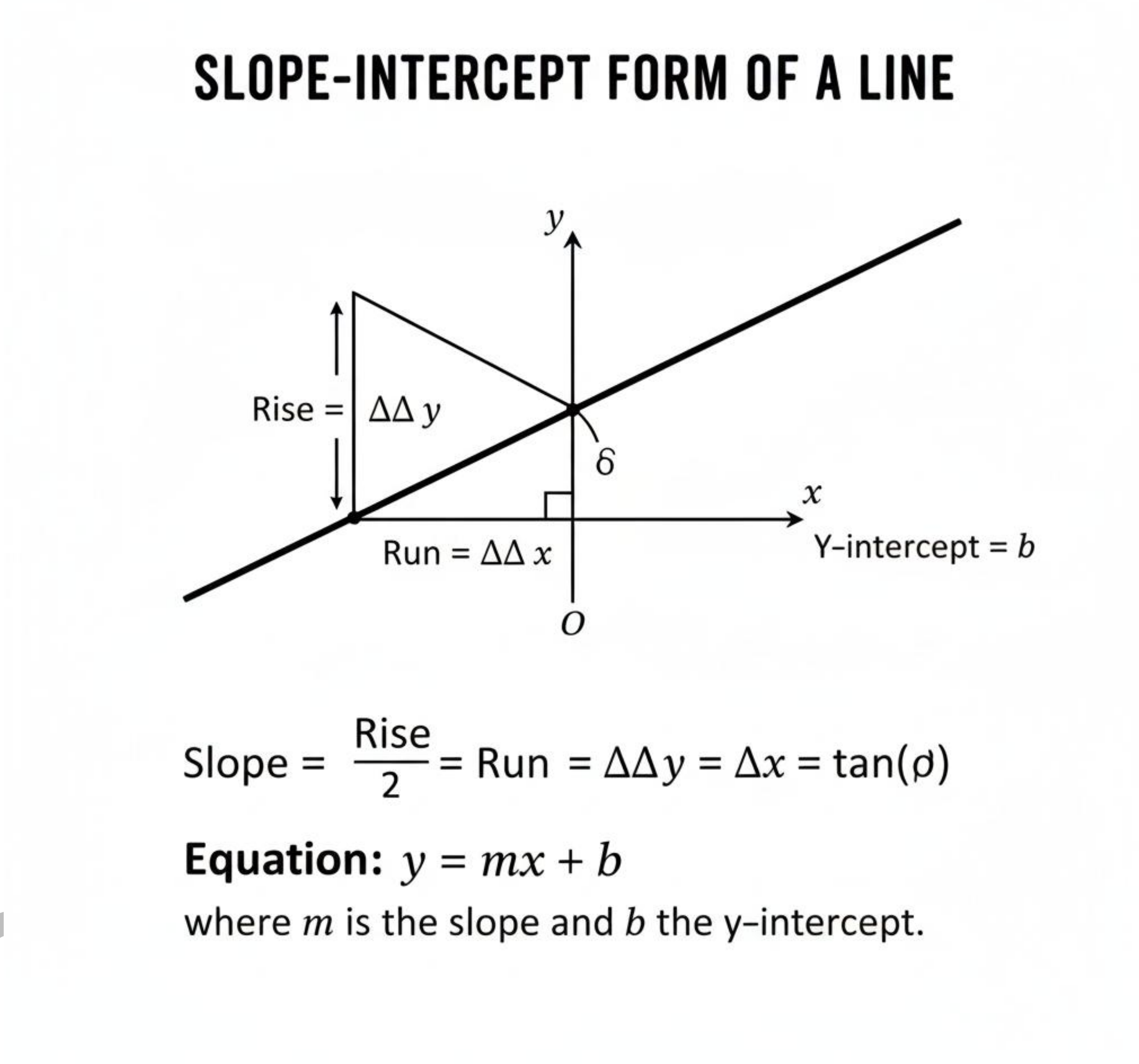


General representation of straight lines in two dimensions

Fill in the blank: The general equation of a straight line in two dimensions is ______.

2.1.2 Slope and intercept forms

The **slope** of a line measures its steepness, defined as the ratio of vertical change to horizontal change. If a line passes through points y1and y2, its slope is m=y2-y1x2-x1. The **slope-intercept form** of a line is y=mx+c, where mis the slope and cis the y-intercept. Another useful form is the **point-slope form**, given by y-y1=m(x-x1).



Slope and intercept of a straight line

Fill in the blank: The slope of a line passing through y1and y2is given by ______.

2.1.3 Distance between points and lines

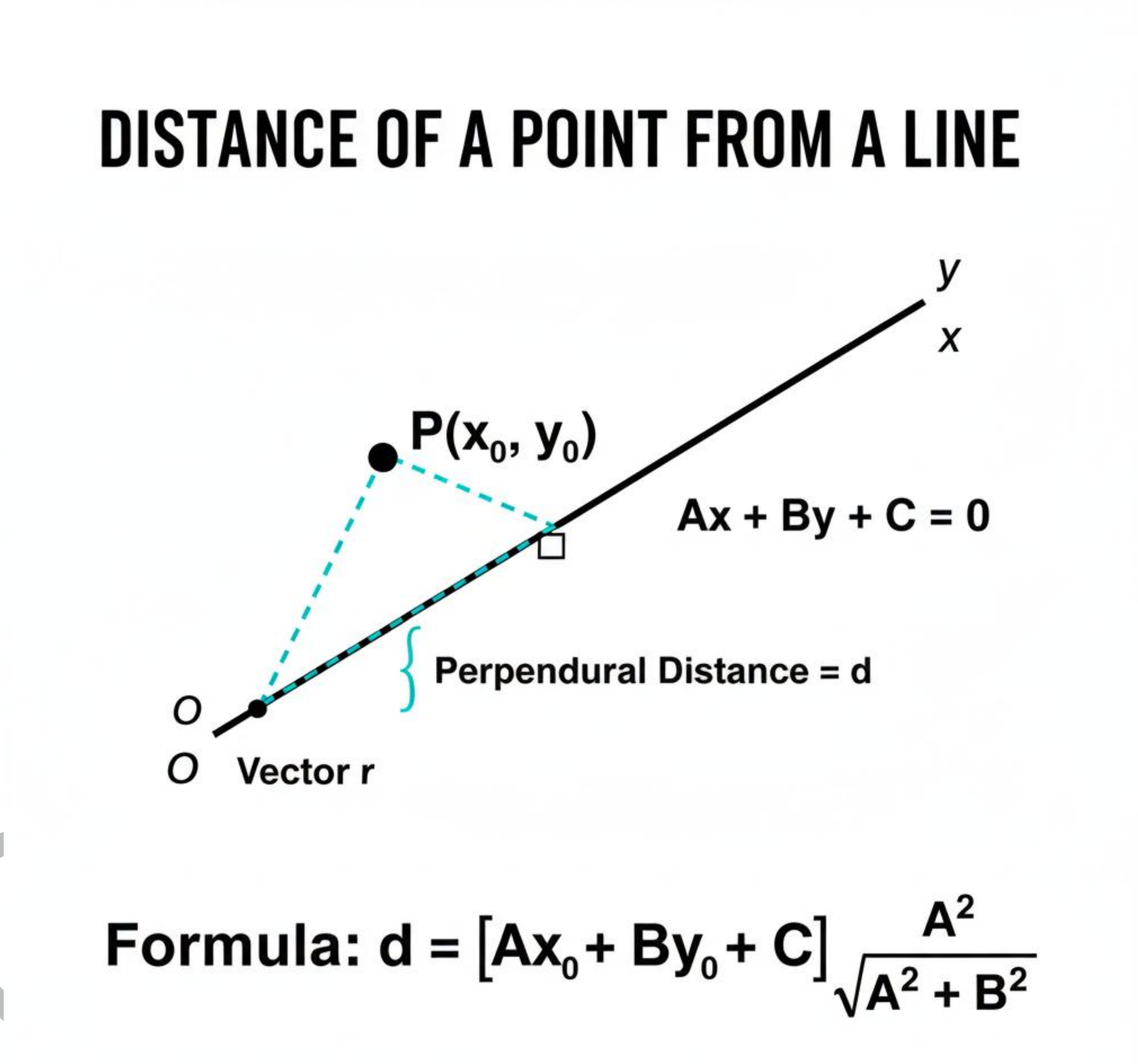
The **distance between two points** y1and y2is given by

$$d=(x2-x1)2+(y2-y1)2.$$

The **distance of a point** y0from a line ax+by+c=0is

$$d=|ax0+by0+c|a2+b2.$$

These formulas are essential for solving problems involving shortest paths and perpendicular distances in geometry.



Distance of a point from a line AI prompt suggestion: “

Fill in the blank: The distance between two points $y1$and $y2$is given by the ______ formula.

Pilot questions 2.1

The general equation of a straight line in two dimensions is: A. $ax2+by+c=0$B. $ax+by+c=0$C. $ax+by2+c=0$D. $ax+by+cz=0$

Find the slope of the line passing through points $3$and 11.

Write the slope-intercept form of a line with slope 2 and y-intercept -3.

State the formula for the distance between two points in a plane.

Calculate the distance of the point $1$from the line $3x+4y-5=0$.

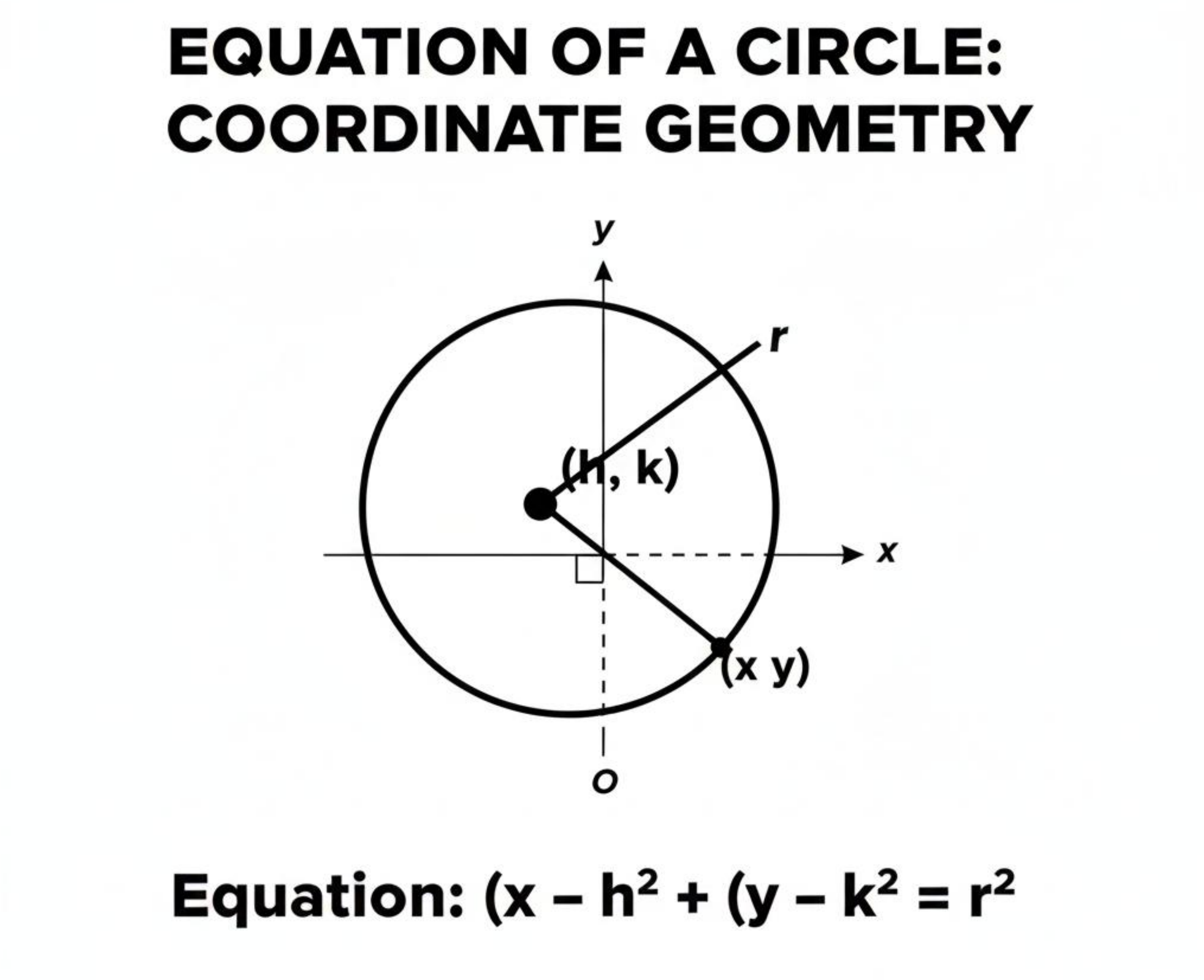
2.2 Circles

2.2.1 Equation of a circle

A **circle** is defined as the set of all points in a plane that are equidistant from a fixed point called the **centre**. The general equation of a circle with centre $k$and radius $r$is

$$(x-h)2+(y-k)2=r2.$$

If the circle is centred at the origin, the equation simplifies to $x2+y2=r2$.

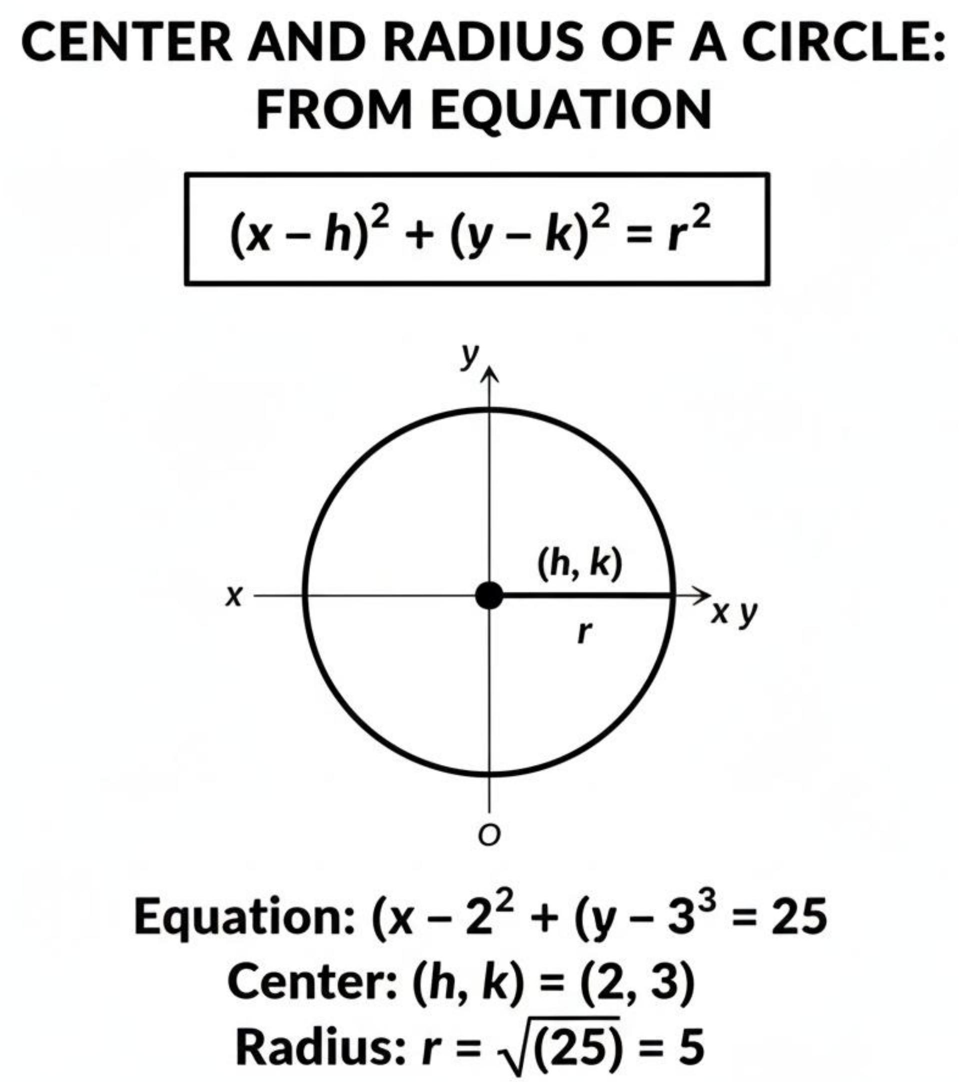


equation of a circle in two dimensions

Fill in the blank: The equation of a circle with centre kand radius ris ______.

2.2.2 Centre and radius

The **centre** of a circle is the fixed point from which all points on the circle are equidistant. The **radius** is the constant distance from the centre to any point on the circle. From the general equation x2+y2+2gx+2fy+c=0, the centre is -g,-fand the radius is g2+f2-c.

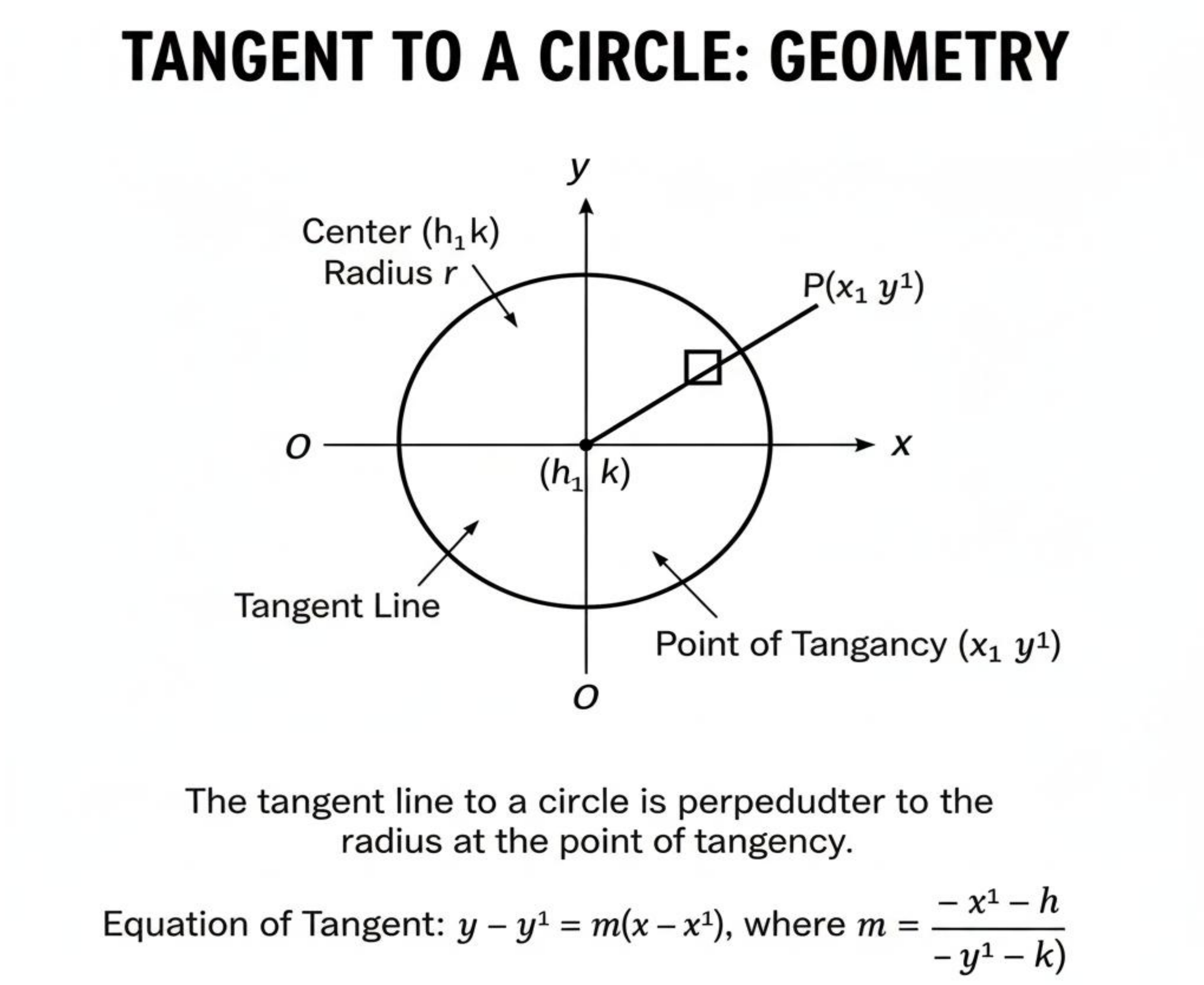


Centre and radius of a circle from its equation

Fill in the blank: In the equation x2+y2+2gx+2fy+c=0, the centre is ______ and the radius is ______.

2.2.3 Properties of circles

Circles have several important properties. All radii are equal, and the diameter is twice the radius. The tangent to a circle is perpendicular to the radius at the point of contact. The equation of a tangent at point y1on the circle x2+y2=r2is xx1+yy1=r2. Circles also exhibit symmetry about their centre, and they are widely used in geometry and dynamics to model rotational motion.



Tangent to a circle at a point

Fill in the blank: The tangent to a circle is always ______ to the radius at the point of contact.

Pilot questions 2.2

The equation of a circle with centre 0and radius 5 is: A. x2+y2=25B. x2+y2=5C. x2+y2=10D. x2+y2=50

Find the centre and radius of the circle given by $x2+y2-4x+6y-3=0$.

State the formula for the radius of a circle from the general equation $x2+y2+2gx+2fy+c=0$.

Write the equation of the tangent to the circle $x2+y2=16$at point 0.

What is the relationship between the tangent to a circle and the radius at the point of contact?

2.3 Conic Sections: Parabola, Ellipse, And Hyperbola

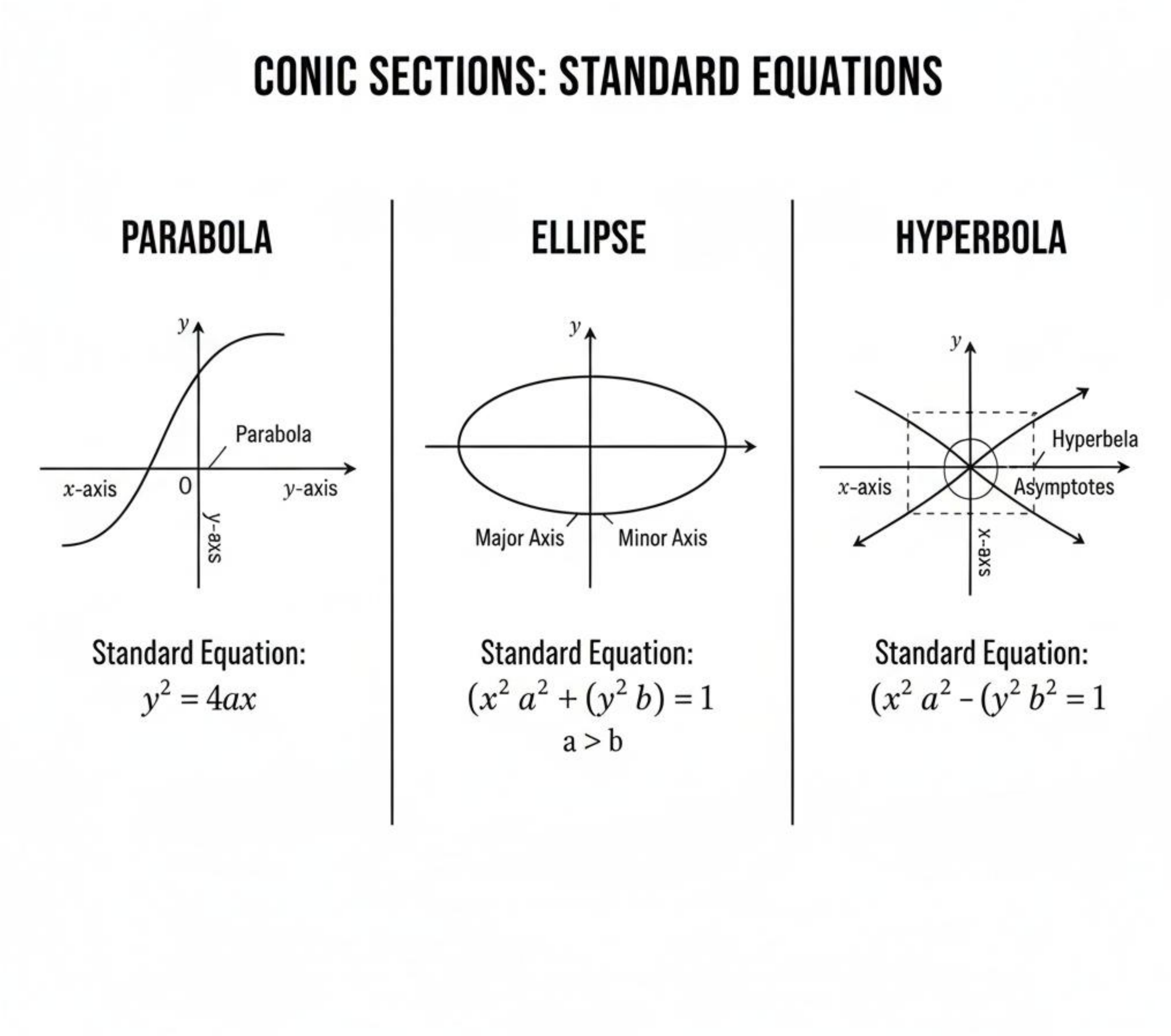
2.3.1 Standard equations of conic sections

A **conic section** is a curve obtained by intersecting a plane with a double-napped cone. The three main conic sections are the **parabola**, **ellipse**, and **hyperbola**. Their standard equations are:

Parabola: $y2=4ax$or $x2=4ay$depending on orientation.

Ellipse: $x2a2+y2b2=1$.

Hyperbola: $x2a2-y2b2=1$.



Standard equations of conic sections AI prompt suggestion: "

Fill in the blank: The standard equation of an ellipse is \frac{x^2}{a^2} + \frac{y^2}{b^2} = ______.

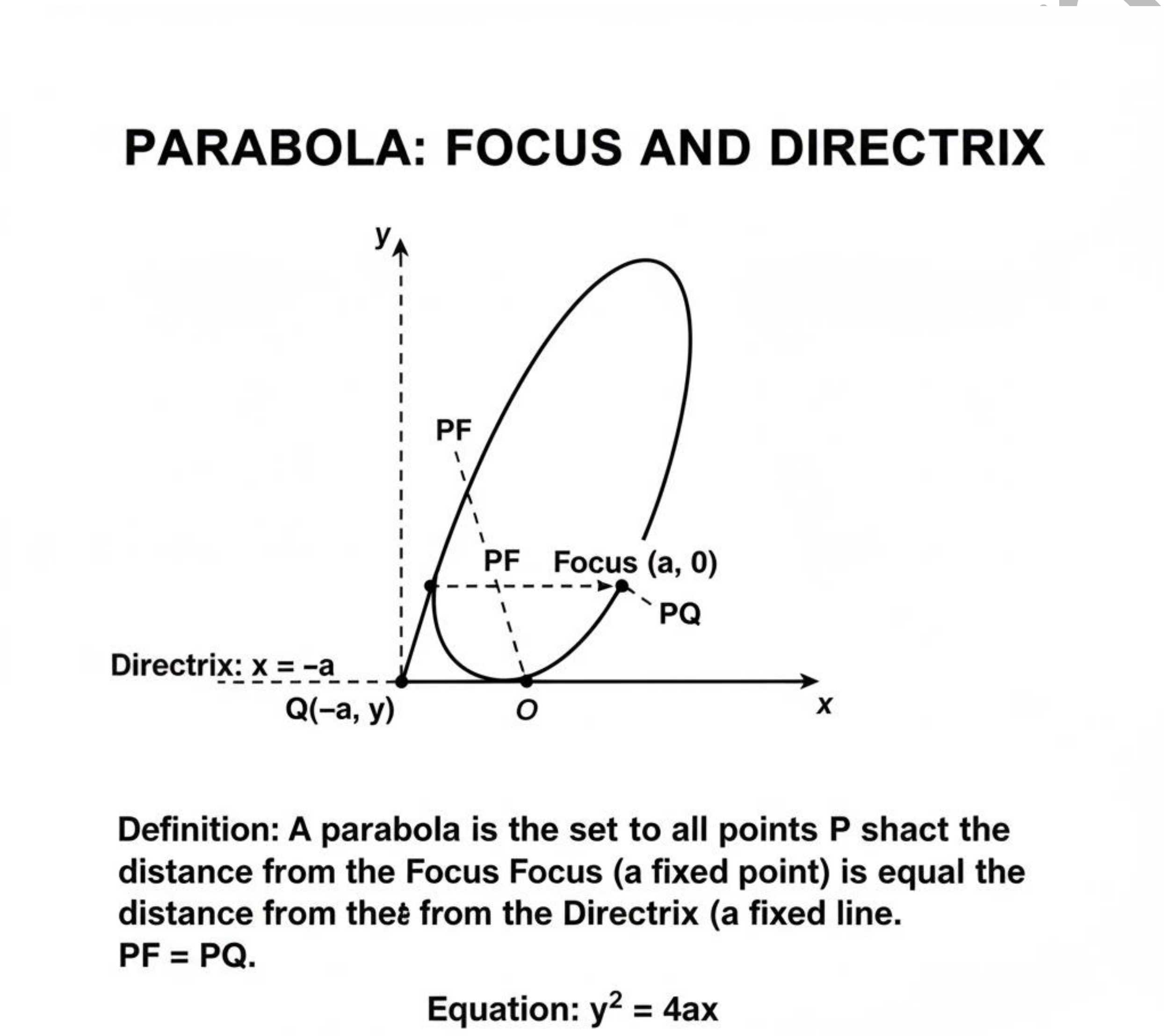
2.3.2 Geometric properties of each conic

Each conic section has unique properties:

Parabola: All points are equidistant from a fixed point (focus) and a fixed line (directrix).

Ellipse: The sum of distances from any point on the ellipse to two fixed points (foci) is constant.

Hyperbola: The difference of distances from any point on the hyperbola to two fixed points (foci) is constant.



Geometric property of a parabola AI prompt suggestion: "

Fill in the blank: In a parabola, each point is equidistant from the focus and the ______.

2.3.3 Applications of conic sections

Conic sections have wide applications in science and engineering. Parabolas are used in the design of satellite dishes and headlights because they reflect rays to a single focus. Ellipses describe planetary orbits in astronomy. Hyperbolas are used in navigation systems and in the design of cooling towers. These applications show the practical importance of conic sections beyond pure mathematics.

Applications of conic sections Short description:

Conic Section	Key Applications
Parabola	**Satellite Dishes & Reflectors:** Due to the reflective property of parabolas, all incoming parallel rays (like signals from space) are directed to a single **focus**. This is also used in car headlights and solar furnaces. **Projectile Motion:** In a vacuum, the path of any object thrown into the air follows a parabolic trajectory under the influence of gravity.
Ellipse	**Planetary Orbits:** According to Kepler's First Law, all planets move in elliptical orbits with the Sun at one focus. **Lithotripsy:** Medical devices use elliptical reflectors to focus ultrasonic shock waves at one focus to break up kidney stones located at the other focus.

Conic Section	Key Applications
	Whispering Galleries: Sound waves generated at one focus of an elliptical room reflect off the walls and converge at the other focus.
Hyperbola	**Navigation Systems (LORAN):** Long-range navigation systems use the constant difference in distance between two radio stations to determine a ship's position along a hyperbolic path. **Cooling Towers:** The hyperbolic shape of nuclear power plant cooling towers provides high structural strength with minimal material. **Sonic Booms:** The shock wave produced by a supersonic aircraft forms a cone that intersects the ground in a hyperbolic curve.

Fill in the blank: Planetary orbits are described mathematically by the ______.

Pilot questions 2.3

The standard equation of a parabola opening to the right is: A. y2=4axB. x2=4ayC. x2a2+y2b2=1D. x2a2-y2b2=1

State the geometric property of an ellipse involving its foci.

What is the geometric property of a hyperbola involving its foci?

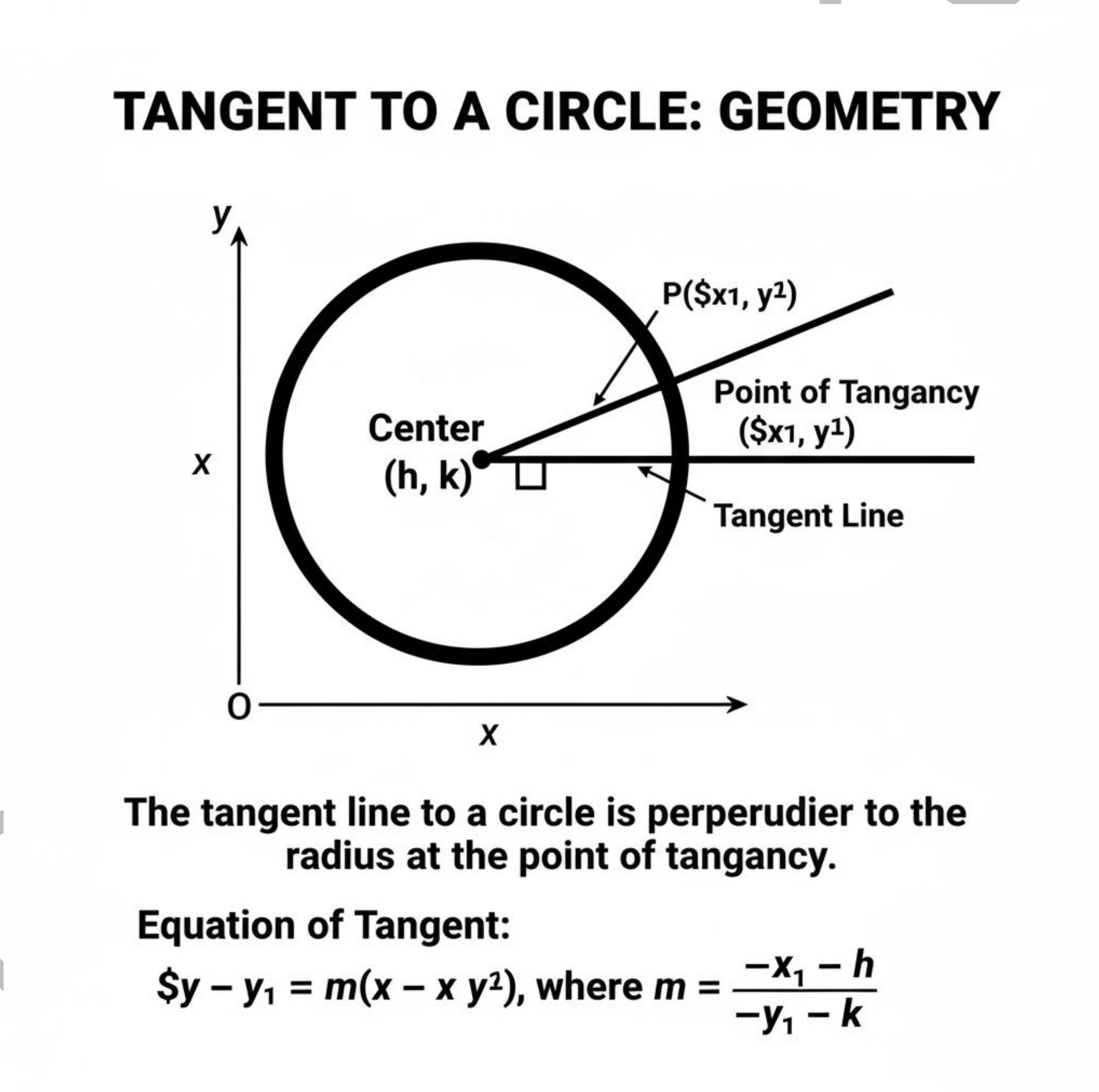
Give one real-world application of a parabola.

Which conic section is used to describe planetary orbits?

2.4 Tangents And Normals

2.4.1 Tangents to straight lines and circles

A **tangent** is a line that touches a curve at exactly one point without crossing it. For a straight line, the tangent coincides with the line itself. For a circle, the tangent at point $y_1$on the circle $x^2+y^2=r^2$is given by the equation $xx_1+yy_1=r^2$. The tangent is always perpendicular to the radius drawn to the point of contact.

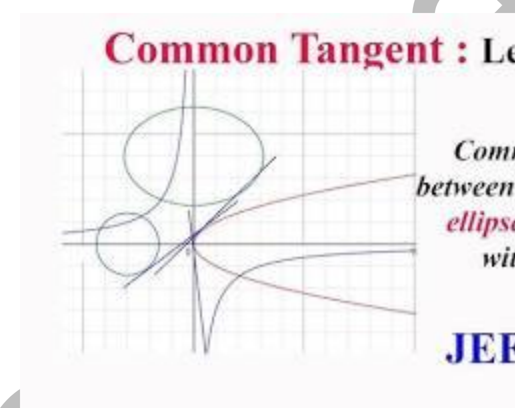


Tangent to a circle at a point

Fill in the blank: The tangent to a circle at a point is always ______ to the radius at that point.

2.4.2 Tangents to conic sections

Tangents to conic sections are important in geometry and physics. For a parabola $y2=4ax$, the tangent at point $y1$is $yy1=2a(x+x1)$. For an ellipse $x2a2+y2b2=1$, the tangent at point $y1$is $xx1a2+yy1b2=1$. For a hyperbola $x2a2-y2b2=1$, the tangent at point $y1$is $xx1a2-yy1b2=1$.



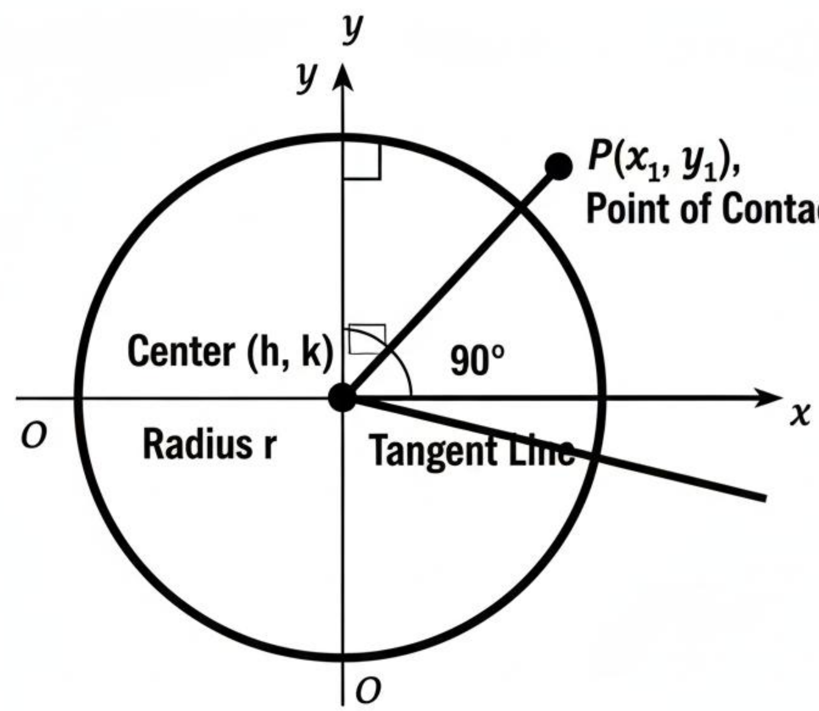
Tangents to conic sections

Fill in the blank: The tangent to the ellipse $x2a2+y2b2=1$at point $y1$is given by ______.

2.4.3 Normals and their applications

A **normal** is a line perpendicular to the tangent at the point of contact. For a circle, the normal at point $y1$passes through the centre. Normals are useful in physics and engineering, for example in calculating forces acting perpendicular to surfaces. In conic sections, normals are used in optics to study reflection and refraction.

TANGENT AND NORMAL TO A CIRCLE: GEOMETRY



The tangent line is perperuder to the normal line (and the radius) at the point of contact P.

Equation of Tangent: $y - y_1 = m_2\text{tangent}(x - y_1)$

Equation of Normal: $y - y_1 = \; m_2\text{nonmal}(x - y_1)$

Normal to a circle at a point

Fill in the blank: The normal to a circle at a point passes through the ______ of the circle.

Pilot questions 2.4

- The tangent to a circle at point y1on x2+y2=r2is: A. xx1+yy1=r2B. xx1+yy1=rC. xx1+yy1=2rD. xx1+yy1=r3
- Write the equation of the tangent to the parabola y2=4axat point y1.
- State the equation of the tangent to the ellipse x2a2+y2b2=1at point y1.
- What is the relationship between the tangent and the normal at a point on a curve?
- In a circle, where does the normal at a point always pass through?

Series Summary

This Series has guided you through the essential concepts of coordinate geometry in two dimensions. We began with straight lines, exploring their equations, slopes, intercepts, and distances. We then examined circles, focusing on their equations, centres, radii, and properties. Next, we studied conic sections—parabola, ellipse, and hyperbola—covering their standard equations, geometric properties, and real-world applications. Finally, we concluded with tangents and normals, learning how to construct and apply them to straight lines, circles, and conic sections.

By completing this Series, you should now be able to derive and interpret equations of straight lines and circles, evaluate properties of conic sections, and construct tangents and normals. These abilities directly align with the Series Learning Outcomes, ensuring you are well prepared to apply coordinate geometry in both mathematical problem-solving and practical contexts.

Glossary of Terms

- **Centre of a circle**: The fixed point equidistant from all points on the circle.
- **Conic section**: A curve obtained by intersecting a plane with a double-napped cone, including parabola, ellipse, and hyperbola.
- **Ellipse**: A conic section where the sum of distances from any point to two fixed points (foci) is constant.
- **Hyperbola**: A conic section where the difference of distances from any point to two fixed points (foci) is constant.
- **Intercept**: The point where a line crosses the x-axis or y-axis.
- **Normal**: A line perpendicular to the tangent at the point of contact on a curve.

Parabola: A conic section where each point is equidistant from a fixed point (focus) and a fixed line (directrix).

Radius: The distance from the centre of a circle to any point on the circle.

Slope: The measure of steepness of a line, defined as the ratio of vertical change to horizontal change.

Tangent: A line that touches a curve at exactly one point without crossing it.

Concept Check Questions 2

Objective questions

The general equation of a straight line in two dimensions is: A. ax+by+c=0B. ax2+by+c=0C. ax+by2+c=0D. ax+by+cz=0(Linked to SLO 3.1)

The tangent to a circle at point y1on x2+y2=r2is: A. xx1+yy1=r2B. xx1+yy1=rC. xx1+yy1=2rD. xx1+yy1=r3(Linked to SLO 3.4)

Short-answer questions

Write the slope-intercept form of a line with slope 3 and y-intercept -2. (Linked to SLO 3.1)

Find the centre and radius of the circle given by x2+y2-6x+4y-12=0. (Linked to SLO 3.2)

Long-answer questions

Derive the equation of the tangent to the ellipse x2a2+y2b2=1at point y1. (Linked to SLO 3.4)

Basic response: Show the derivation using substitution and tangent condition.

Value-added response: Extend the explanation to applications in optics, such as reflection properties of ellipses.

Explain the geometric properties of parabola, ellipse, and hyperbola, and discuss one real-world application of each. (Linked to SLO 3.3)

Basic response: Define each property clearly.

Value-added response: Connect each property to practical applications in engineering, astronomy, and navigation.

Answers to Concept Check Questions 2

A. ax+by+c=0

A. xx1+yy1=r2

y=3x-2

Centre 3,-2, radius 5

Basic response: xx1a2+yy1b2=1. **Value-added response**: Elliptical mirrors use this property to focus light at one of the foci.

Basic response: Parabola—equidistant from focus and directrix; Ellipse—sum of distances to foci constant; Hyperbola—difference of distances to foci constant. **Value-added response**: Parabola in satellite dishes, ellipse in planetary orbits, hyperbola in navigation systems.

Answers to Pilot Questions 2

From Part 2.1

ax+by+c=0

Slope = 11-35-2=83

y=2x-3

d=(x2-x1)2+(y2-y1)2

d=|3(2)+4(1)-5|32+42=|6+4-5|5=1

From Part 2.2

x2+y2=25

Centre 2,-3, radius 4

Radius = g2+f2-c

Tangent: xx1+yy1=r2⇒4x=16

Perpendicular to the radius

From Part 2.3

y2=4ax

Sum of distances from any point to the two foci is constant

Difference of distances from any point to the two foci is constant

Satellite dishes

Ellipse

From Part 2.4

$xx1+yy1=r2$

$yy1=2a(x+x1)$

$xx1a2+yy1b2=1$

They are perpendicular at the point of contact

The centre of the circle

Series 3: Coordinate Geometry In Three Dimensions

Series Outline

This Series builds upon your knowledge of two-dimensional coordinate geometry and extends it into three dimensions. It introduces the representation of points, lines, and planes in space, along with their equations and relationships. Learners will also explore distances, angles, and applications in geometry and physics.

Proposed Parts and Sub-Parts

3.1 Points and distances in three dimensions

3.1.1 Coordinates of a point in 3D space

3.1.2 Distance between two points

3.1.3 Section formula (internal and external division)

3.2 Straight lines in three dimensions

3.2.1 Equation of a line in vector form

3.2.2 Equation of a line in parametric form

3.2.3 Angle between two lines

3.3 Planes in three dimensions

3.3.1 General equation of a plane

3.3.2 Normal form of a plane

3.3.3 Angle between two planes

Introduction

Three-dimensional coordinate geometry extends the concepts of points, lines, and curves into space, allowing us to represent and analyse geometric figures beyond the plane. This branch of mathematics is essential for understanding spatial relationships and is widely applied in physics, engineering, and computer graphics.

The aim of this Series is to enable you to represent points, lines, and planes in three dimensions, calculate distances and angles between them, and apply these concepts to solve geometric and physical problems.

We shall begin by studying points and distances in three dimensions, including the section formula. Next, we will examine straight lines in space, focusing on their equations and angles. Following that, we will explore planes, learning their equations and relationships. Finally, we will conclude with distances and angles involving points, lines, and planes, highlighting their applications in geometry and dynamics.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 4.1 identify and represent points in three-dimensional space, calculate distances between points, and apply the section formula for internal and external division,

SLO 4.2 derive equations of straight lines in vector and parametric forms, and determine the angle between two lines,

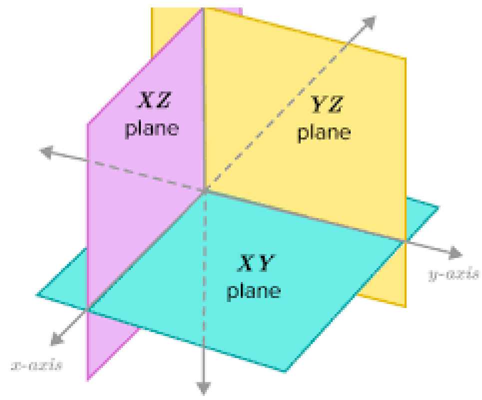
SLO 4.3 formulate equations of planes in general and normal forms, and evaluate the angle between two planes,

SLO 4.4 calculate distances involving points and planes, determine distances between parallel planes, and analyse the angle between a line and a plane.

3.1 Points And Distances In Three Dimensions

3.1.1 Coordinates of a point in 3D space

In three-dimensional geometry, a point is represented by an ordered triple z, where x, y, and zare its coordinates along the three mutually perpendicular axes. These axes are usually labelled as the **x-axis**, **y-axis**, and **z-axis**, forming the 3D Cartesian coordinate system.



Representation of a point in three-dimensional space AI prompt suggestion: “

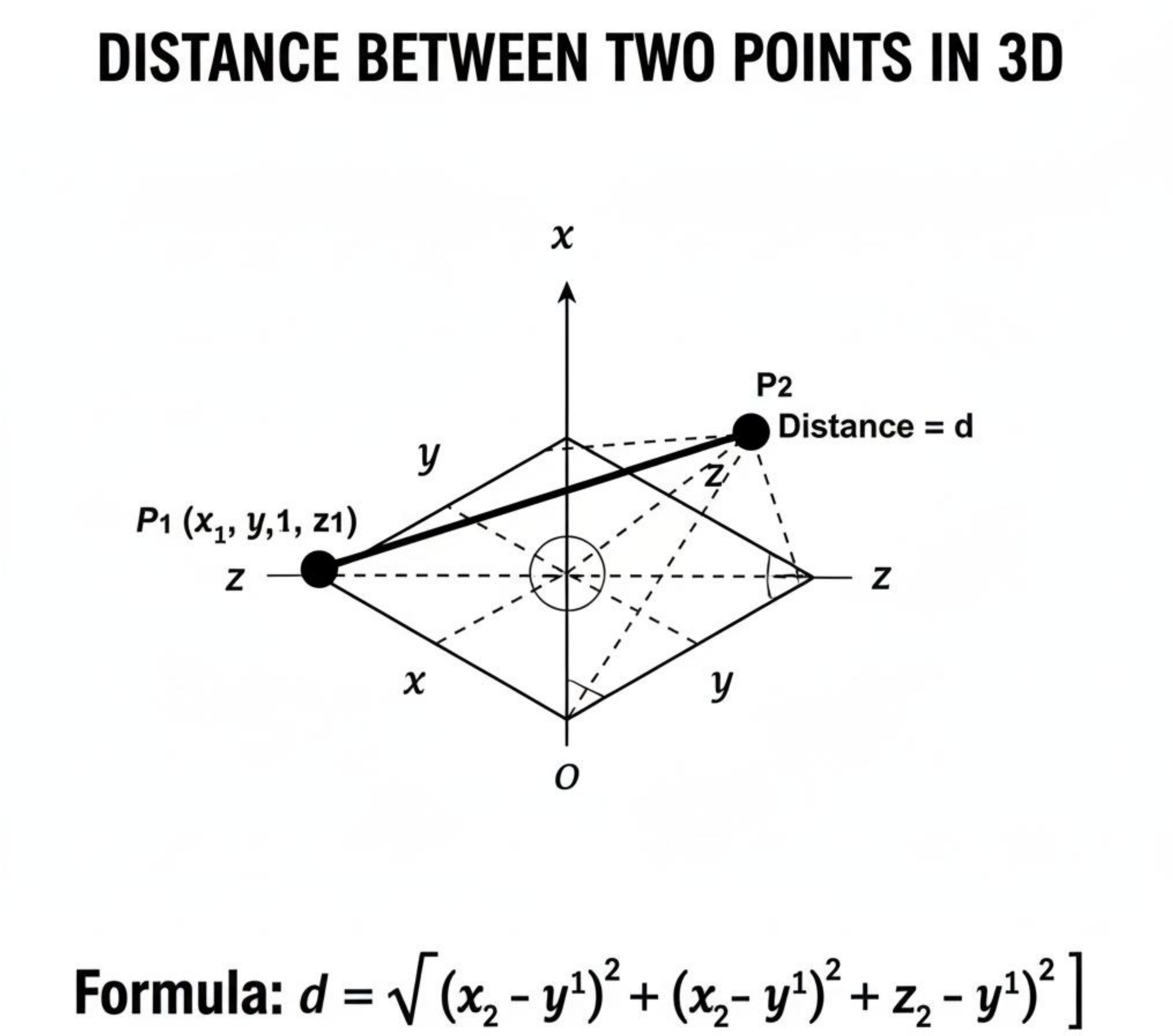
Fill in the blank: A point in three-dimensional space is represented by the ordered triple ______.

3.1.2 Distance between two points

The **distance between two points** z1and z2in 3D space is given by the formula:

$$d=(x2-x1)2+(y2-y1)2+(z2-z1)2.$$

This formula extends the two-dimensional distance formula by including the z-coordinate.



Distance between two points in three-dimensional space AI prompt suggestion: "

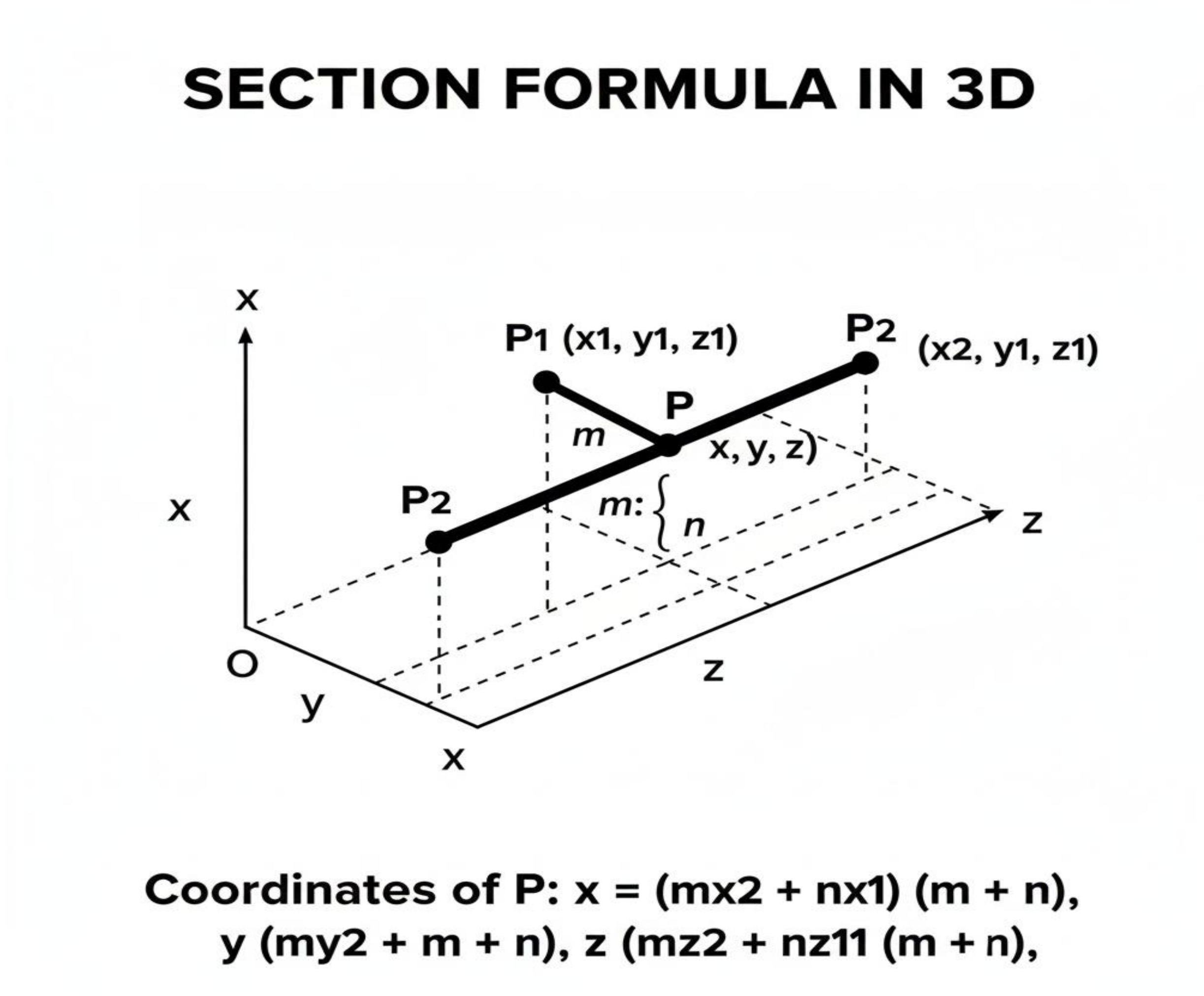
Fill in the blank: The distance between two points in 3D space is found using the ______ formula.

3.1.3 Section formula (internal and external division)

The **section formula** is used to find the coordinates of a point dividing a line segment between two points in a given ratio. If point P(x,y,z)divides the line segment joining z1and z2in the ratio m:n, then

$$P=mz2+nz1m+n.$$

This is the **internal division formula**. For external division, the denominators change to m-n.



Section formula in three-dimensional space AI prompt suggestion: “”

Fill in the blank: The section formula gives the coordinates of a point dividing a line segment in a given ______.

Pilot Questions 3.1

A point in three-dimensional space is represented by: A. yB. zC. wD. t

Find the distance between points 3and 8.

State the formula for the distance between two points in 3D space.

If point Pdivides the line segment joining 2,-1,3and 4,3,-1internally in the ratio 2:3, find the coordinates of P.

What is the difference between internal and external division in the section formula?

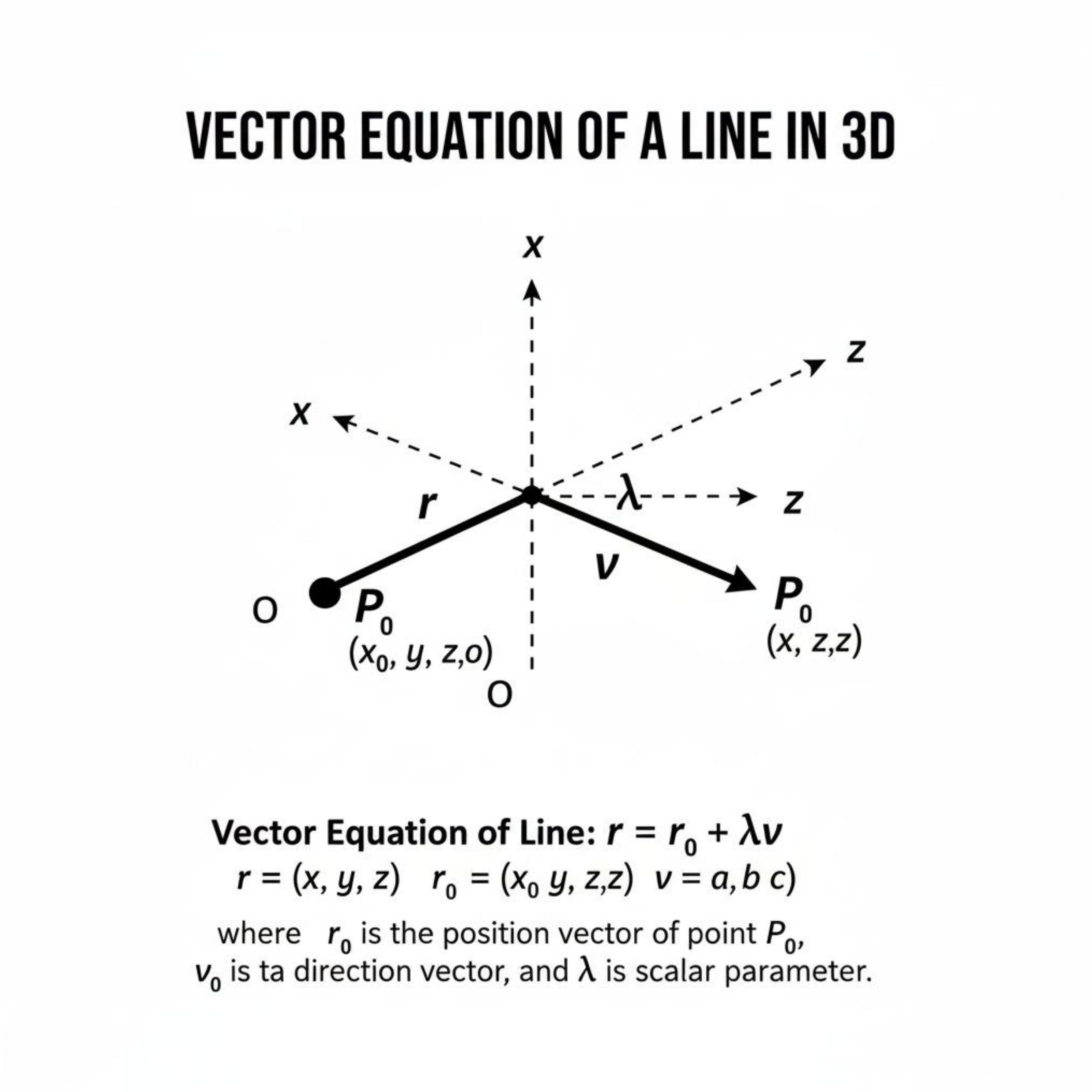
3.2 Straight Lines In Three Dimensions

3.2.1 Equation of a line in vector form

In three-dimensional space, a straight line can be represented using vectors. If a line passes through a point $A(x1,y1,z1)$and is parallel to a vector $d=(l,m,n)$, then its vector equation is:

$$r=a+\lambda d,$$

where $a=x1i+y1j+z1k$is the position vector of point A, dis the direction vector, and is a scalar parameter.



Vector form of a line in three dimensions AI prompt suggestion: "

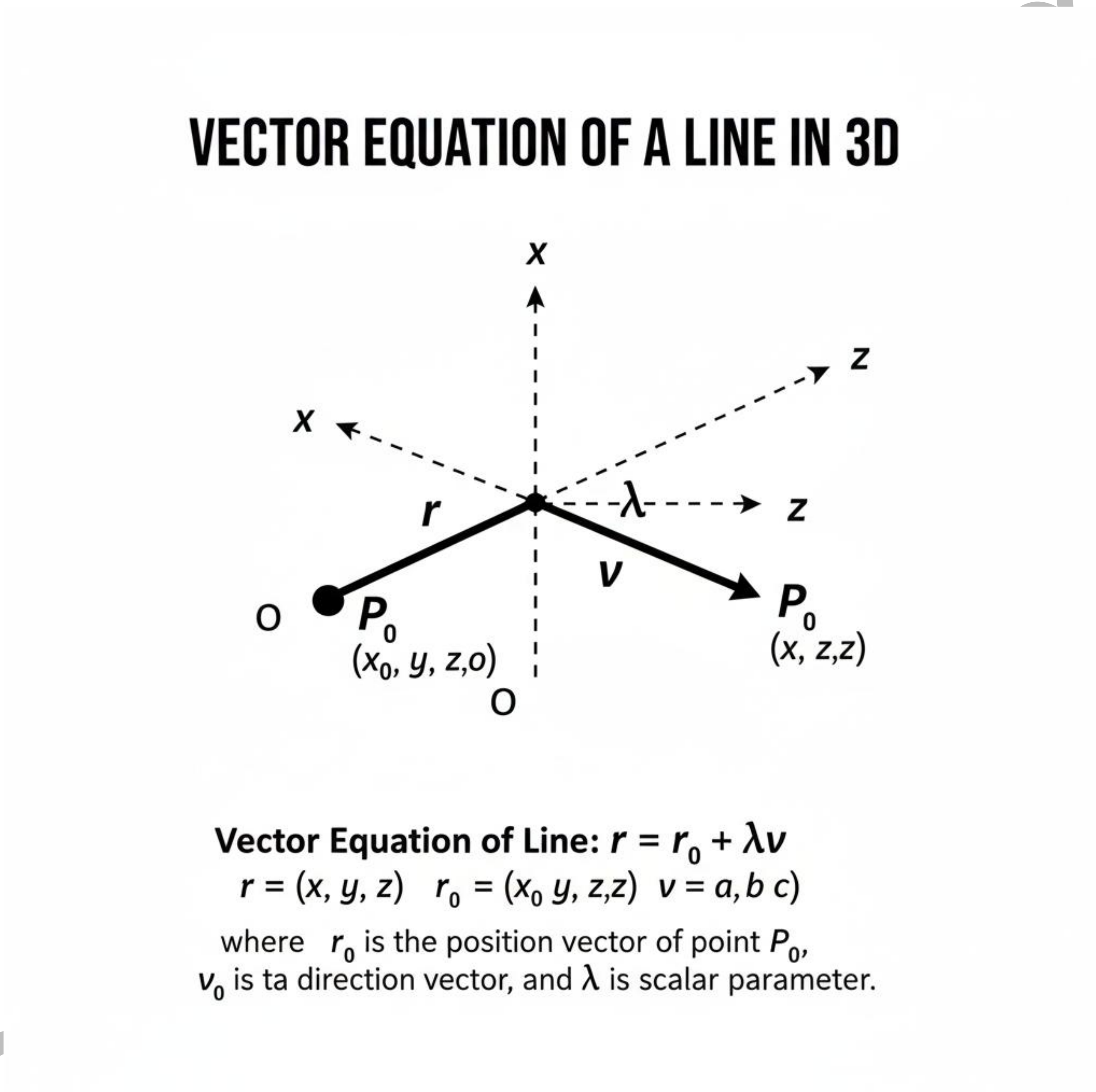
Fill in the blank: The vector equation of a line is r=a+λd, where ais the ______ vector.

3.2.2 Equation of a line in parametric form

The parametric form of a line expresses the coordinates of points on the line in terms of a parameter. If a line passes through point z1and has direction ratios n, then its parametric equations are:

$$x=x1+\lambda l,y=y1+\lambda m,z=z1+\lambda n.$$

This form is particularly useful for solving problems involving intersections and projections.



Parametric form of a line in three dimensions “

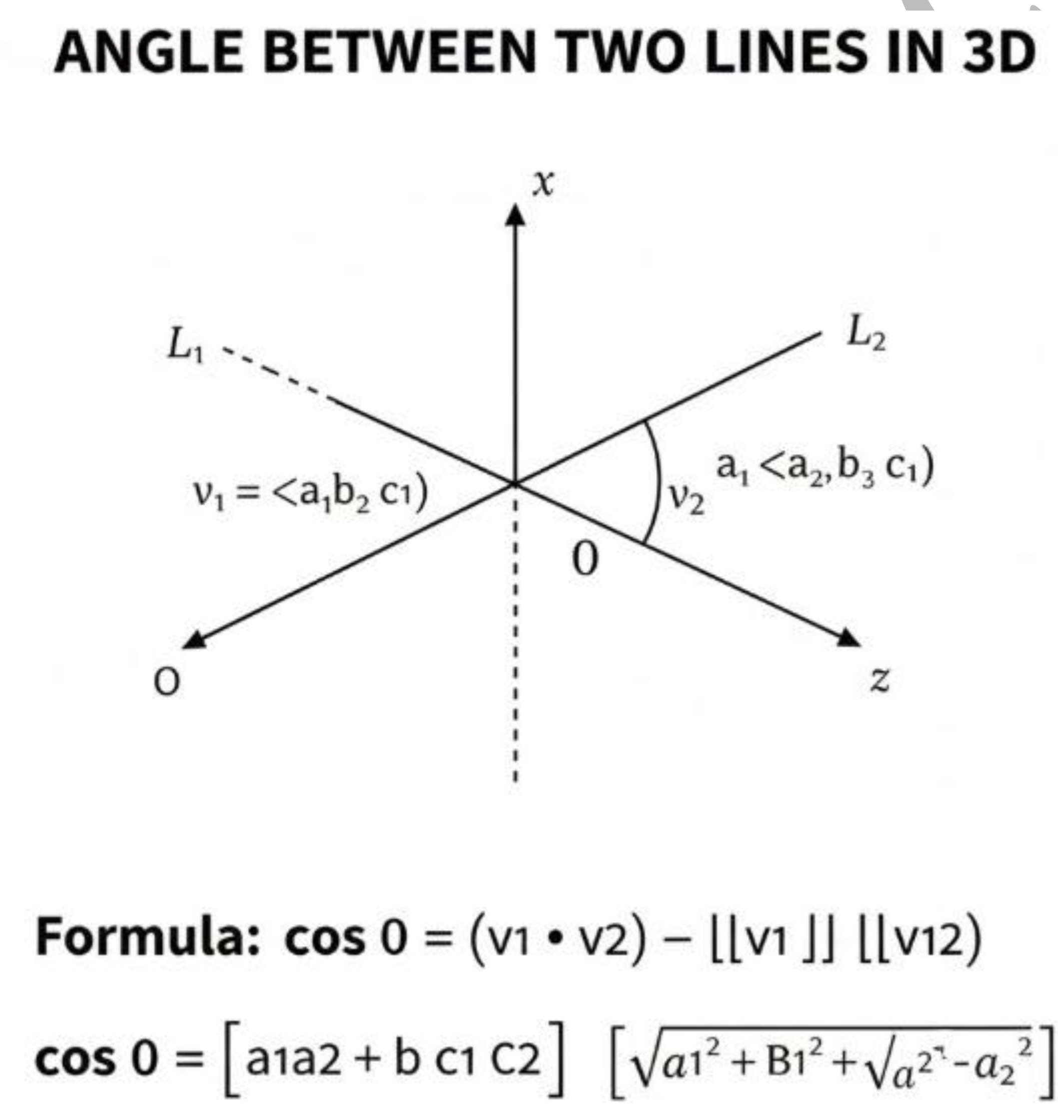
Fill in the blank: The parametric equation of a line passing through z1with direction ratios nis x=x1+λl,y=y1+λm,z=z1+λn.

3.2.3 Angle between two lines

The angle between two lines in 3D space can be found using their direction vectors. If the lines have direction vectors d1=(l1,m1,n1)and d2=(l2,m2,n2), then the cosine of the angle between them is:

$$\cos\theta=l1l2+m1m2+n1n2l12+m12+n12l22+m22+n22.$$

This formula is derived from the dot product of the direction vectors.



Angle between two lines in three dimensions

Fill in the blank: The angle between two lines in 3D space is calculated using the ______ product of their direction vectors.

Pilot Questions 3.2

The vector equation of a line passing through point 3and parallel to vector 2,-1,4is: A. r=(1i+2j+3k)+λ(2i-j+4k)B. r=(2i-j+4k)+λ(1i+2j+3k)C. r=(1i+2j+3k)-λ(2i-j+4k)D. None of the above

Write the parametric equations of the line passing through 0,1,-2with direction ratios 1.

State the formula for the cosine of the angle between two lines in 3D space.

Find the angle between lines with direction vectors 0and 0.

Explain why the dot product is used to calculate the angle between two lines.

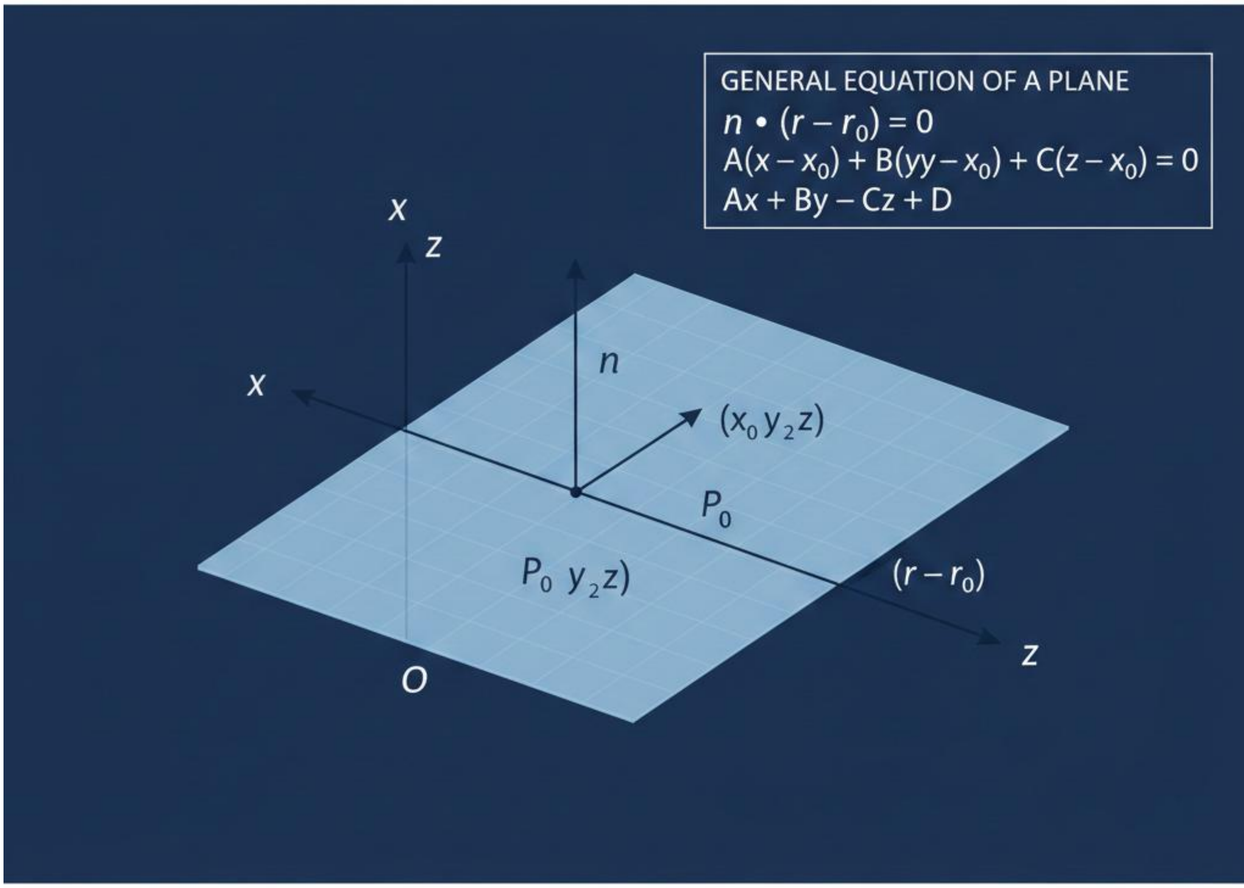
3.3 Planes In Three Dimensions

3.3.1 General equation of a plane

A **plane** in three-dimensional space can be represented by the general equation:

$$ax+by+cz+d=0,$$

where a,b,care the direction ratios of the normal to the plane, and dis a constant. This equation describes all points zlying on the plane.



General equation of a plane in three dimensions

Fill in the blank: The general equation of a plane in 3D space is ______.

3.3.2 Normal form of a plane

The **normal form** of a plane equation expresses the plane in terms of its perpendicular distance from the origin and the direction cosines of the normal. It is given by:

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p,$$

where pis the perpendicular distance from the origin, and $\cos\alpha,\cos\beta,\cos\gamma$are the direction cosines of the normal to the plane.

Normal form of a plane equation AI prompt suggestion: "

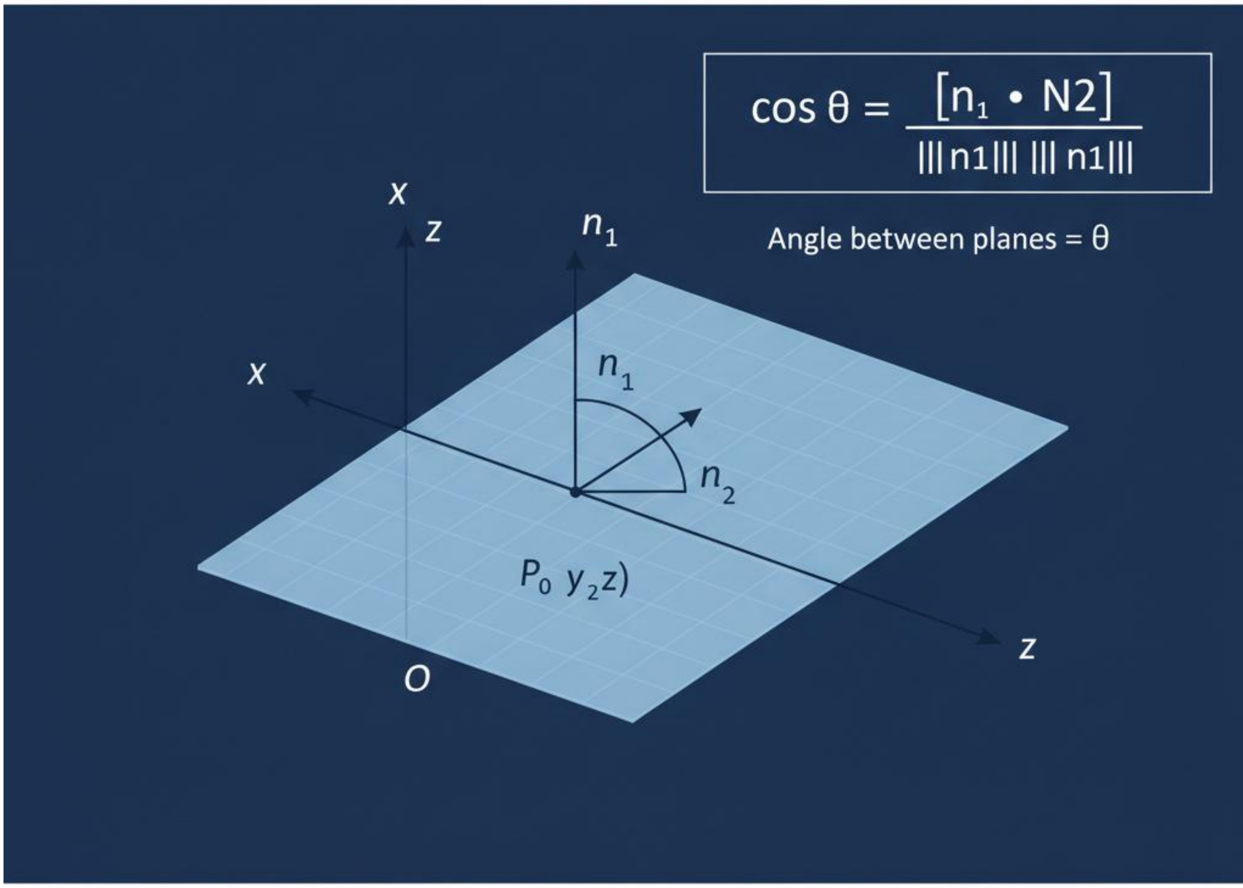
Fill in the blank: In the normal form of a plane, prepresents the ______ from the origin to the plane.

3.3.3 Angle between two planes

The angle between two planes is determined by the angle between their normal vectors. If the planes have equations $a1x+b1y+c1z+d1=0$and $a2x+b2y+c2z+d2=0$, then the cosine of the angle between them is:

$$\cos\theta=a1a2+b1b2+c1c2a12+b12+c12a22+b22+c22.$$

This formula is derived from the dot product of the normal vectors of the planes.



Angle between two planes in three dimensions

Fill in the blank: The angle between two planes is calculated using the ______ product of their normal vectors.

Pilot Questions 3.3

The general equation of a plane in 3D space is: A. ax+by+cz+d=0B. ax+by+cz=1C. ax2+by2+cz2=0D. ax+by+cz+d=1

Write the normal form of a plane at distance 5 units from the origin with direction cosines 0.

State the formula for the cosine of the angle between two planes.

Find the angle between the planes x+y+z=0and 2x-y+2z=5.

Explain why the dot product is used to calculate the angle between two planes.

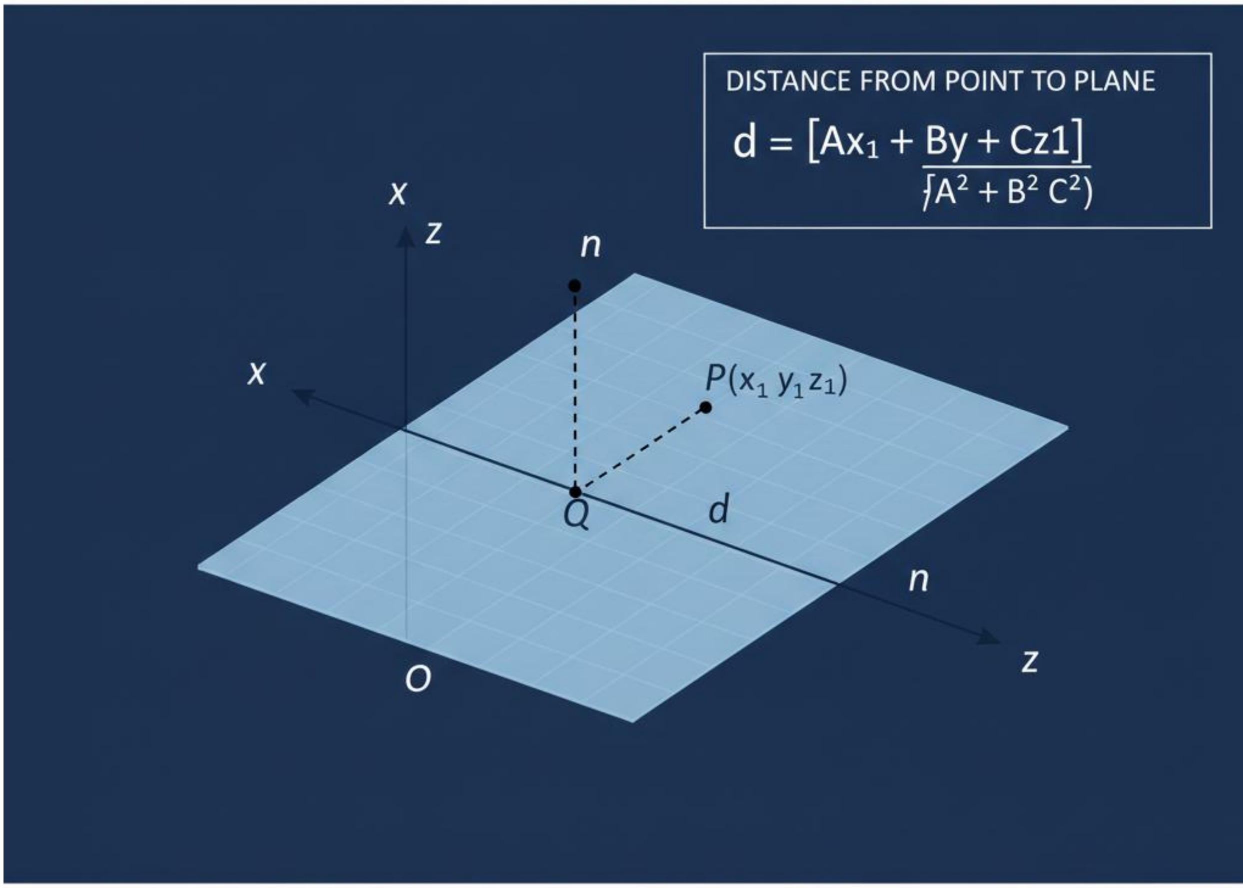
3.4 Distances And Angles In Three Dimensions

3.4.1 Distance of a point from a plane

The **distance of a point** z1from a plane ax+by+cz+d=0is given by:

$$d=|ax1+by1+cz1+d|a2+b2+c2.$$

This formula calculates the perpendicular distance from the point to the plane.



Distance of a point from a plane

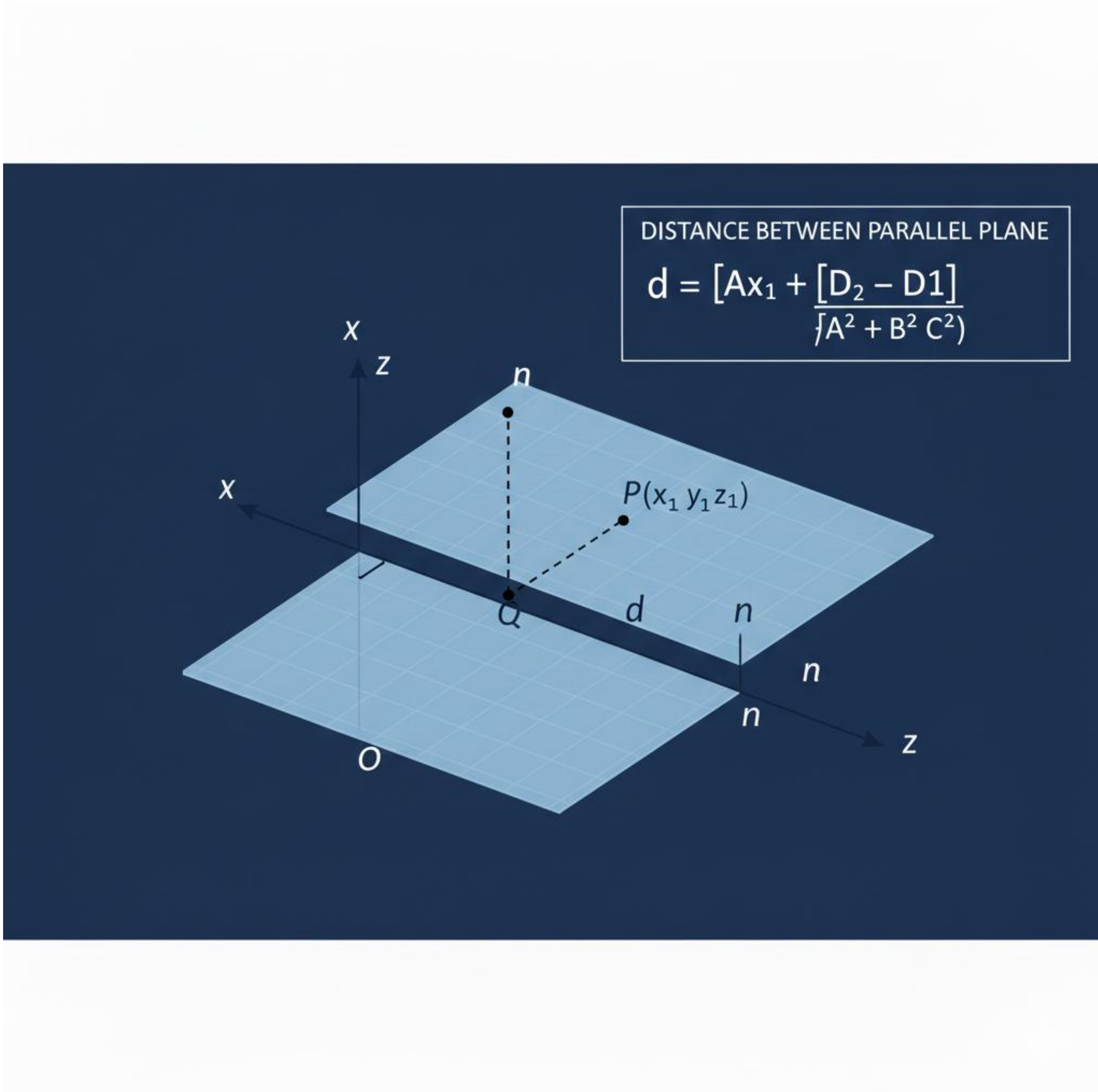
Fill in the blank: The distance of a point from a plane is calculated using the ______ formula.

3.4.2 Distance between parallel planes

If two planes are parallel, their normal vectors are proportional. The distance between two parallel planes ax+by+cz+d1=0and ax+by+cz+d2=0is:

$$d=|d1-d2|a2+b2+c2.$$

This formula gives the shortest distance between the two planes.



Distance between parallel planes

Fill in the blank: The distance between two parallel planes is given by |d1-d2|a2+b2+c2.

3.4.3 Angle between a line and a plane

The angle between a line and a plane is the complement of the angle between the line's direction vector and the plane's normal vector. If the line has direction vector nand the plane has normal vector c, then:

$$\sin\theta=|al+bm+cn|a2+b2+c2l2+m2+n2,$$

where is the angle between the line and the plane.

[Insert figure here] Short description: Diagram showing a line intersecting a plane with angle between them. Caption: Figure 3.12 Angle between a line and a plane AI prompt suggestion: "Generate a diagram of a line intersecting a plane in 3D space with angle θ between them." Search string: "angle between line and plane 3D coordinate geometry diagram OER"

Fill in the blank: The angle between a line and a plane is the complement of the angle between the line's direction vector and the plane's ______ vector.

Pilot Questions 3.4

The distance of point 3from plane 2x+y-2z+5=0is: A. |2(1)+2-2(3)+5|22+12+(-2)2B. |2(1)+1(2)-2(3)+5|22+12+(-2)2C. |2+2-6+5|3D. Both A and B

State the formula for the distance between two parallel planes.

Find the distance between planes x+y+z-4=0and x+y+z+2=0.

Write the formula for the sine of the angle between a line and a plane.

Explain why the angle between a line and a plane is related to the dot product of their vectors.

Series Summary

This Series extended coordinate geometry into three dimensions, enabling representation and analysis of points, lines, and planes in space. We began with points and distances, including the section formula for internal and external division. Next, we studied straight lines in 3D, deriving their vector and parametric equations and calculating angles between lines. We then explored planes, learning their general and normal forms and determining angles between planes. Finally, we examined distances and angles in 3D, including the distance of a point from a plane, distance between parallel planes, and the angle between a line and a plane.

By completing this Series, you should now be able to represent points, lines, and planes in three dimensions, calculate distances and angles between them, and apply these concepts to solve geometric and physical problems.

Glossary of Terms

Direction ratios: Numbers proportional to the components of a vector that indicate the direction of a line.

Normal vector: A vector perpendicular to a plane, used to define its orientation.

Parametric equations: Equations that express coordinates of points on a line in terms of a parameter.

Plane: A flat, two-dimensional surface extending infinitely in three-dimensional space.

Position vector: A vector representing the location of a point relative to the origin.

Section formula: Formula used to find coordinates of a point dividing a line segment in a given ratio.

Vector equation of a line: Representation of a line using a position vector and a direction vector.

Concept Check Questions 3

Objective questions

The vector equation of a line is: A. r=a+λdB. r=d+λaC. r=a-λdD. None of the above (Linked to SLO 4.2)

The general equation of a plane is: A. ax+by+cz+d=0B. ax+by+cz=1C. ax2+by2+cz2=0D. None of the above (Linked to SLO 4.3)

Short-answer questions

Write the distance formula between two points in 3D space. (Linked to SLO 4.1)

State the formula for the distance between two parallel planes. (Linked to SLO 4.4)

Long-answer questions

Derive the parametric equations of a line passing through a point with given direction ratios. (Linked to SLO 4.2)

Basic response: Show the derivation using coordinates and parameter .

Value-added response: Explain how parametric equations are applied in intersection problems.

Explain the geometric meaning of the normal form of a plane and discuss one application in physics or engineering. (Linked to SLO 4.3)

Basic response: Define the normal form and its components.

Value-added response: Connect the concept to applications such as calculating forces perpendicular to surfaces.

Answers to Concept Check Questions 3

A. r=a+λd

A. ax+by+cz+d=0

d=(x2-x1)2+(y2-y1)2+(z2-z1)2

d=|d1-d2|a2+b2+c2

Basic response: x=x1+λl,y=y1+λm,z=z1+λn. **Value-added response**: Used to find intersection points of lines with planes.

Basic response: xcos α+ycos β+zcos γ=p, where pis perpendicular distance from origin. **Value-added response**: In physics, used to calculate pressure forces acting perpendicular to surfaces.

Answers to Pilot Questions 3

From Part 3.1

z

d=(4-1)2+(6-2)2+(8-3)2=9+16+25=50=52

d=(x2-x1)2+(y2-y1)2+(z2-z1)2

P=2(-1)+3(3)5=(145,35,75)

Internal division uses m+nin denominator, external division uses m-n.

From Part 3.2

A. r=(1i+2j+3k)+λ(2i-j+4k)

x=0+3λ,y=1+2λ,z=-2+λ

cos θ=l1l2+m1m2+n1n2l12+m12+n12l22+m22+n22

Angle = 90

Because the dot product measures the cosine of the angle between vectors.

From Part 3.3

A. $ax+by+cz+d=0$

$x\cdot 35+y\cdot 45+z\cdot 0=5$

$\cos\theta=a1a2+b1b2+c1c2a12+b12+c12a22+b22+c22$

Normals: 1and 2,-1,2. Dot product = $1(2)+1(-1)+1(2)=3$. Magnitudes = 3and 3. So $\cos\theta=333=13$. Angle = 60.

Because the angle between planes is defined by the angle between their normals.

From Part 3.4

Distance = $|2(1)+1(2)-2(3)+5|9=|2+2-6+5|3=33=1$.

$d=|d1-d2|a2+b2+c2$

Distance = $|(-4)-(2)|12+12+12=|-6|3=63=23$.

$\sin\theta=|al+bm+cn|a2+b2+c2l2+m2+n2$

Because the dot product relates the orientation of the line’s direction vector to the plane’s normal vector.

Series 4: Differential Calculus

Series Outline

This Series introduces the fundamental concepts of **differential calculus**, focusing on limits, continuity, derivatives, and applications. It builds on your knowledge of vectors and coordinate geometry, preparing you to analyse rates of change and solve optimization problems.

Proposed Parts and Sub-Parts

4.1 Limits and continuity

4.1.1 Concept of a limit

4.1.2 Evaluating limits

4.1.3 Continuity of functions

4.2 Derivatives of functions

4.2.1 Definition of derivative

4.2.2 Techniques of differentiation (power rule, product rule, quotient rule, chain rule)

4.2.3 Higher-order derivatives

4.3 Applications of derivatives

4.3.1 Tangents and normals to curves

4.3.2 Increasing and decreasing functions

4.3.3 Maxima and minima

4.4 Special functions and derivatives

4.4.1 Derivatives of trigonometric functions

4.4.2 Derivatives of exponential and logarithmic functions

4.4.3 Implicit differentiation

Introduction

Differential calculus is the study of how functions change. It provides tools to measure instantaneous rates of change, slopes of curves, and sensitivity of systems. The concept of a derivative lies at the heart of calculus, connecting algebraic functions with geometric interpretations.

The aim of this Series is to enable you to understand limits and continuity, compute derivatives using standard techniques, and apply derivatives to solve problems involving tangents, normals, and optimization.

We shall begin with limits and continuity, establishing the foundation for differentiation. Next, we will define derivatives and explore techniques of differentiation. Following that, we will apply derivatives to tangents, normals, and maxima/minima problems. Finally, we will study derivatives of special functions, including trigonometric, exponential, and logarithmic functions, as well as implicit differentiation.

Series Learning Outcomes – Differential Calculus

Upon completing this Series, you should be able to:

SLO 5.1 explain the concept of limits, evaluate limits of functions, and determine continuity at a point or over an interval,

SLO 5.2 define the derivative of a function, apply standard differentiation techniques (power rule, product rule, quotient rule, chain rule), and compute higher-order derivatives,

SLO 5.3 apply derivatives to geometric problems, including finding tangents and normals to curves, and to analytical problems such as identifying increasing/decreasing functions and determining maxima and minima,

SLO 5.4 differentiate special functions, including trigonometric, exponential, and logarithmic functions, and perform implicit differentiation when functions are not explicitly defined.

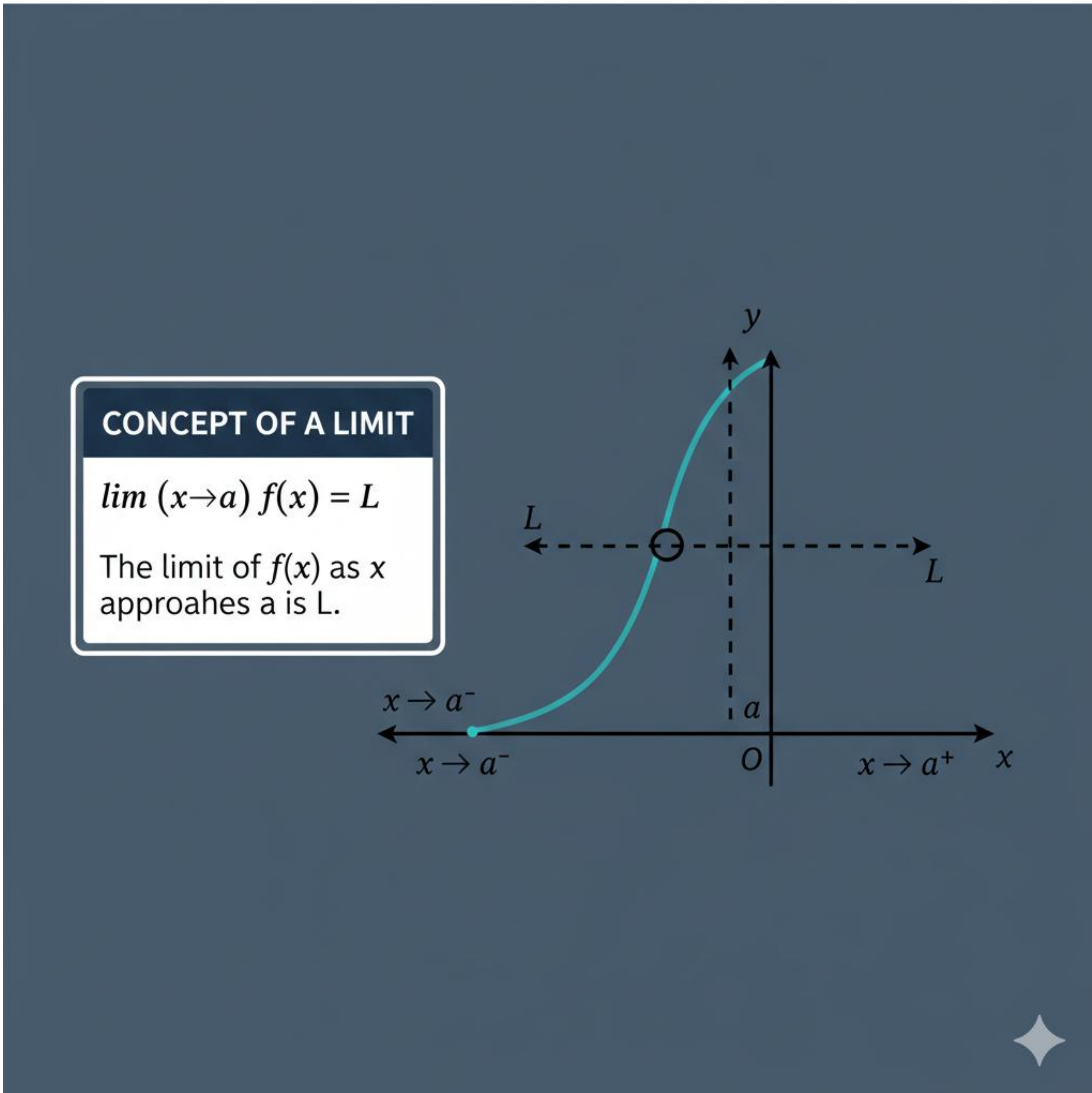
4.1 Limits And Continuity

4.1.1 Concept of a limit

A **limit** describes the value a function approaches as the input approaches a particular point. Formally,

$$\lim_{x \to a} f(x) = L$$

means that as x gets arbitrarily close to a, the function $f(x)$ gets arbitrarily close to L. Limits are the foundation of calculus, allowing us to define derivatives and integrals.



Concept of a limit

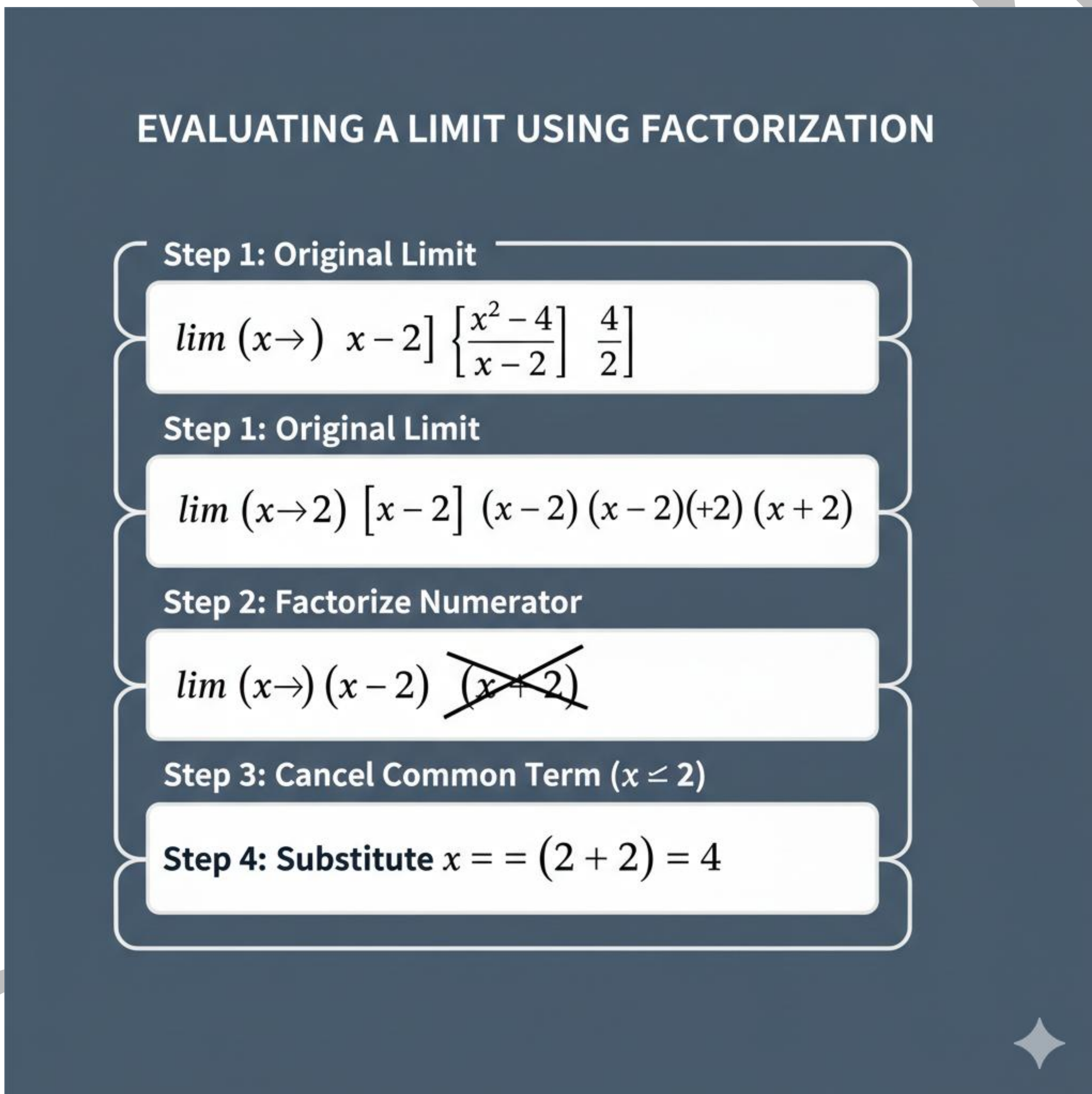
Fill in the blank: A limit describes the value a function ______ as the input approaches a particular point.

4.1.2 Evaluating limits

Limits can be evaluated using substitution, factorization, rationalization, or special techniques like L'Hôpital's Rule. For example:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4.$$

This shows how algebraic manipulation helps resolve indeterminate forms.



Evaluating a limit by factorization

Fill in the blank: L'Hôpital's Rule is used to evaluate limits of the form ______.

4.1.3 Continuity of functions

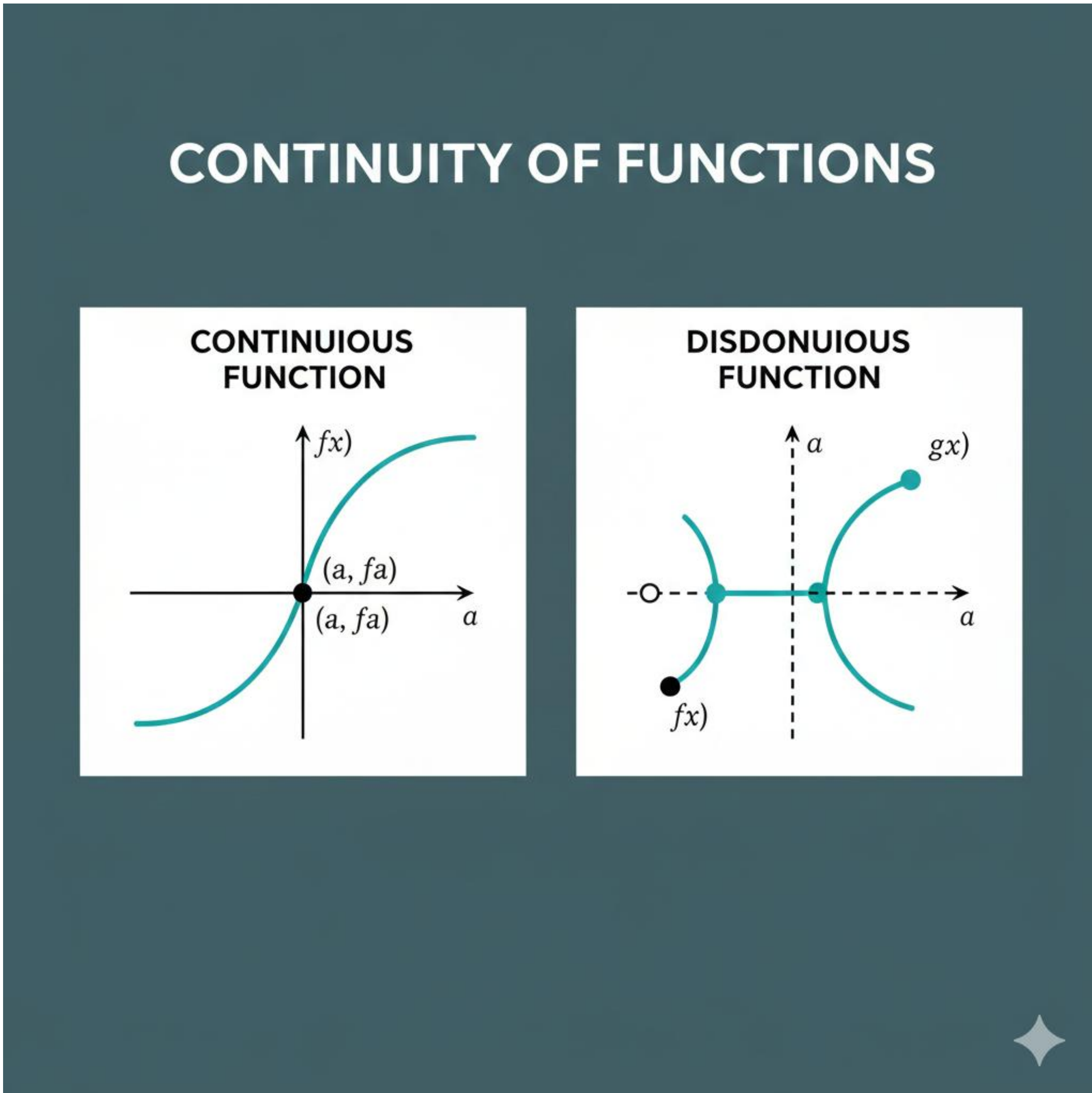
A function is **continuous** at a point $x=a$ if:

$f(a)$ is defined,

$\lim_{x\to a} f(x)$ exists,

$\lim_{x\to a} f(x)=f(a)$.

If these conditions hold for all points in an interval, the function is continuous on that interval. Discontinuities occur when one or more conditions fail.



Continuity and discontinuity of functions

Fill in the blank: A function is continuous at $x=a$ if $\lim_{x \to a} f(x) =$ ______.

Pilot Questions 4.1

The limit $\lim_{x\to 3}(2x+1)$ is: A. 5 B. 7 C. 6 D. 4

Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

State the three conditions for continuity of a function at a point.

Give an example of a function that is discontinuous at x=0.

Explain why limits are essential for defining derivatives.

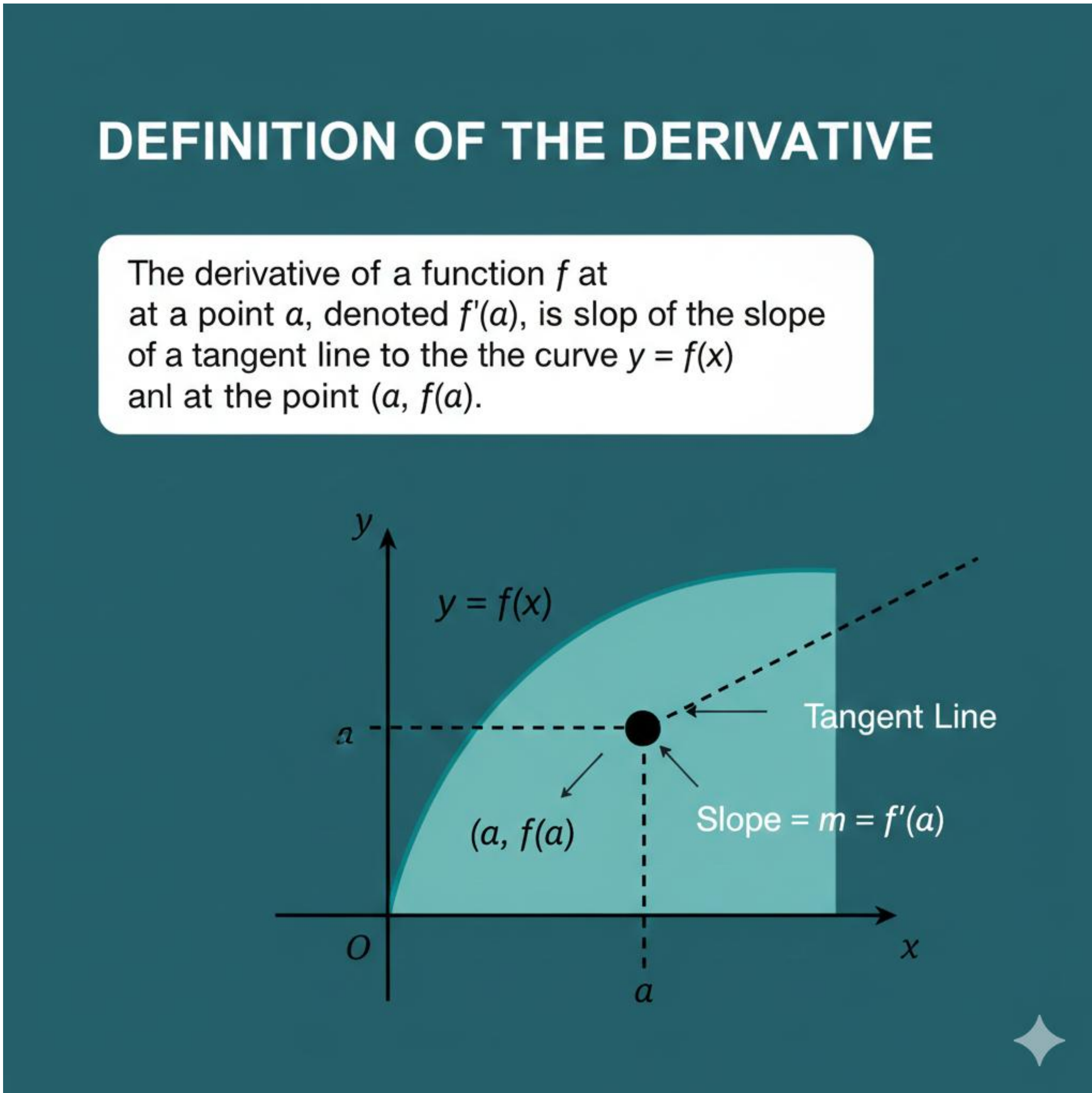
4.2 Derivatives Of Functions

4.2.1 Definition of derivative

The **derivative** of a function measures its instantaneous rate of change. Formally, the derivative of f(x)at x=ais defined as:

$$f'(a)=\lim h\to 0f(a+h)-f(a)h.$$

Geometrically, this represents the slope of the tangent to the curve at the point a,f(a).



Definition of derivative as slope of tangent

Fill in the blank: The derivative of a function at a point is the slope of the ______ at that point.

4.2.2 Techniques of differentiation

Several rules simplify the process of differentiation:

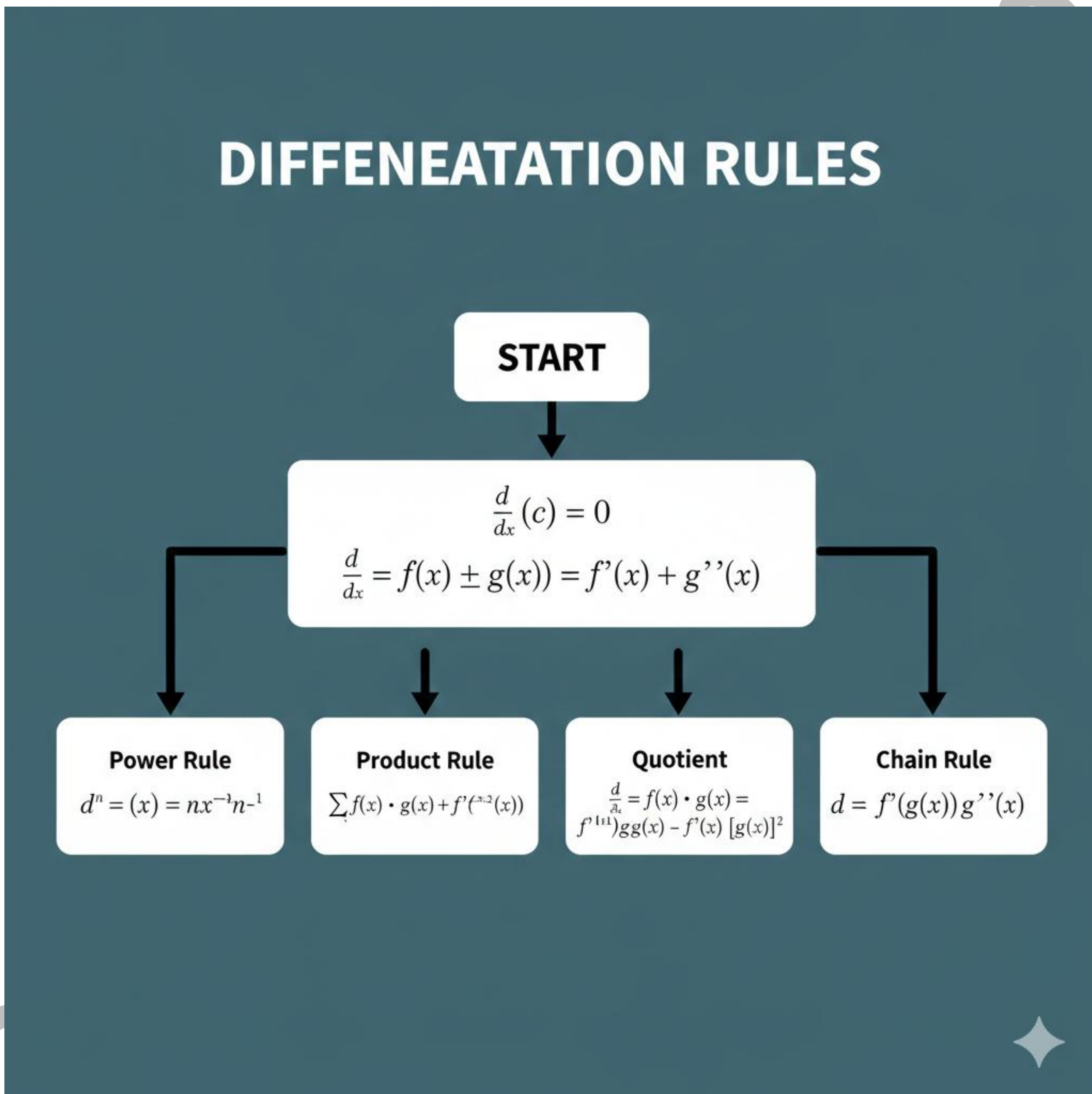
Power rule: ddx(xn)=nxn-1.

Product rule: ddx[uv]=u'v+uv'.

Quotient rule: ddxuv=u'v-uv'v2.

Chain rule: dydx=dydududx.

These rules allow us to differentiate complex functions efficiently.



Techniques of differentiation

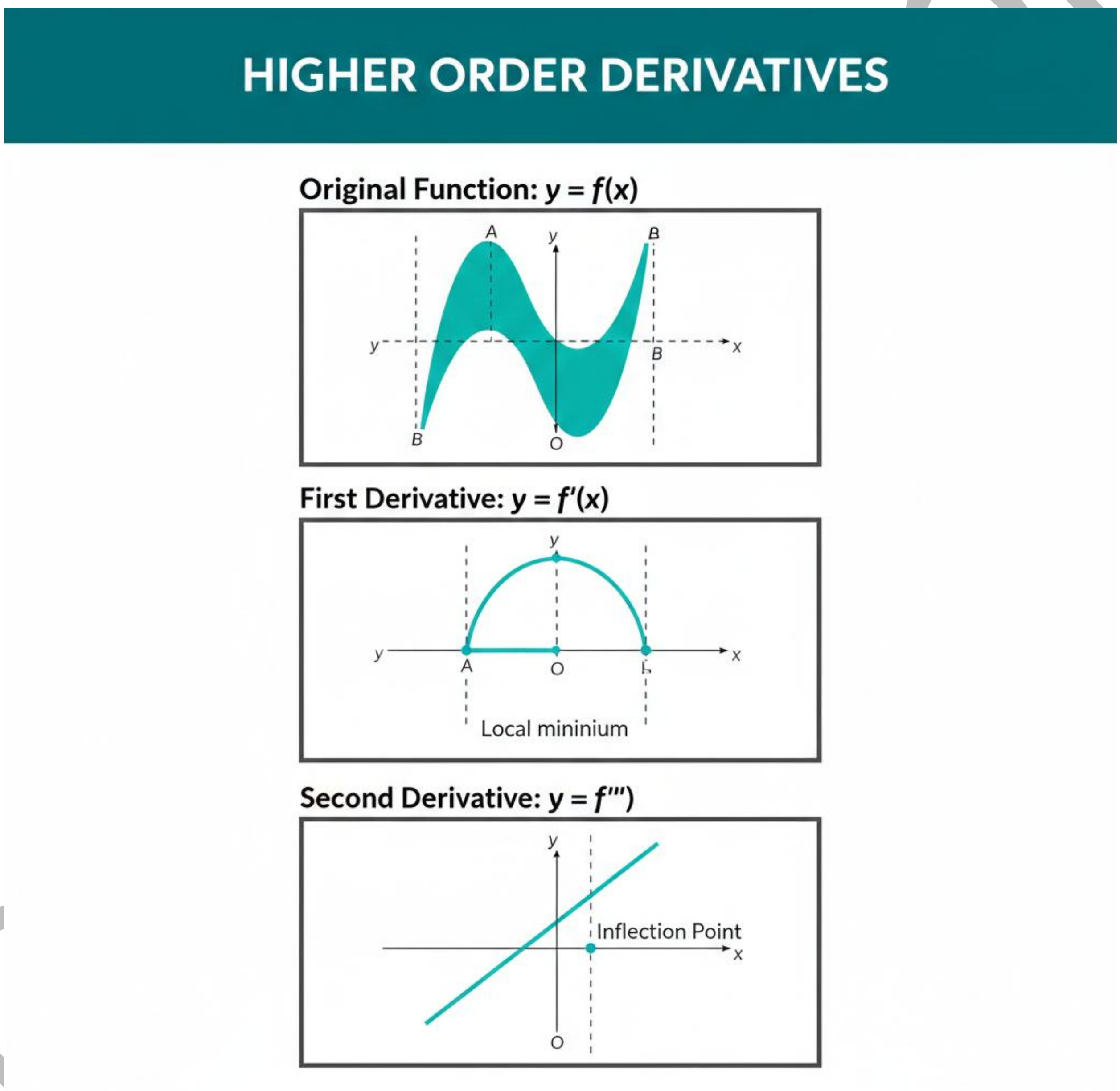
Fill in the blank: The derivative of xnwith respect to xis ______.

4.2.3 Higher-order derivatives

The **second derivative** is the derivative of the derivative, denoted $f''(x)$. It measures the rate of change of the slope and is used to analyse concavity and acceleration. Higher-order derivatives ($fn(x)$) are obtained by differentiating repeatedly.

Example: If $f(x)=x3$, then

$$f'(x)=3x2, f''(x)=6x, f'''(x)=6.$$



Higher-order derivatives of a cubic function

Fill in the blank: The second derivative of a function measures the rate of change of the ______.

Pilot Questions 4.2

The derivative of $f(x)=x2$is: A. 2xB. xC. x3D. 2

Differentiate $f(x)=x3+5x$.

State the product rule of differentiation.

Find the second derivative of $f(x)=x4$.

Explain the geometric meaning of the derivative at a point.

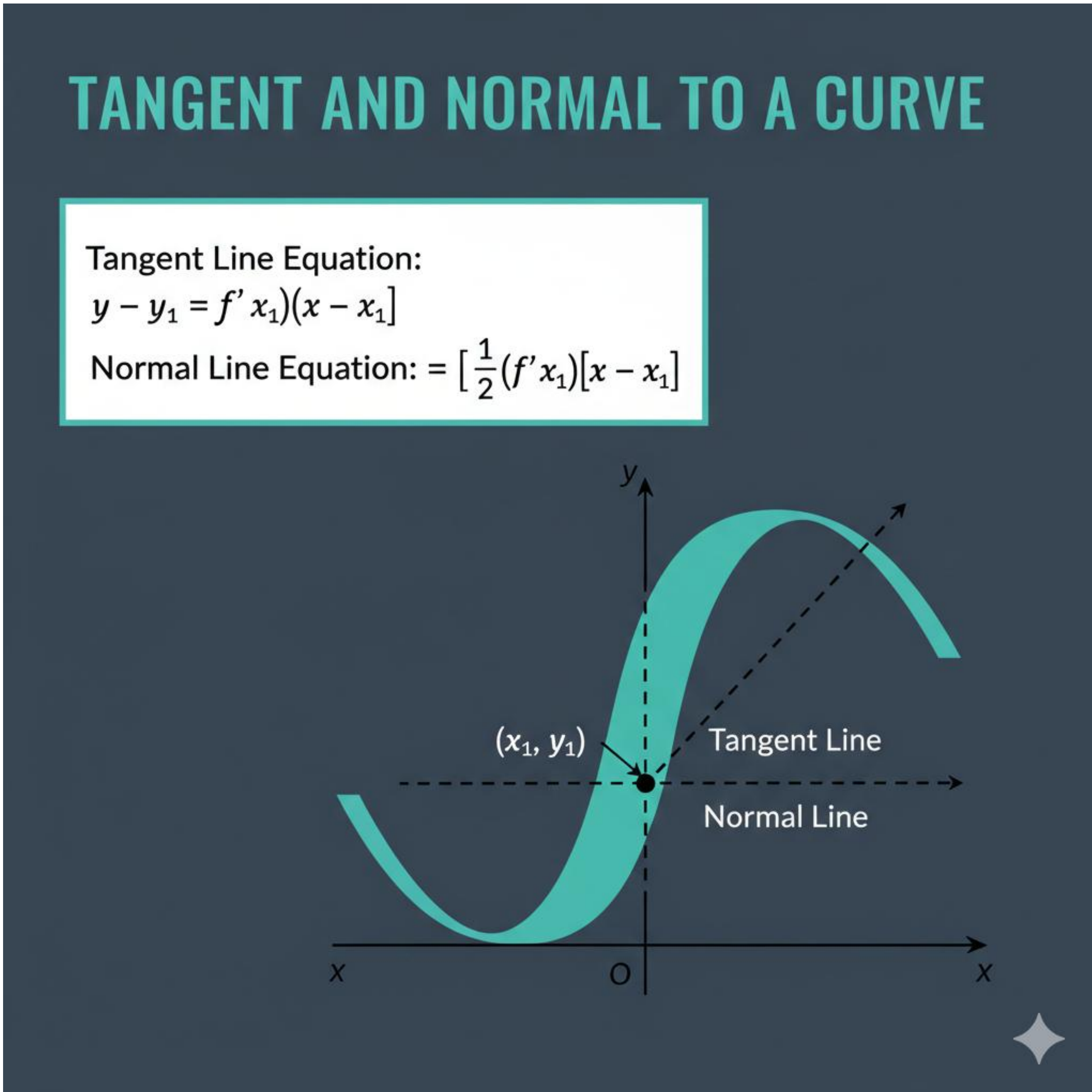
4.3 Applications Of Derivatives

4.3.1 Tangents and normals to curves

The derivative of a function at a point gives the slope of the tangent line to the curve at that point. If $y=f(x)$and $f'(x1)$is the derivative at x1, then:

Equation of tangent: $y-y1=f'(x1)(x-x1)$.

Equation of normal: $y-y1=-1f'(x1)(x-x1)$.



Tangent and normal to a curve

Fill in the blank: The slope of the normal to a curve at a point is the negative ______ of the slope of the tangent.

4.3.2 Increasing and decreasing functions

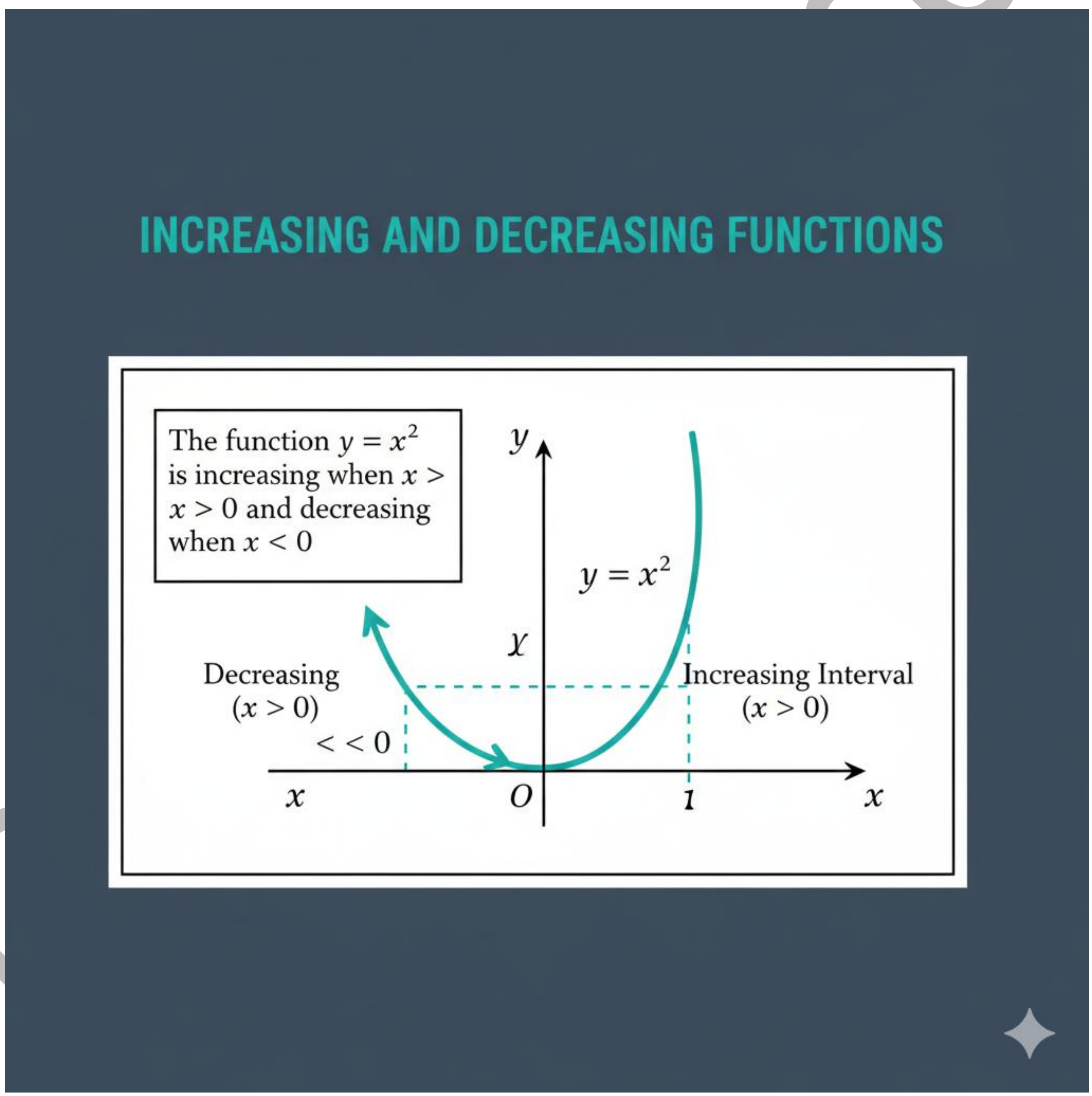
A function is **increasing** on an interval if $f'(x)>0$for all xin that interval, and **decreasing** if $f'(x)<0$. These conditions allow us to determine where a function rises or falls.

Example: For $f(x)=x2$,

$$f'(x)=2x.$$

Increasing when $x>0$.

Decreasing when $x<0$.



Increasing and decreasing functions

Fill in the blank: A function is increasing on an interval if its derivative is ______ throughout that interval.

4.3.3 Maxima and minima

Maxima and minima are points where a function reaches its highest or lowest value locally.

At these points, $f'(x)=0$.

The **second derivative test** helps classify them:

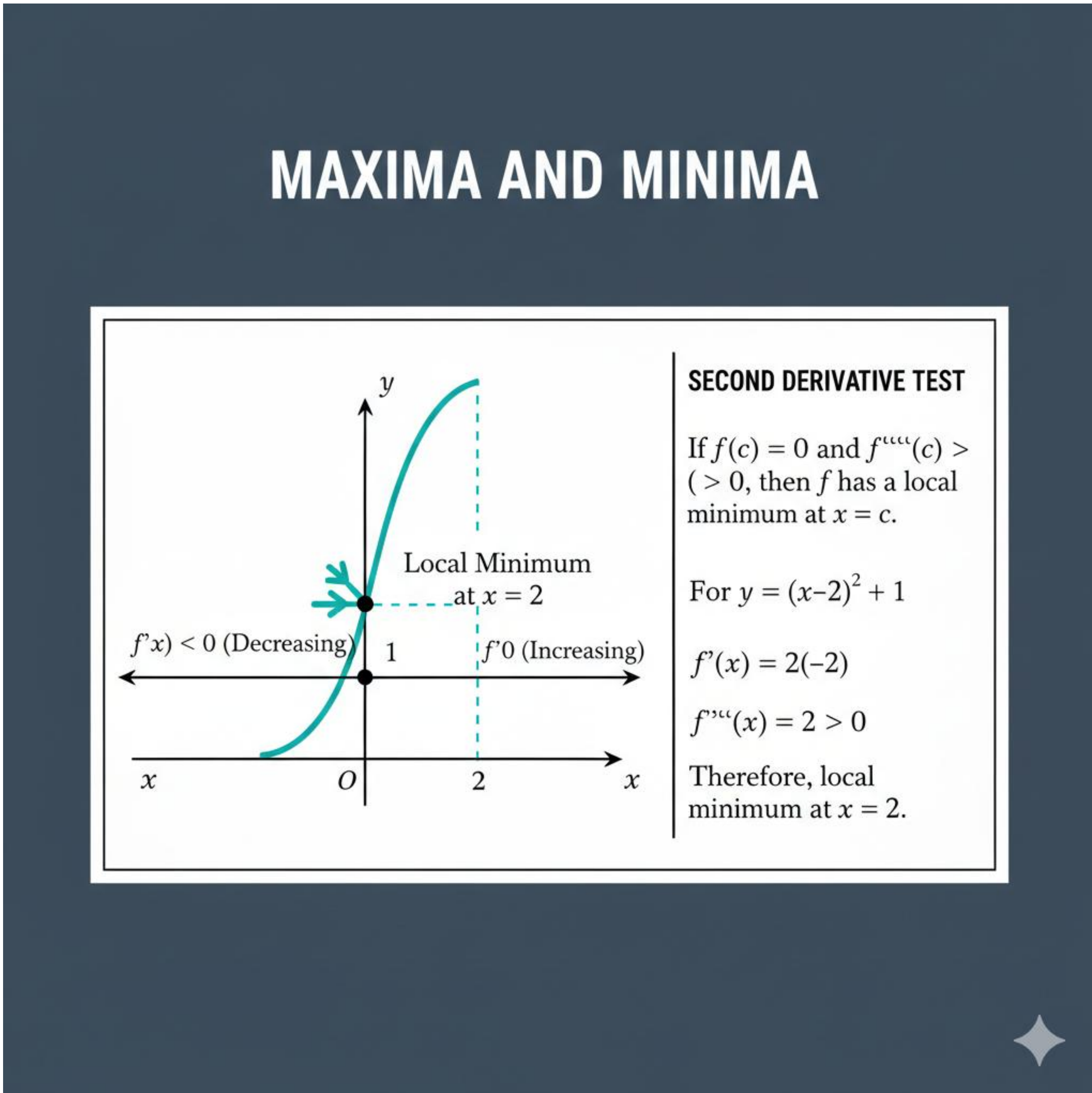
If $f''(x)>0$, the point is a local minimum.

If $f''(x)<0$, the point is a local maximum.

Example: For $f(x)=x2-4x+3$,

$$f'(x)=2x-4=0 \Rightarrow x=2.$$

$$f''(x)=2>0 \Rightarrow \text{local minimum at } x=2.$$



Maxima and minima using second derivative test

Fill in the blank: If f''(x)<0at a stationary point, the function has a local ______.

Pilot Questions 4.3

Write the equation of the tangent to the curve y=x2at point 1.

State the condition for a function to be increasing on an interval.

Find the intervals where f(x)=x3is increasing and decreasing.

Use the second derivative test to classify the stationary point of $f(x)=x^2-6x+8$.

Explain the difference between maxima and minima in terms of the second derivative.

4.4 Special Functions And Derivatives

4.4.1 Derivatives of trigonometric functions

Trigonometric functions have well-defined derivatives:

$\frac{d}{dx}(\sin x)=\cos x$

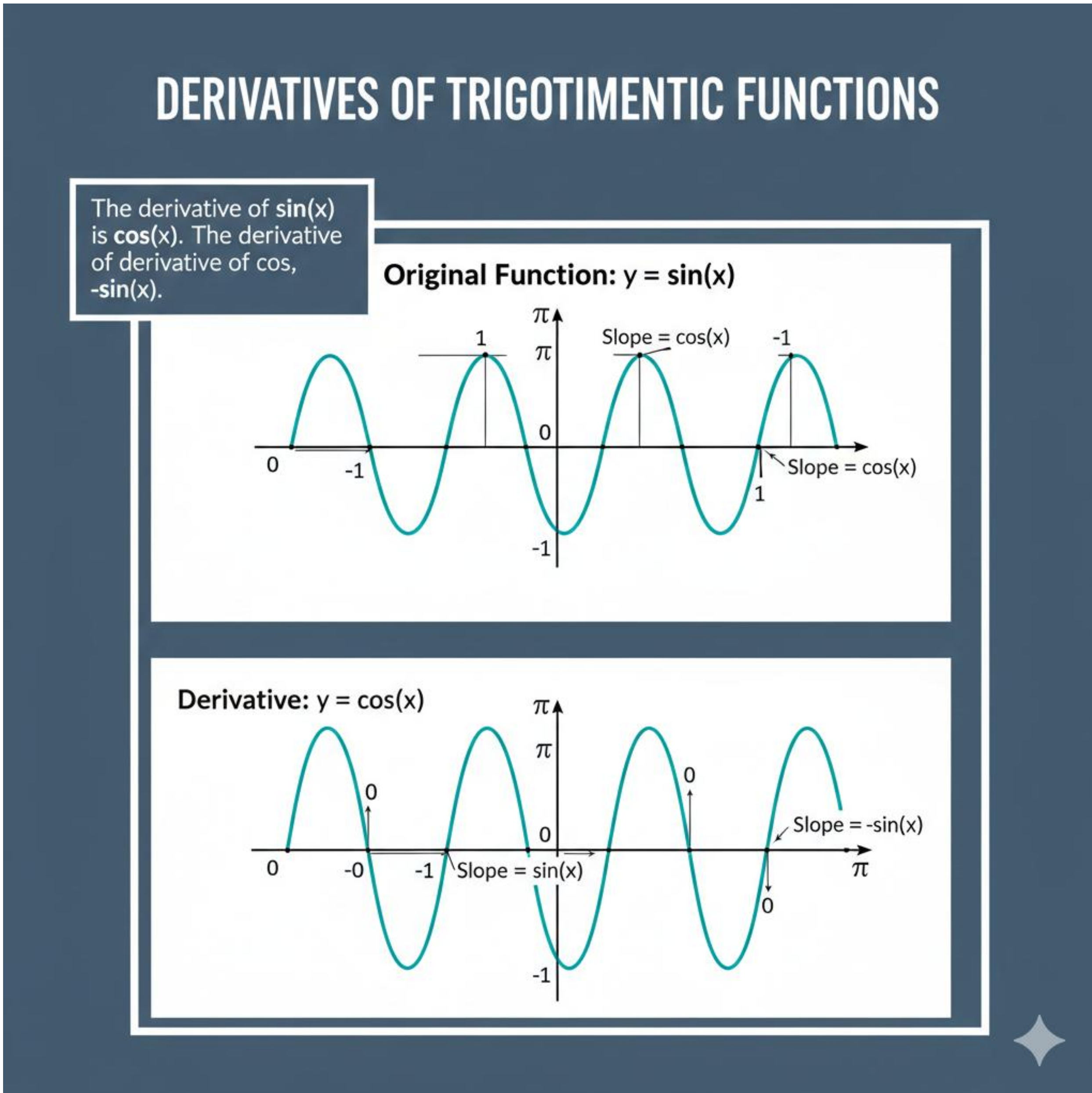
$\frac{d}{dx}(\cos x)=-\sin x$

$\frac{d}{dx}(\tan x)=\sec^2 x$

$\frac{d}{dx}(\cot x)=-\csc^2 x$

$\frac{d}{dx}(\sec x)=\sec x\tan x$

$\frac{d}{dx}(\csc x)=-\csc x\cot x$



Derivatives of trigonometric functions

Fill in the blank: The derivative of sin x with respect to xis ______.

4.4.2 Derivatives of exponential and logarithmic functions

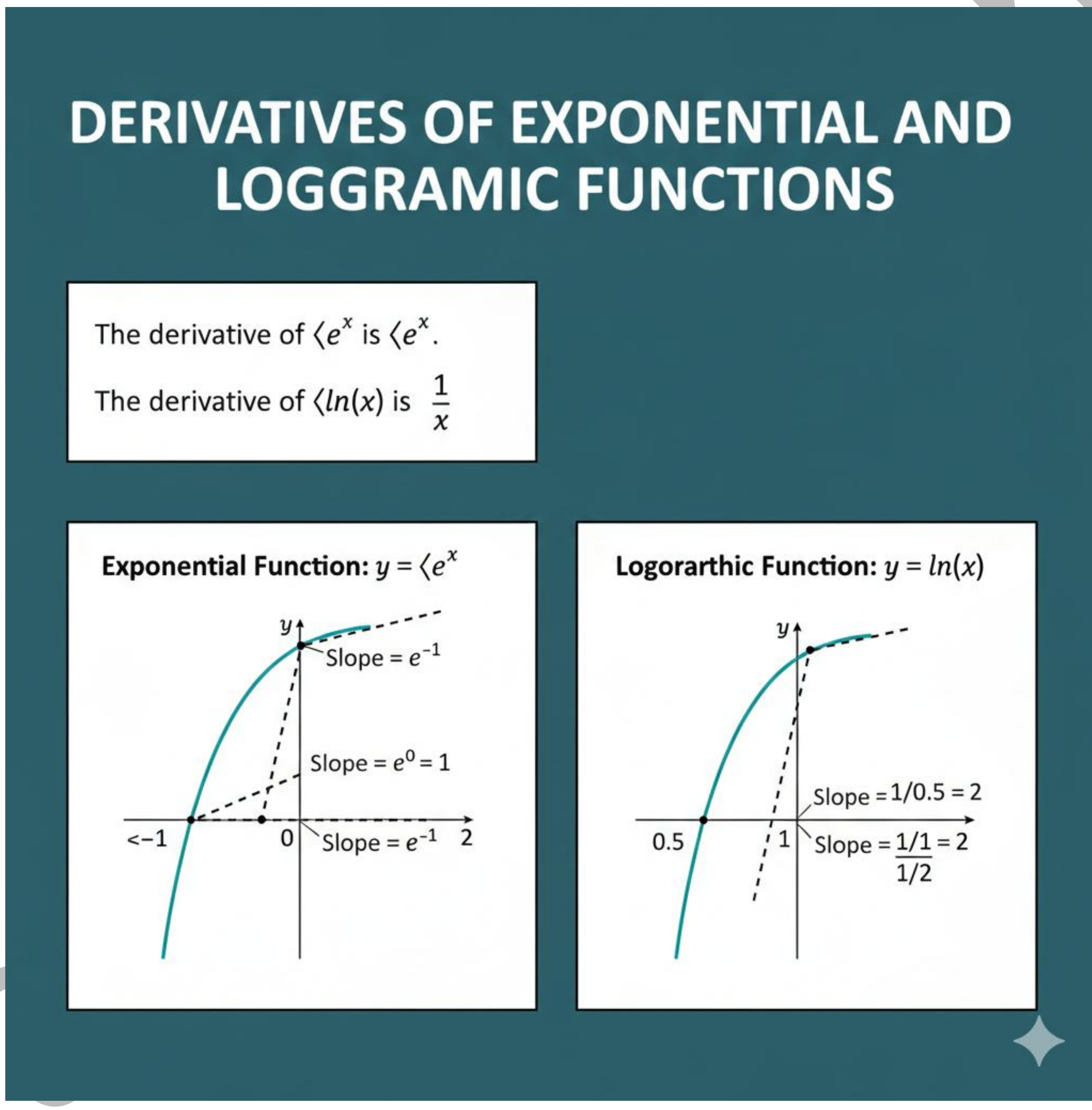
Exponential and logarithmic functions are central in calculus:

ddx(ex)=ex

$ddx(ax)=ax\ln a$

$ddx(\ln x)=1x$

These derivatives are widely used in growth/decay models, finance, and natural sciences.



Derivatives of exponential and logarithmic

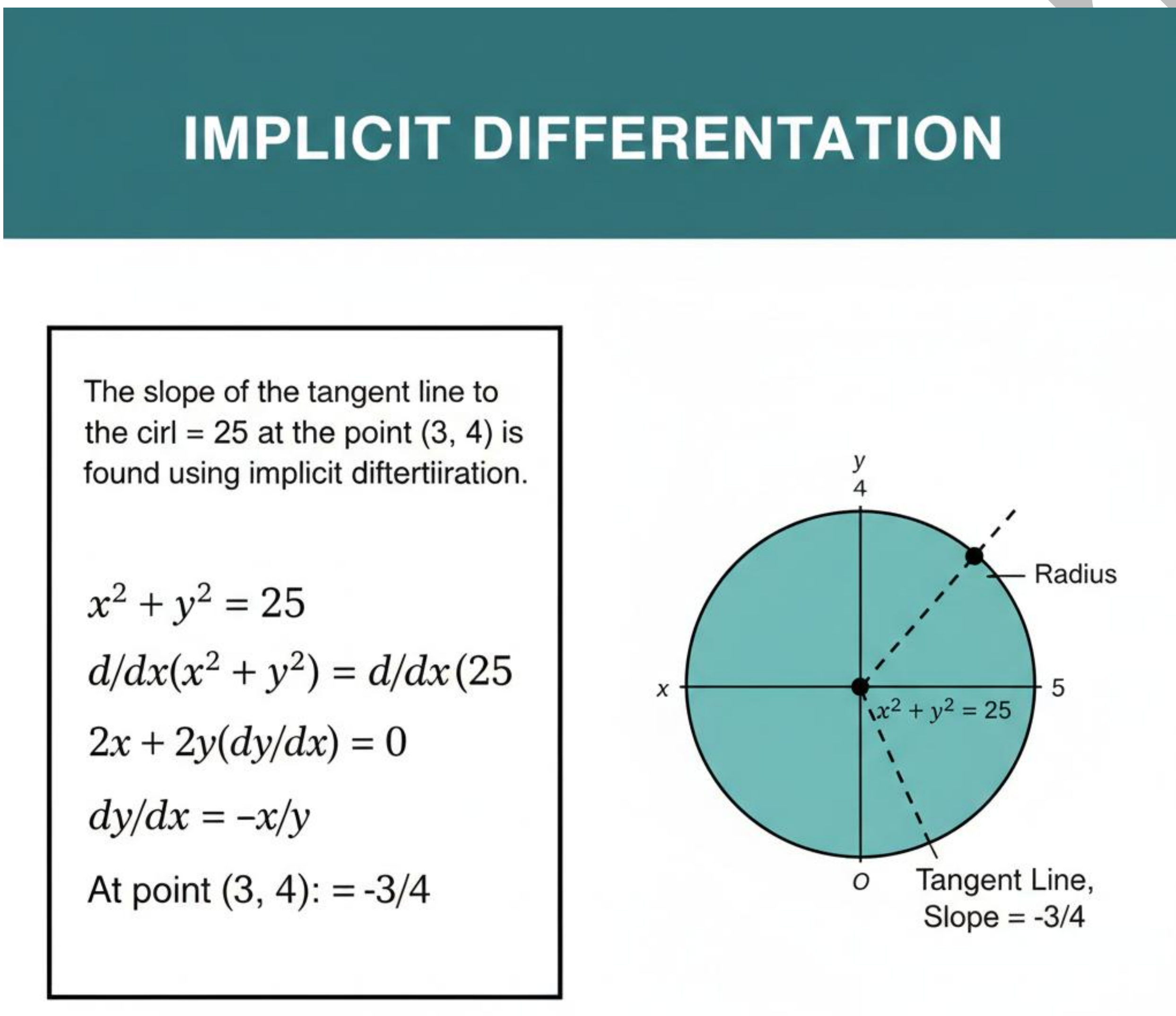
Fill in the blank: The derivative of ln x with respect to xis ______.

4.4.3 Implicit differentiation

When functions are not explicitly defined, we use **implicit differentiation**. For example, if $x2+y2=25$, differentiating both sides with respect to x:

$$2x+2ydydx=0 \Rightarrow dydx=-xy.$$

This technique is essential for curves defined by equations involving both xand y.



Implicit differentiation applied to a circle

Fill in the blank: Implicit differentiation is used when functions are not ______ defined.

Pilot Questions 4.4

Differentiate sin x .

State the derivative of ex.

Find dydxif x2+y2=16.

Write the derivative of ln x .

Explain why implicit differentiation is necessary for equations like x2+y2=r2.

Series Summary

This Series introduced the foundations of **differential calculus**, beginning with limits and continuity, which establish the groundwork for derivatives. We then defined derivatives formally and explored techniques of differentiation, including power, product, quotient, and chain rules, as well as higher-order derivatives. Next, we applied derivatives to tangents, normals, increasing/decreasing functions, and maxima/minima problems. Finally, we studied derivatives of special functions—trigonometric, exponential, and logarithmic—and learned implicit differentiation for equations not explicitly defined.

By completing this Series, you should now be able to evaluate limits, test continuity, compute derivatives using standard rules, apply derivatives to optimization and curve analysis, and differentiate special functions.

Glossary of Terms

Limit: The value a function approaches as the input approaches a specific point.

Continuity: A property of functions where the limit at a point equals the function's value at that point.

Derivative: The instantaneous rate of change of a function, or the slope of its tangent line.

Product rule: Differentiation rule for the product of two functions.

Quotient rule: Differentiation rule for the quotient of two functions.

Chain rule: Differentiation rule for composite functions.

Higher-order derivatives: Derivatives taken repeatedly, such as second or third derivatives.

Tangent: A line that touches a curve at one point with slope equal to the derivative.

Normal: A line perpendicular to the tangent at the point of contact.

Maxima/Minima: Points where a function reaches local highest or lowest values.

Implicit differentiation: Differentiation method for equations where yis not explicitly expressed in terms of x.

Concept Check Questions 4

Objective questions

The derivative of f(x)=x2is: A. 2xB. xC. x3D. 2(Linked to SLO 5.2)

The derivative of sin x is: A. cos x B. -sinxC. sec2xD. tan x (Linked to SLO 5.4)

Short-answer questions

State the three conditions for continuity of a function at a point. (Linked to SLO 5.1)

Differentiate f(x)=ex+lnx. (Linked to SLO 5.4)

Long-answer questions

Derive the equation of the tangent and normal to the curve y=x2at point 1. (Linked to SLO 5.3)

Basic response: Show tangent slope and equations.

Value-added response: Explain geometric meaning of tangent and normal.

Use the second derivative test to classify the stationary points of f(x)=x3-3x2+2. (Linked to SLO 5.3)

Basic response: Compute derivatives and classify points.

Value-added response: Connect results to graph shape and optimization problems.

Answers to Concept Check Questions 4

A. 2x

A. cos x

(i) f(a)is defined, (ii) $\lim x \to a$ f(x)exists, (iii) $\lim x \to a$ f(x)=f(a).

ddx(ex+ln x)=ex+1x.

Basic response: f'(x)=2x. At x=1, slope = 2. Tangent: y-1=2(x-1). Normal slope = -12. Normal: y-1=-12(x-1). **Value-added response**: Tangent represents instantaneous slope; normal represents perpendicular direction.

f'(x)=3x2-6x. Stationary points: x=0,x=2. f''(x)=6x-6. At x=0, f''(0)=-6<0→ local maximum. At x=2, f''(2)=6>0→ local minimum.

Answers to Pilot Questions 4

From Part 4.1

$\lim x \to 3$(2x+1)=7.

$\lim x \to 1$x2-1x-1=$\lim x \to 1$(x+1)=2.

Conditions: f(a)defined, limit exists, limit equals f(a).

Example: f(x)=1xdiscontinuous at x=0.

Limits define instantaneous change, essential for derivatives.

From Part 4.2

2x.

f'(x)=3x2+5.

Product rule: (uv)'=u'v+uv'.

f''(x)=12x2.

Geometric meaning: slope of tangent at a point.

From Part 4.3

Tangent: slope = 2 at 1. Equation: y-1=2(x-1).

Condition: f'(x)>0.

f'(x)=3x2. Always positive except at x=0. So increasing everywhere, no decreasing interval.

f'(x)=2x-6. Stationary point at x=3. f''(x)=2>0→ local minimum.

Maxima: slope zero, concavity negative. Minima: slope zero, concavity positive.

From Part 4.4

cos x .

ex.

2x+2ydydx=0⇒dydx=-xy.

1x.

Because yis defined implicitly, not explicitly.

Series 5: Integral Calculus

Series Outline

This Series introduces the fundamental concepts of integral calculus, focusing on indefinite and definite integrals, techniques of integration, and applications. It builds directly on differential calculus, showing how integration serves as the reverse process of differentiation and as a powerful tool for measuring areas, volumes, and accumulated quantities.

Proposed Parts and Sub-Parts

5.1 Indefinite integrals

5.1.1 Concept of integration as reverse differentiation

5.1.2 Standard formulas of integration

5.1.3 Properties of indefinite integrals

5.2 Definite integrals

5.2.1 Definition of definite integral

5.2.2 Evaluation using limits of integration

5.2.3 Properties of definite integrals

5.3 Techniques of integration

5.3.1 Integration by substitution

5.3.2 Integration by parts

5.3.3 Partial fractions method

5.4 Applications of integrals

5.4.1 Area under a curve

5.4.2 Area between two curves

5.4.3 Volumes of solids of revolution

Introduction

Integral calculus is the branch of mathematics concerned with accumulation and area. While differential calculus focuses on rates of change, integral calculus focuses on summing infinitely small quantities to find totals. The two are connected through the Fundamental Theorem of Calculus, which states that differentiation and integration are inverse processes.

The aim of this Series is to enable you to understand indefinite and definite integrals, apply techniques of integration, and use integrals to solve problems involving areas, volumes, and other applications in geometry and physics.

We shall begin with indefinite integrals, introducing integration as reverse differentiation and exploring standard formulas. Next, we will study definite integrals and their properties. Following that, we will examine techniques of integration, including substitution, parts, and partial fractions. Finally, we will apply integrals to practical problems such as finding areas under curves and volumes of solids of revolution.

Series Learning Outcomes – Integral Calculus

Upon completing this Series, you should be able to:

SLO 6.1 explain the concept of integration as reverse differentiation, apply standard formulas, and use properties of indefinite integrals to simplify expressions,

SLO 6.2 define definite integrals, evaluate them using limits of integration, and apply their properties to solve problems,

SLO 6.3 apply techniques of integration, including substitution, integration by parts, and partial fractions, to evaluate complex integrals,

SLO 6.4 use integrals to calculate areas under curves, areas between curves, and volumes of solids of revolution.

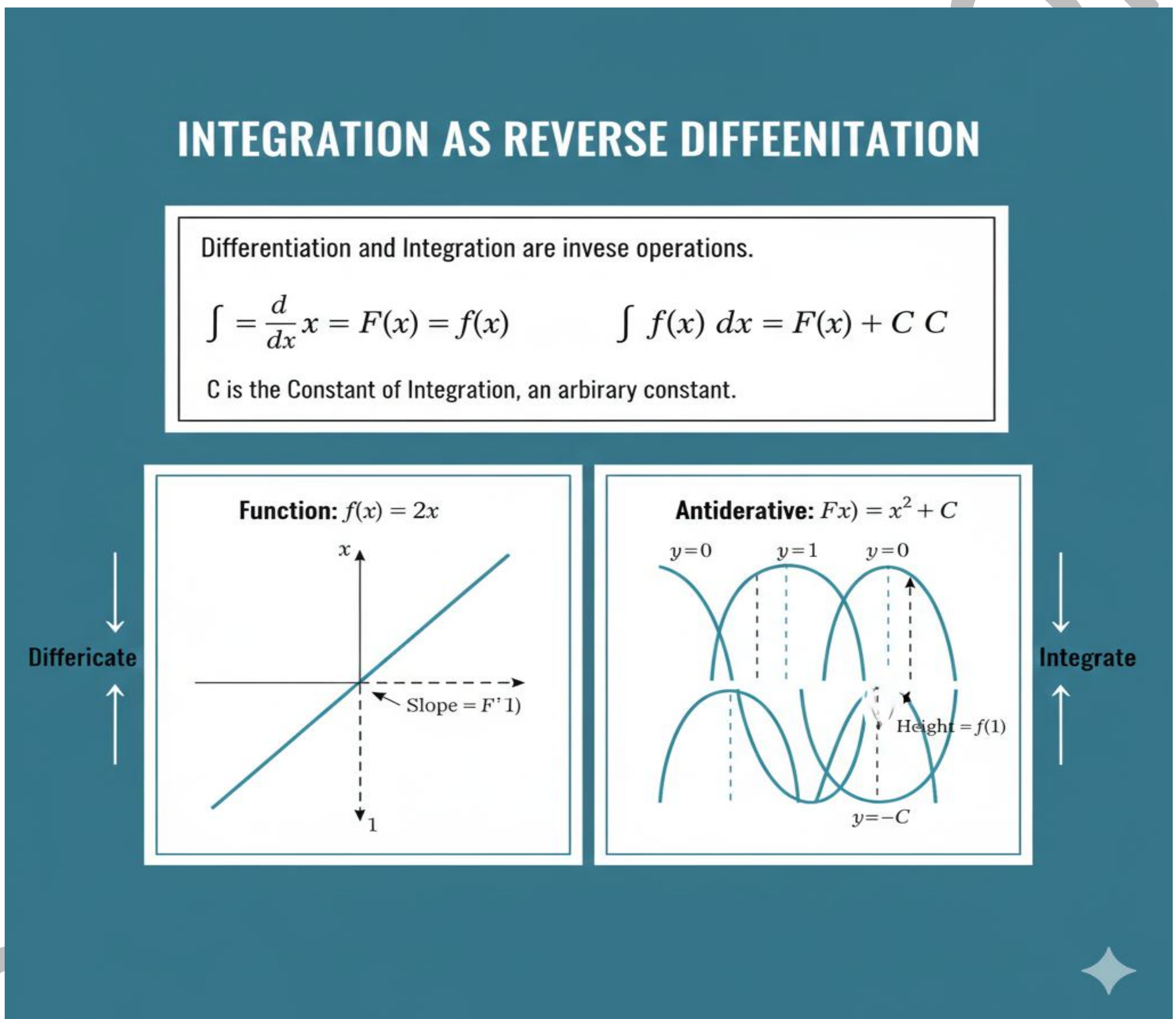
5.1 Indefinite Integrals

5.1.1 Concept of integration as reverse differentiation

Integration is the reverse process of differentiation. If ddxF(x)=f(x), then

$$\int f(x)\,dx = F(x) + C,$$

where Cis the constant of integration. This constant accounts for the fact that differentiation of any constant is zero.



Integration as reverse differentiation

Fill in the blank: Integration is the reverse process of ______.

5.1.2 Standard formulas of integration

Some standard integrals include:

$\int xndx=xn+1n+1+C,n-1$

$\int 1xdx=\ln|x|+C$

$\int exdx=ex+C$

$\int \sin xdx=-\cos x+C$

$\int \cos xdx=\sin x+C$

Fill in the blank: The integral of exwith respect to xis ______.

5.1.3 Properties of indefinite integrals

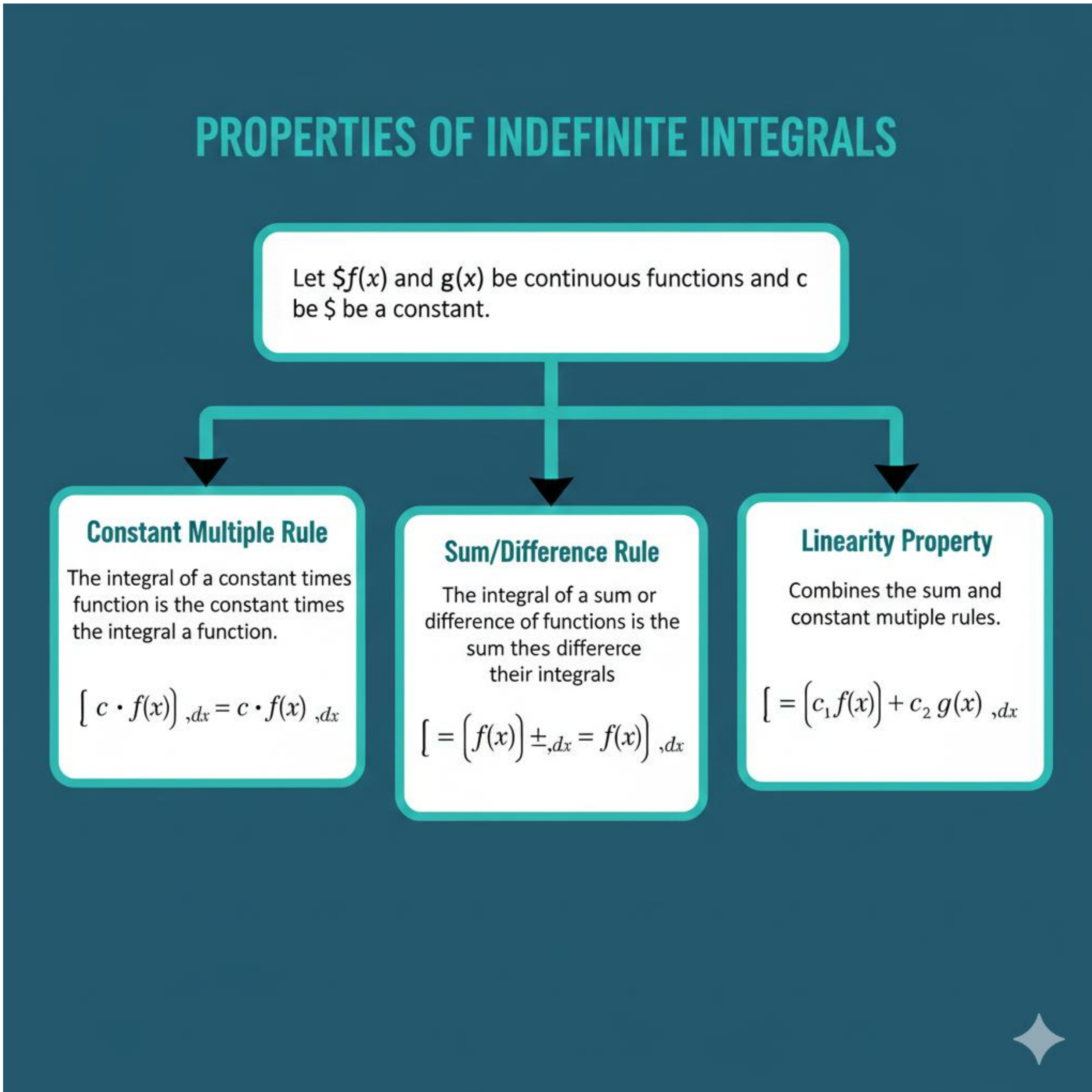
Indefinite integrals satisfy several properties:

Linearity: $\int[af(x)+bg(x)]dx=a\int f(x)dx+b\int g(x)dx$.

Constant multiple rule: $\int kf(x)dx=k\int f(x)dx$.

Sum rule: The integral of a sum is the sum of the integrals.

These properties simplify complex integrals by breaking them into manageable parts.



Properties of indefinite integrals

Fill in the blank: The integral of a sum of functions equals the ______ of their integrals.

Pilot Questions 5.1

The integral of x3is: A. x44+CB. 3x2+CC. x22+CD. None of the above

Evaluate ∫(2x+3)dx.

State the constant of integration and explain its significance.

Write the property of linearity of indefinite integrals.

Differentiate your answer to Question 2 to verify the result.

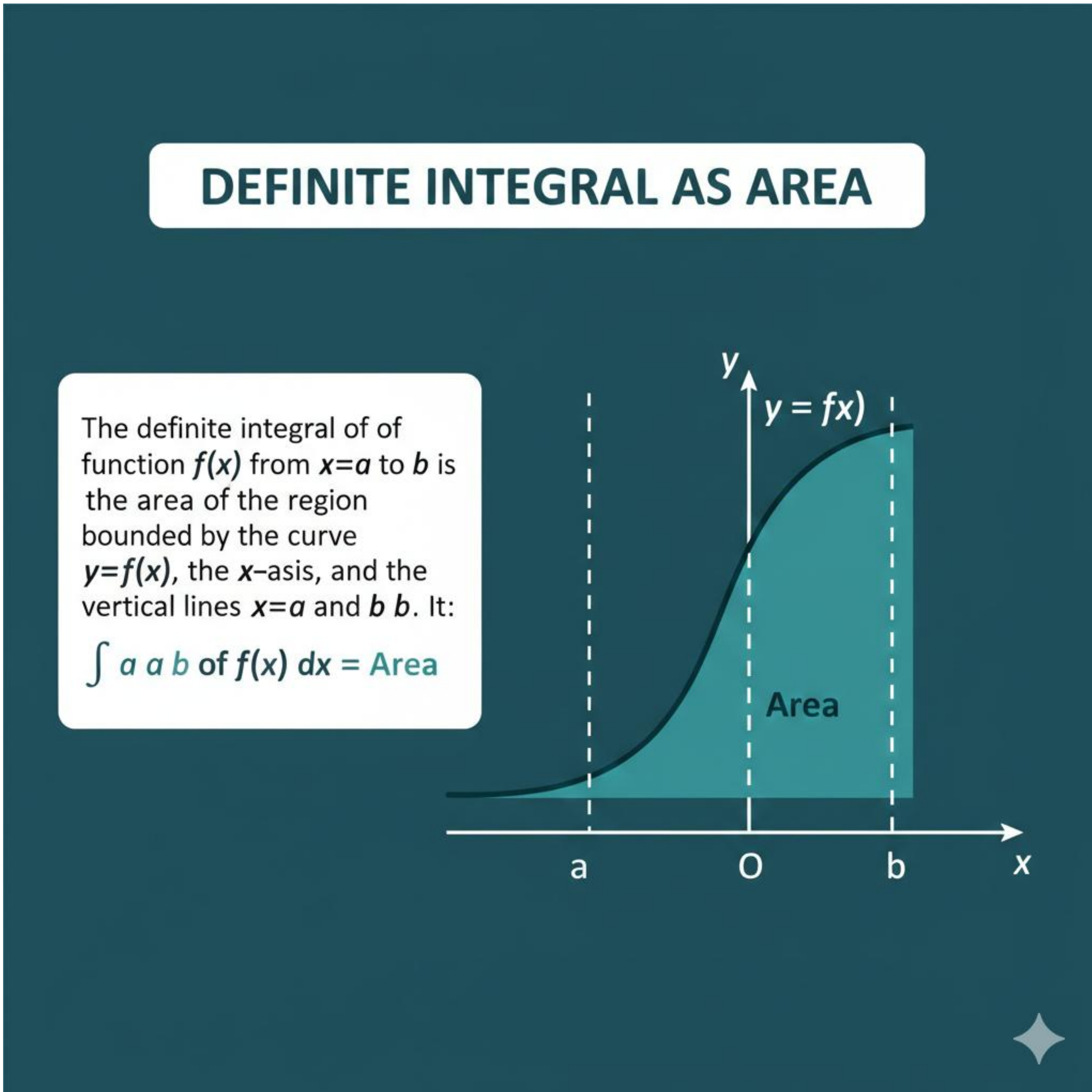
5.2 Definite Integrals

5.2.1 Definition of definite integral

A **definite integral** represents the accumulation of values of a function over an interval b. It is defined as:

$$abf(x)\ dx=F(b)-F(a),$$

where F(x)is any antiderivative of f(x). This definition connects integration with limits and provides a way to calculate exact areas under curves.



Definition of definite integral as area under a curve

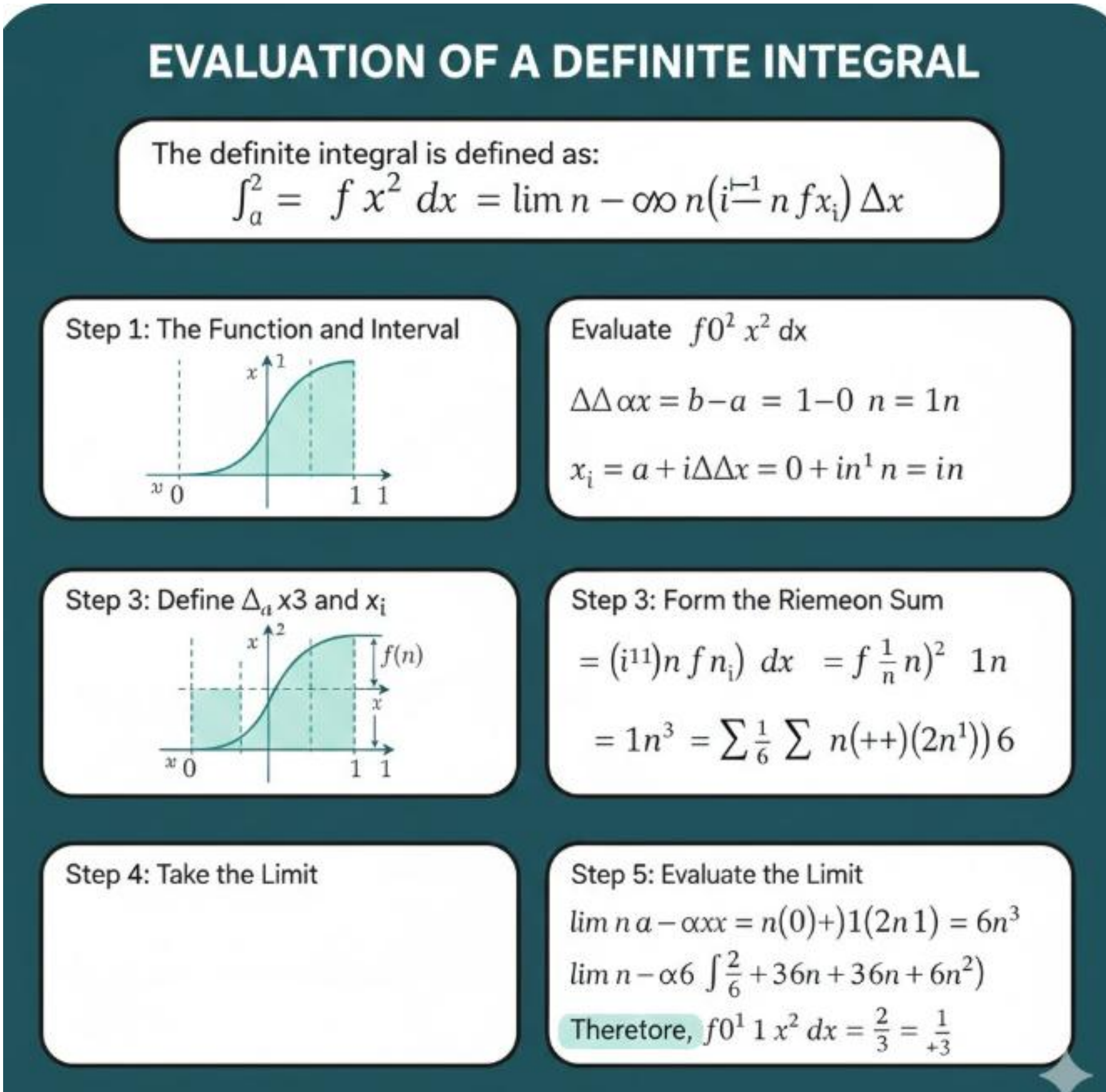
Fill in the blank: A definite integral represents the accumulation of values of a function over an ______.

5.2.2 Evaluation using limits of integration

To evaluate a definite integral, we compute the antiderivative and apply the limits: Example:

02x2dx=x3302=83-0=83.

This process is known as applying the **Fundamental Theorem of Calculus**.



Evaluating a definite integral with limits of integration

Fill in the blank: The Fundamental Theorem of Calculus connects differentiation and ______.

5.2.3 Properties of definite integrals

Definite integrals satisfy several useful properties:

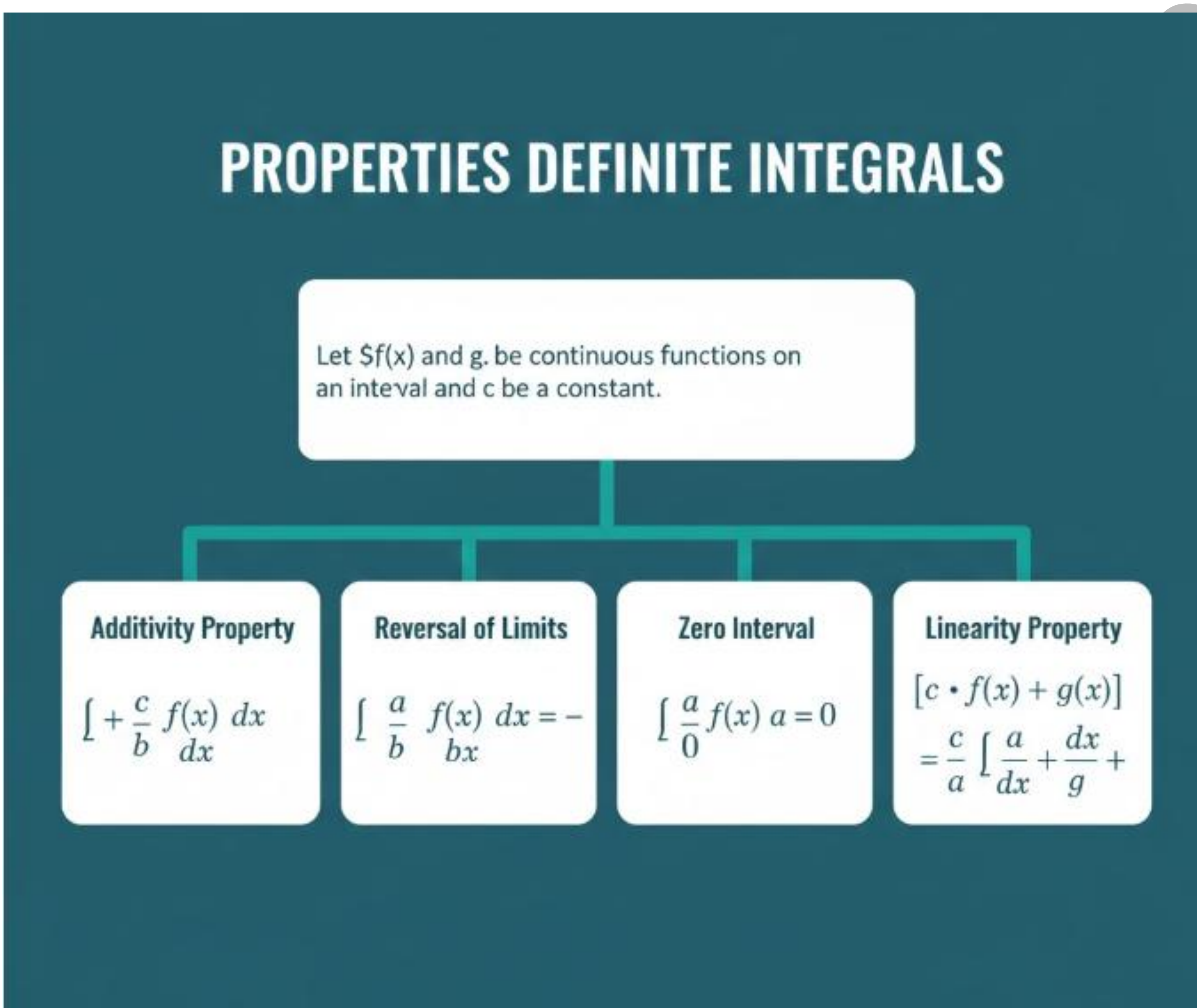
Additivity: abf(x) dx+bcf(x) dx=acf(x) dx.

Reversal of limits: abf(x) dx=-baf(x) dx.

Zero interval: aaf(x) dx=0.

Linearity: ab[af(x)+bg(x)]dx=aabf(x)dx+babg(x)dx.

These properties simplify calculations and allow breaking integrals into smaller parts.



Properties of definite integrals

Fill in the blank: The integral from ato aof any function is always ______.

Pilot Questions 5.2

Evaluate 13xdx.

State the Fundamental Theorem of Calculus.

Write the property of additivity of definite integrals.

Find 01(2x+1)dx.

Explain why reversing the limits of integration changes the sign of the integral.

5.3 Techniques Of Integration

5.3.1 Integration by substitution

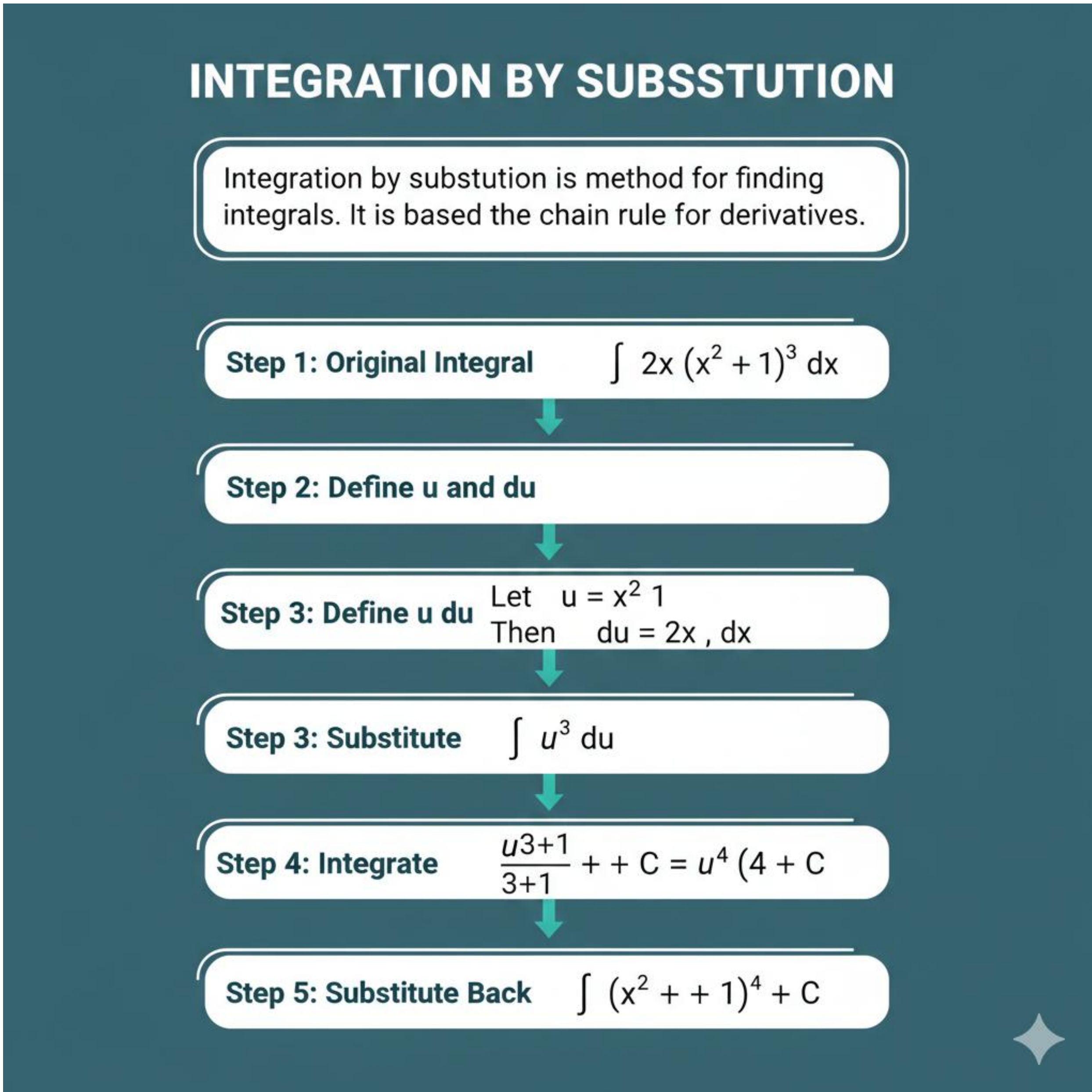
This method is used when an integral can be simplified by changing variables. If u=g(x), then

$$\int f(g(x))g'(x)\,dx=\int f(u)\,du.$$

Example:

$$\int 2x\cos(x2)\,dx \text{Let } u=x2 \Rightarrow du=2x\,dx.$$

$$=\int \cos(u)\,du=\sin(u)+C=\sin(x2)+C.$$



Integration by substitution

Fill in the blank: In substitution, we replace the variable xwith a new variable ______.

5.3.2 Integration by parts

This method is based on the product rule of differentiation. If uand vare functions of x, then:

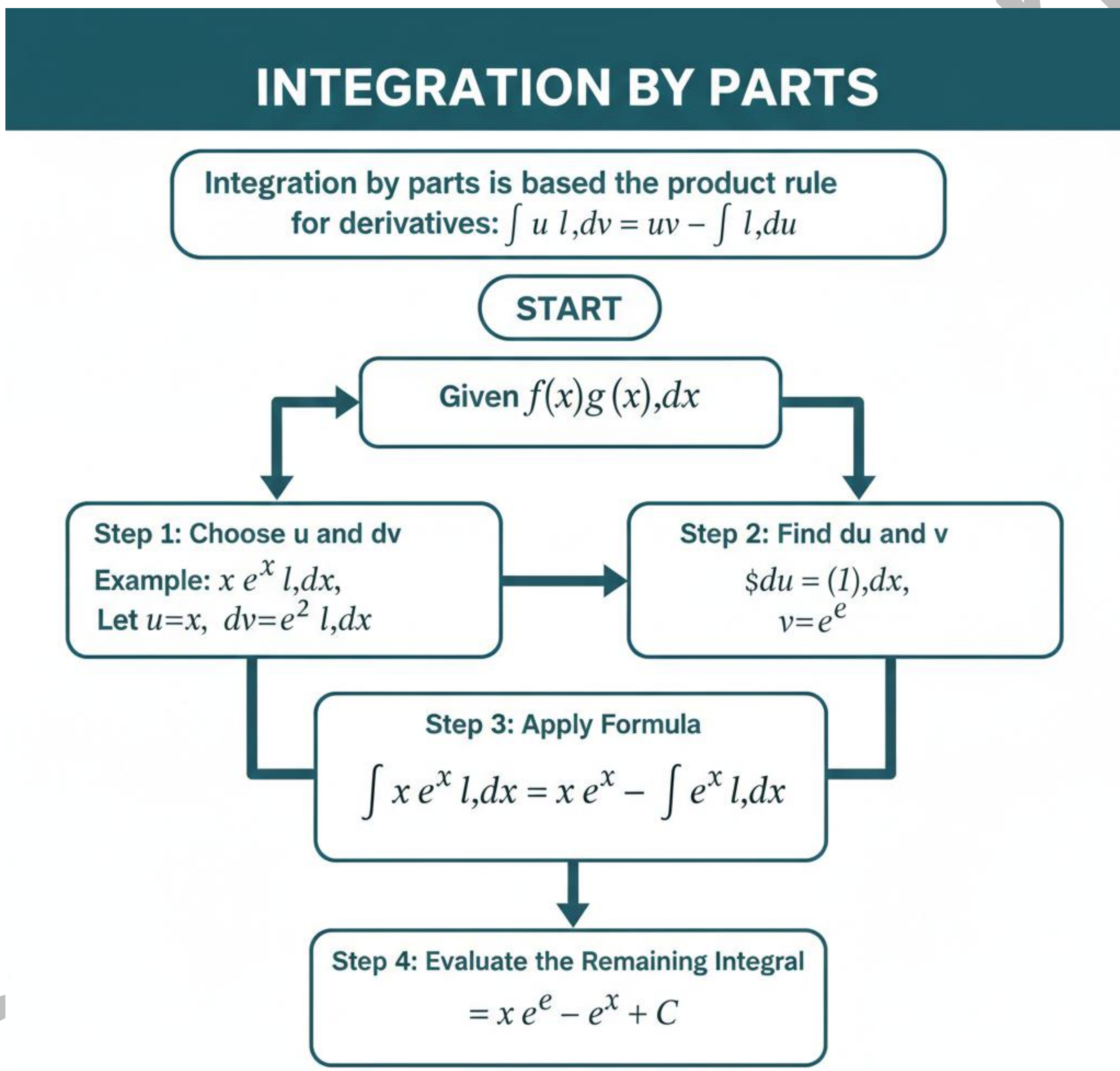
$$\int u\,dv = uv - \int v\,du.$$

Example:

∫xexdxLet u=x,dv=exdx.

du=dx,v=ex.

=xex-∫exdx=xex-ex+C.



Integration by parts

Fill in the blank: The formula for integration by parts is ∫u dv=uv-∫v du.

5.3.3 Partial fractions method

This method is used to integrate rational functions by expressing them as sums of simpler fractions.

Example:

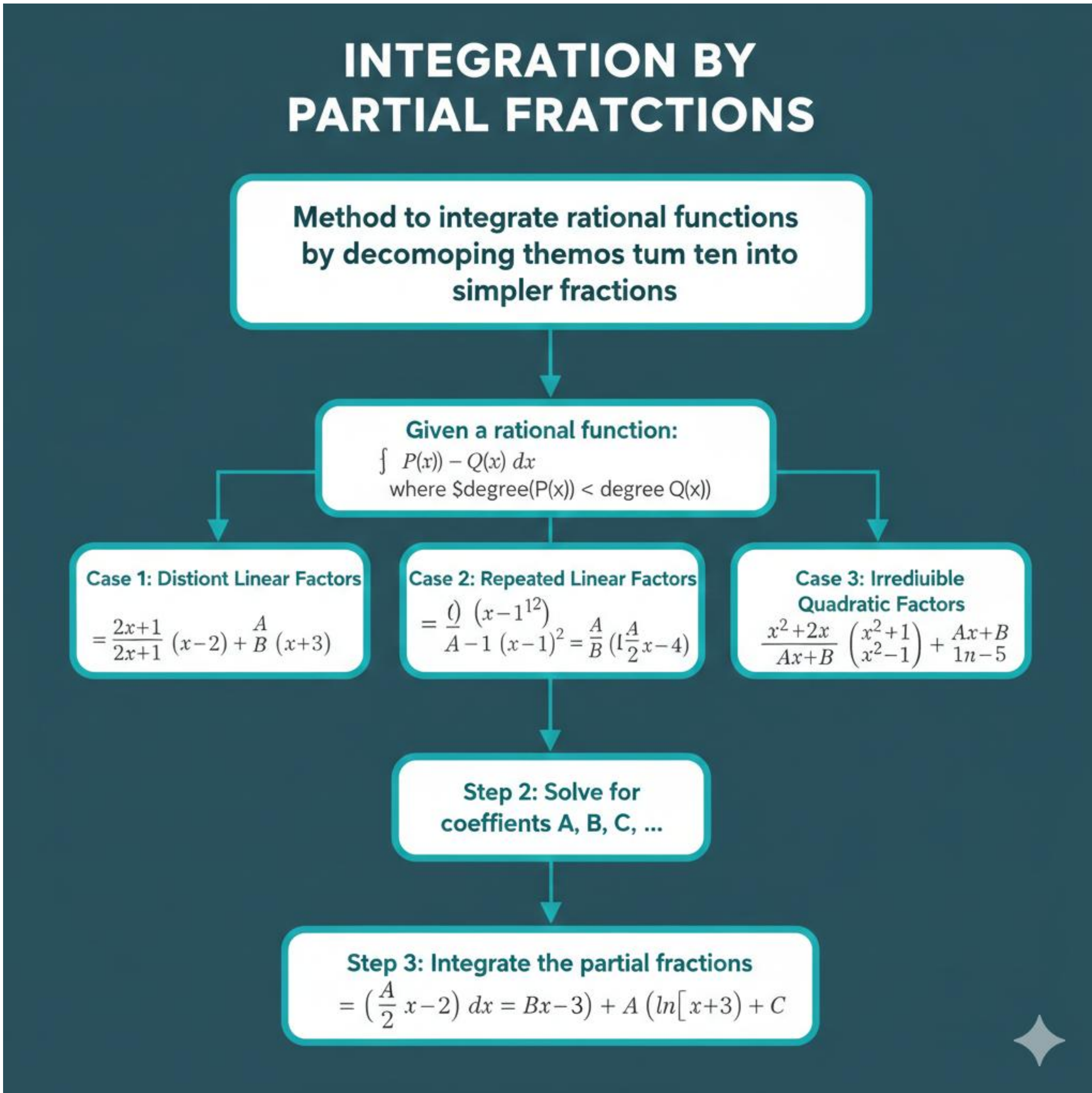
$$\int 1x2\text{-}1dx=\int 1x\text{-}1)(x+1dx.$$

Decompose:

$$1x\text{-}1)(x+1=121x+1.$$

So,

$$\int 1x2\text{-}1dx=12\ln|x\text{-}1|\text{-}12\ln|x+1|+C.$$



Integration using partial fractions

Fill in the blank: The partial fractions method decomposes a rational function into a sum of _______ fractions.

Pilot Questions 5.3

Evaluate ∫(3x2)(x3+1)4dxusing substitution.

State the formula for integration by parts.

Find ∫xsinxdxusing integration by parts.

Decompose 2xx2-1into partial fractions.

Explain why partial fractions are useful in integration.

5.4 Applications Of Integrals

5.4.1 Area under a curve

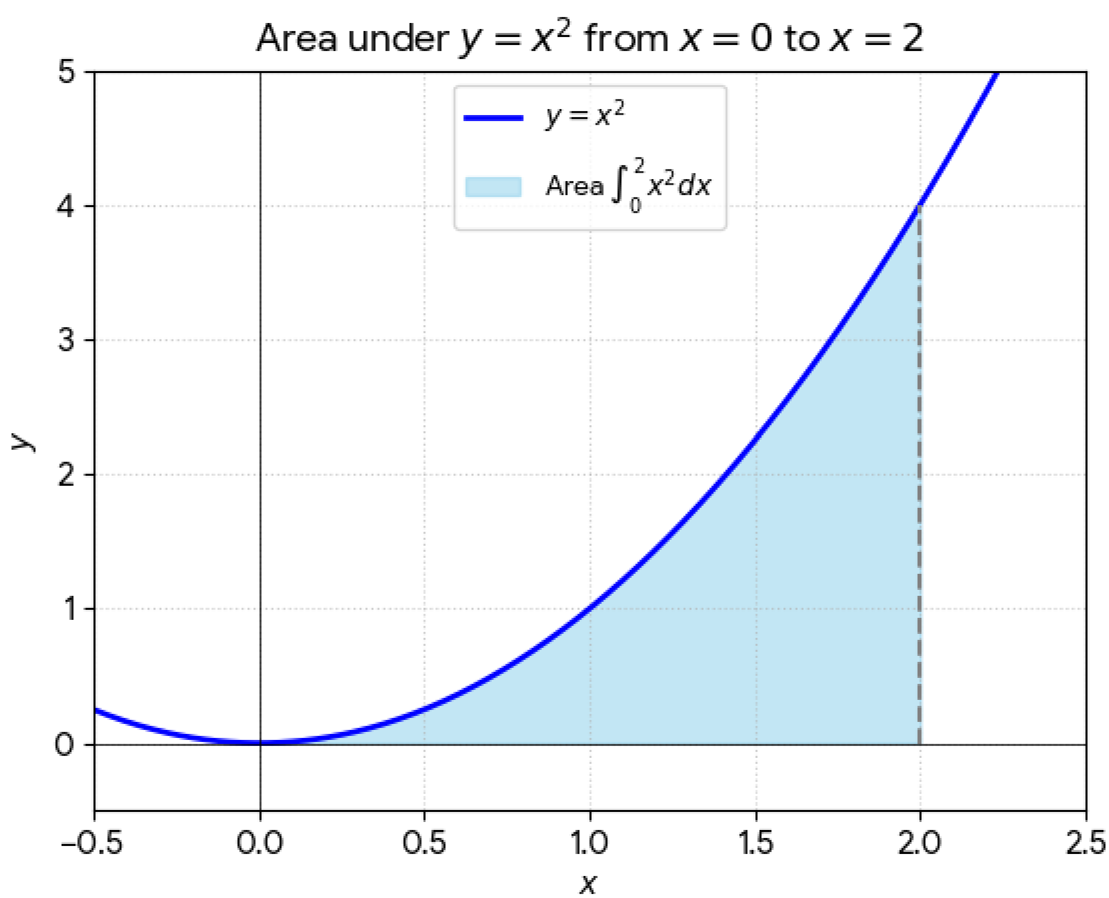
The definite integral of a function between two points gives the area under the curve:

A=abf(x) dx.

This is one of the most important applications of integrals in geometry and physics.

Example:

A=02x2dx=x3302=83.



Area under a curve

Fill in the blank: The area under a curve from ato bis given by the definite integral \int_a^b _______ dx.

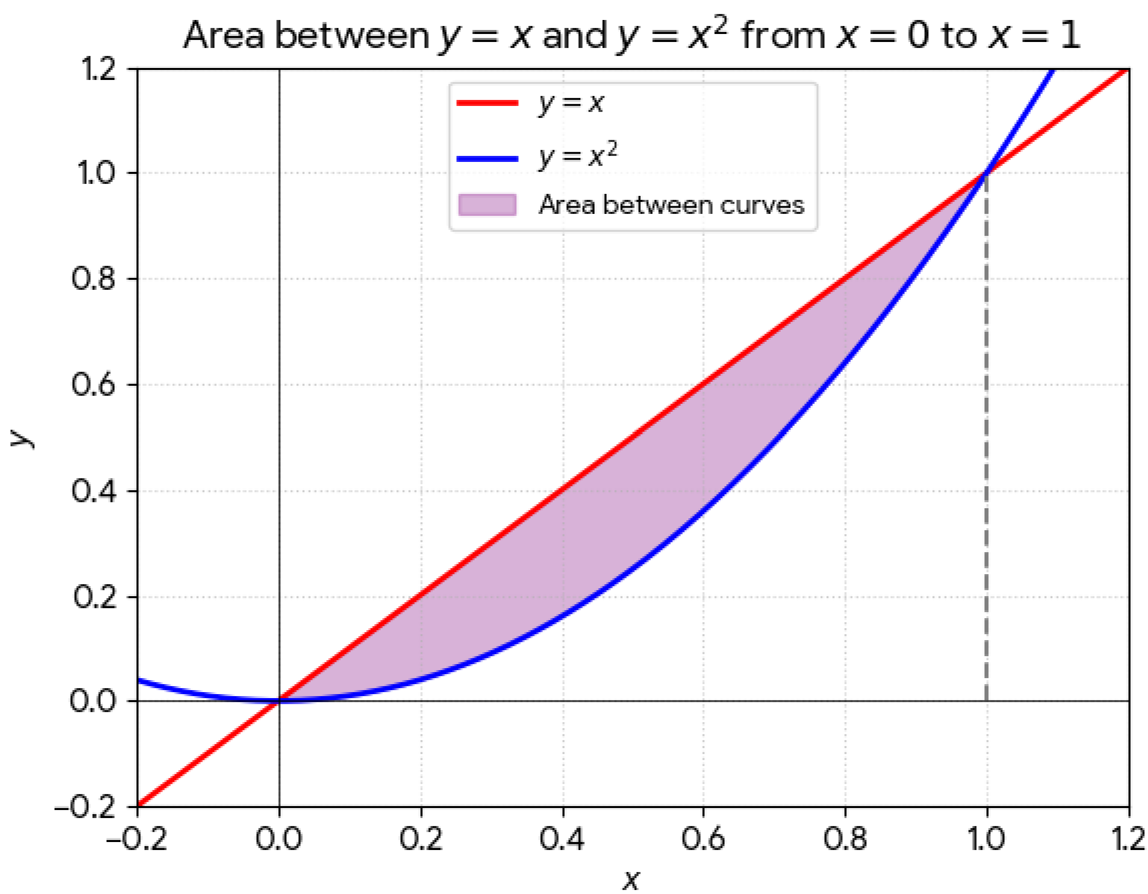
5.4.2 Area between two curves

If two curves are given by y=f(x)and y=g(x), where f(x)≥g(x)on b, then the area between them is:

A=ab[f(x)-g(x)]dx.

Example: Find the area between y=x2and y=xfrom x=0to x=1:

A=01(x-x2)dx=x3301=12-13=16.



Area between two curves

Fill in the blank: The area between two curves is found by integrating the _______ of their functions.

5.4.3 Volumes of solids of revolution

Rotating a curve around an axis generates a solid. The volume of such a solid is given by:

About the x-axis:

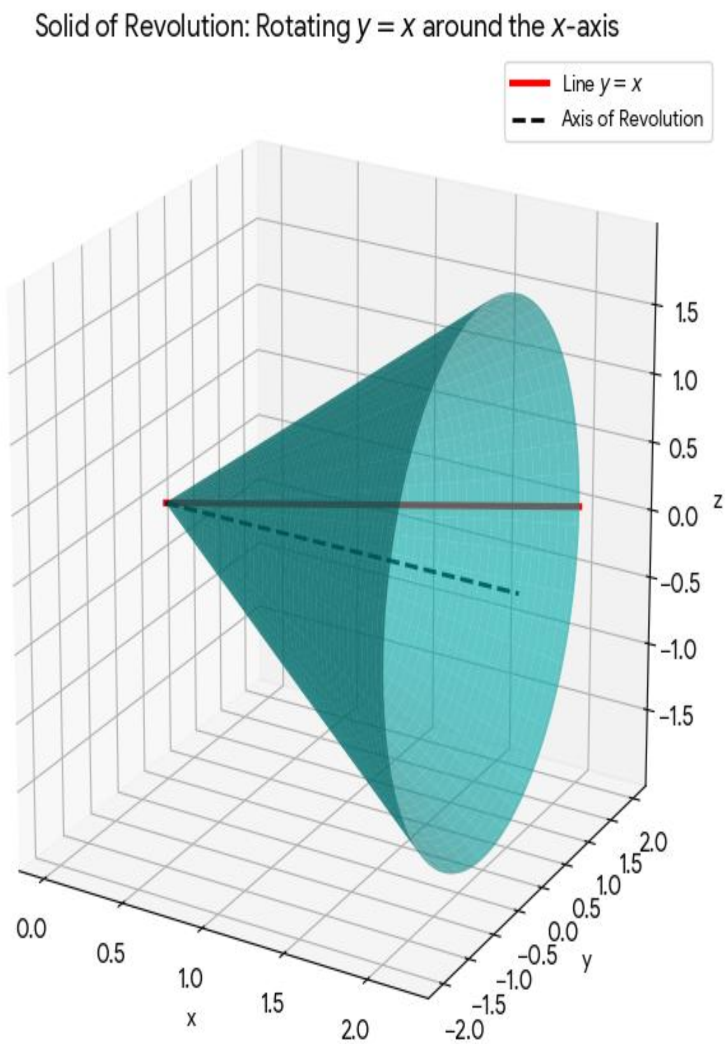
$$V=\pi ab[f(x)]2dx.$$

About the y-axis:

$$V=\pi cd[g(y)]2dy.$$

Example: Volume of solid formed by rotating $y=x$from $x=0$to $x=1$about the x-axis:

$$V=\pi 01x2dx=\pi x3301=3.$$



Volume of solid of revolution AI prompt suggestion: "

Fill in the blank: The volume of a solid of revolution about the x-axis is given by ab[f(x)]2dx.

Pilot Questions 5.4

Find the area under y=3xfrom x=0to x=2.

State the formula for the area between two curves.

Calculate the area between y=sin xand y=cos xfrom x=0to x=4.

Write the formula for the volume of a solid of revolution about the x-axis.

Explain how integrals are used to calculate physical quantities like work or volume.

Series Summary

This Series introduced **integral calculus**, the counterpart to differential calculus. We began with indefinite integrals, defining integration as reverse differentiation and learning standard formulas and properties. Next, we studied definite integrals, connecting them to limits and the Fundamental Theorem of Calculus, and explored their properties. We then examined techniques of integration—substitution, parts, and partial fractions—to handle more complex problems. Finally, we applied integrals to geometry and physics, calculating areas under curves, areas between curves, and volumes of solids of revolution.

By completing this Series, you should now be able to compute indefinite and definite integrals, apply integration techniques, and use integrals to solve real-world problems involving areas, volumes, and accumulation.

Glossary of Terms

Indefinite integral: The antiderivative of a function, including a constant of integration.

Definite integral: The accumulation of values of a function over an interval, often interpreted as area under a curve.

Fundamental Theorem of Calculus: Connects differentiation and integration, showing they are inverse processes.

Integration by substitution: Technique that simplifies integrals by changing variables.

Integration by parts: Technique based on the product rule of differentiation.

Partial fractions: Method of decomposing rational functions into simpler fractions for integration.

Solid of revolution: A 3D solid formed by rotating a curve around an axis, with volume calculated using integrals.

Concept Check Questions 5

Objective questions

The integral of x3is: A. x44+CB. 3x2+CC. x22+CD. None of the above (Linked to SLO 6.1)

The definite integral 01xdxequals: A. 12B. 1 C. 0 D. 13(Linked to SLO 6.2)

Short-answer questions

State the Fundamental Theorem of Calculus. (Linked to SLO 6.2)

Write the formula for integration by parts. (Linked to SLO 6.3)

Long-answer questions

Evaluate 01(x2+2x)dxand explain the steps. (Linked to SLO 6.2)

Basic response: Show antiderivative and apply limits.

Value-added response: Interpret result as area under curve.

Derive the formula for the volume of a solid of revolution about the x-axis and apply it to y=xfrom x=0to x=1. (Linked to SLO 6.4)

Basic response: Show derivation and calculation.

Value-added response: Explain geometric meaning of the result.

Answers to Concept Check Questions 5

A. x44+C

A. 12

It states that if F(x)is an antiderivative of f(x), then abf(x)dx=F(b)-F(a).

∫u dv=uv-∫v du.

01(x2+2x)dx=x201=13+1=43.

Value-added response: This represents the area under the curve y=x2+2xfrom 0 to 1.

Formula: V=πab[f(x)]2dx. For y=x, V=π01x2dx=π·13=3.

Value-added response: This is the volume of a cone-like solid formed by rotating the line y=xaround the x-axis.

Answers to Pilot Questions 5

From Part 5.1

x44+C.

∫(2x+3)dx=x2+3x+C.

Constant of integration accounts for family of antiderivatives.

Linearity: ∫[af(x)+bg(x)]dx=a∫f(x)dx+b∫g(x)dx.

Differentiating x2+3x+Cgives 2x+3, verifying result.

From Part 5.2

13xdx=x2213=92-12=4.

Fundamental Theorem: Differentiation and integration are inverse processes.

Additivity: abf(x)dx+bcf(x)dx=acf(x)dx.

01(2x+1)dx=x01=2.

Reversing limits changes sign because direction of accumulation is reversed.

From Part 5.3

Substitution: Let u=x3+1, du=3x2dx. Integral = ∫u4du=u55+C=(1)55+C.

Formula: ∫udv=uv-∫vdu.

∫xsinxdx=-xcosx+∫cosxdx=-xcosx+sinx+C.

2xx2-1=1x-1+1x+1.

Partial fractions simplify rational functions into basic integrals.

From Part 5.4

023xdx=3x2202=6.

Formula: ab[f(x)-g(x)]dx.

0/4(cosx-sinx)dx=[sinx+cosx]0/4=22-(0+1)=2-1.

V=πab[f(x)]2dx.

Integrals sum infinitesimal contributions, allowing calculation of work, volume, and other physical quantities.