La-Net

MTH201

MATHEMATICAL METHODS I

1 Real-Valued Functions of a Real Variable

Series outline

- **Part 1.1: Concept of Real-Valued Functions**
  - **Definition of a real-valued function**
  - **Domain, codomain, and range**
  - **Graphical representation of functions**
- **Part 1.2: Types and Properties of Real-Valued Functions**
  - **Polynomial, rational, trigonometric, exponential, and logarithmic functions**
  - **Properties: continuity, monotonicity, boundedness**
  - **Symmetry and periodicity**
- **Part 1.3: Operations on Real-Valued Functions**
  - **Basic operations: addition, subtraction, multiplication, division**
  - **Composition of functions**
  - **Inverse functions**
- **Part 1.4: Limits and Continuity of Functions**
  - **Concept of limits**
  - **Techniques for evaluating limits**
  - **Continuity and discontinuity**

Introduction

Functions are one of the most important building blocks in mathematics. They describe how one quantity depends on another, and they appear everywhere in science, engineering, and everyday life. For example, when you calculate how distance changes with time, or how profit varies with sales, you are working with functions. In this Series, we focus on **real-valued functions of a real variable**, which form the foundation for calculus and analysis. This knowledge is crucial because it prepares you for later topics such as differentiation, integration, and multivariable calculus.

The aim of this Series is to introduce you to the fundamental concepts, types, and properties of real-valued functions of a real variable, and to equip you with the skills needed to analyse, manipulate, and apply them in mathematical problem solving.

We shall commence this Series by examining the concept of real-valued functions, including their definition, domain, codomain, and graphical representation. Next, we will explore the different types and properties of functions, such as polynomial, trigonometric, exponential, and logarithmic forms, as well as continuity and symmetry. Following that, we will consider operations on functions, including composition and inverses. Finally, we will wrap up by studying limits and continuity, which provide the gateway to calculus and deeper mathematical analysis.

Series Learning Outcomes

Upon completing this Series, you should be able to:

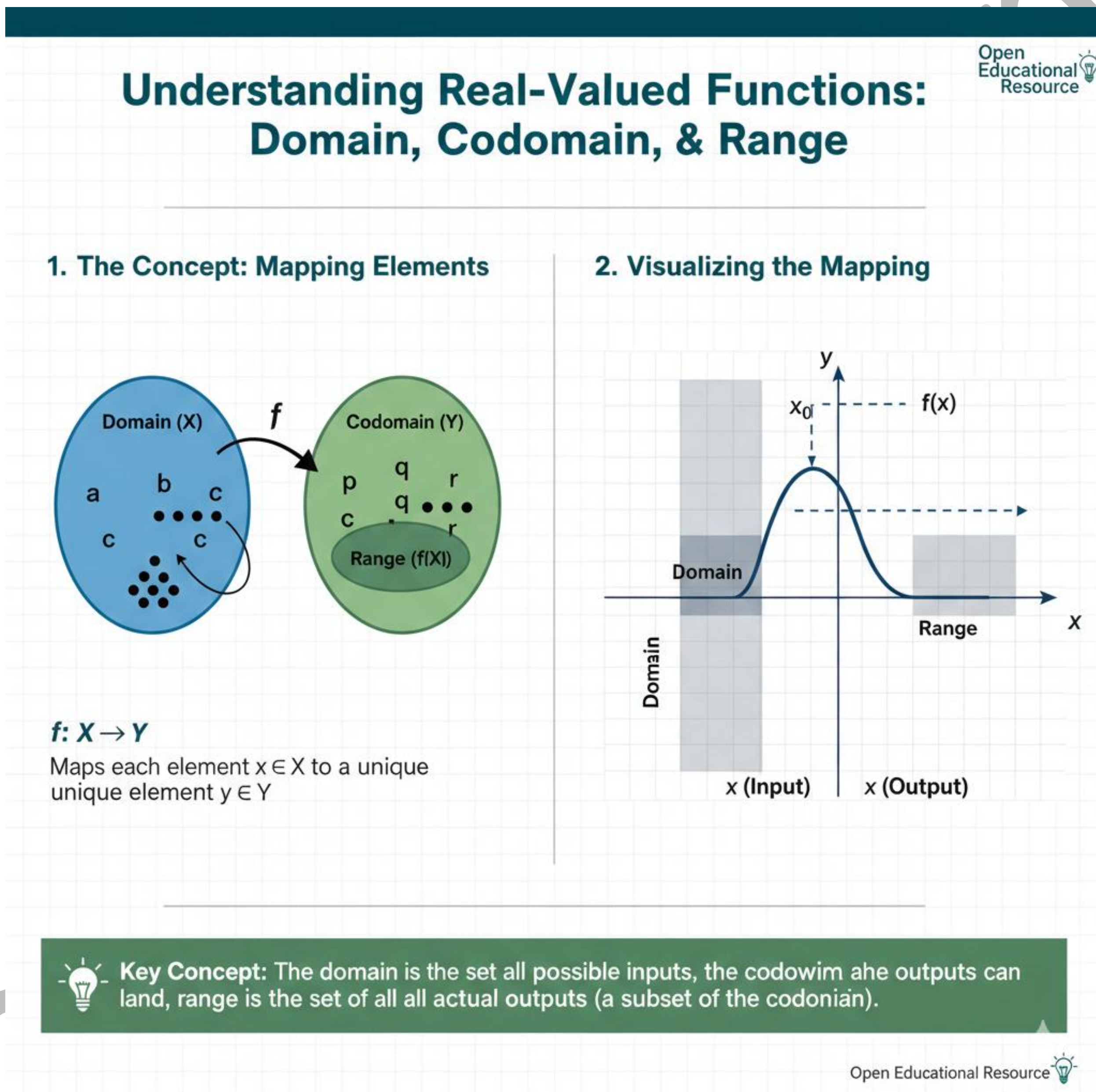
- **SLO 1.1** define real-valued functions of a real variable and identify their domain, codomain, and range,
- **SLO 1.2** explain the main types of real-valued functions, including polynomial, rational, trigonometric, exponential, and logarithmic forms,
- **SLO 1.3** analyse the properties of real-valued functions, such as continuity, monotonicity, boundedness, symmetry, and periodicity,
- **SLO 1.4** demonstrate how to perform operations on functions, including addition, subtraction, multiplication, division, composition, and finding inverses,
- **SLO 1.5** evaluate limits of real-valued functions and assess their continuity and discontinuity.

1.1 Concept of Real-Valued Functions

1.1.1 Definition of a real-valued function

A **real-valued function** is a rule that assigns each element in a set called the **domain** to exactly one real number in a set called the **codomain**. The actual set of outputs produced is called the **range**. For example, the function f(x)=x2takes any real number xand maps it to its square.

Fill in the blank: A real-valued function assigns each element in the domain to exactly one __________ in the codomain.



Mapping of a real-valued function from domain to codomain

1.1.2 Domain, codomain, and range

The **domain** of a function is the set of input values for which the function is defined. The **codomain** is the set of all possible outputs, while the **range** is the actual set of outputs produced. For instance, for f(x)=x, the domain is all non-negative real numbers, the codomain is the set of real numbers, and the range is all non-negative real numbers.

Fill in the blank: The set of all actual outputs of a function is called the __________.

Table showing definitions of domain, codomain, and range with analogies.

Term	Definition	Vending Machine Analogy
Domain	The set of all possible **input** values (usually x) for which the function is defined.	**The Buttons:** The set of all valid codes (like A1, B2) that you are allowed to press.
Codomain	The set of all **potential** output values that *could* theoretically come out of the function.	**The Inventory:** Every item currently sitting inside the glass—all the chips, sodas, and candies.
Range	The set of **actual** output values (y) that the function specifically maps to from the domain.	**The Selection:** Only the specific items that will actually fall into the bin when a valid button is pressed.

1.1.3 Graphical representation of functions

Functions can be represented visually using **graphs**. A graph of a function is the set of points x,f(x)plotted in the Cartesian plane. Graphs help us to understand the behaviour of functions, such as whether they increase, decrease, or remain constant. For example, the graph of f(x)=x2is a parabola opening upwards.

Fill in the blank: The graph of a function is the set of points (x, __________) plotted in the Cartesian plane.

[Insert figure here] Short description: Graph of the function f(x)=x2. Caption: Figure 1.2: Graphical representation of the function f(x)=x2AI prompt: “Generate a graph of the function f(x) = x^2 showing a parabola opening upwards.” Search string: “graph of f(x) = x^2 parabola OER”

Pilot Questions 1.1

- A real-valued function assigns each element in the domain to exactly one: A. Real number in the codomain B. Random value C. Undefined property D. Arbitrary guess (Linked to SLO 1.1)
- The set of input values for which a function is defined is called the: A. Domain B. Codomain C. Range D. Arbitrary set (Linked to SLO 1.1)
- The set of all actual outputs of a function is called the: A. Range B. Domain C. Codomain D. Arbitrary guess (Linked to SLO 1.1)
- The graph of a function is the set of points: A. x,f(x)plotted in the Cartesian plane B. Random points only C. Undefined values D. Arbitrary guesses (Linked to SLO 1.1)
- The function f(x)=x2has a graph shaped as a: A. Parabola opening upwards B. Straight line C. Circle D. Arbitrary shape (Linked to SLO 1.1)

1.2 Types and Properties of Real-Valued Functions

1.2.1 Types of real-valued functions

There are several important categories of **real-valued functions**, each with distinctive features.

- A **polynomial function** is expressed as a finite sum of powers of xwith constant coefficients, for example f(x)=3x2+2x+1.
- A **rational function** is defined as the ratio of two polynomials, such as f(x)=x2+1x-2.
- A **trigonometric function** involves angles and periodic behaviour, for example f(x)=sin x.
- An **exponential function** has the variable in the exponent, such as f(x)=ex.
- A **logarithmic function** is the inverse of an exponential function, for example f(x)=ln x.

Fill in the blank: A function expressed as the ratio of two polynomials is called a __________ function.

Types of real-valued functions with examples

Comparison of Function Types

Function Type	Definition	General Form	Example
Polynomial	A sum of variables raised to non-negative integer powers.	$f(x) = a_n x^n + ... + a_1 x + a_0$	$f(x) = 3x^2 + 2x - 5$
Rational	The ratio of two polynomial functions.	$f(x) = \frac{P(x)}{Q(x)}$	$f(x) = \frac{x+1}{x^2-4}$
Trigonometric	Functions based on the ratios of sides of a right triangle or unit circle coordinates.	$f(x) = \sin(x), \cos(x), \tan(x)$	$f(x) = 2\sin(3x + \pi)$
Exponential	A constant base raised to a variable exponent.	$f(x) = a \cdot b^x$	$f(x) = e^{-0.5x}$
Logarithmic	The inverse of exponential functions; determines the power to which a base must be raised.	$f(x) = \log_b(x)$	$f(x) = \ln(x + 1)$

1.2.2 Properties of real-valued functions

Functions can be studied through their **properties**, which describe how they behave.

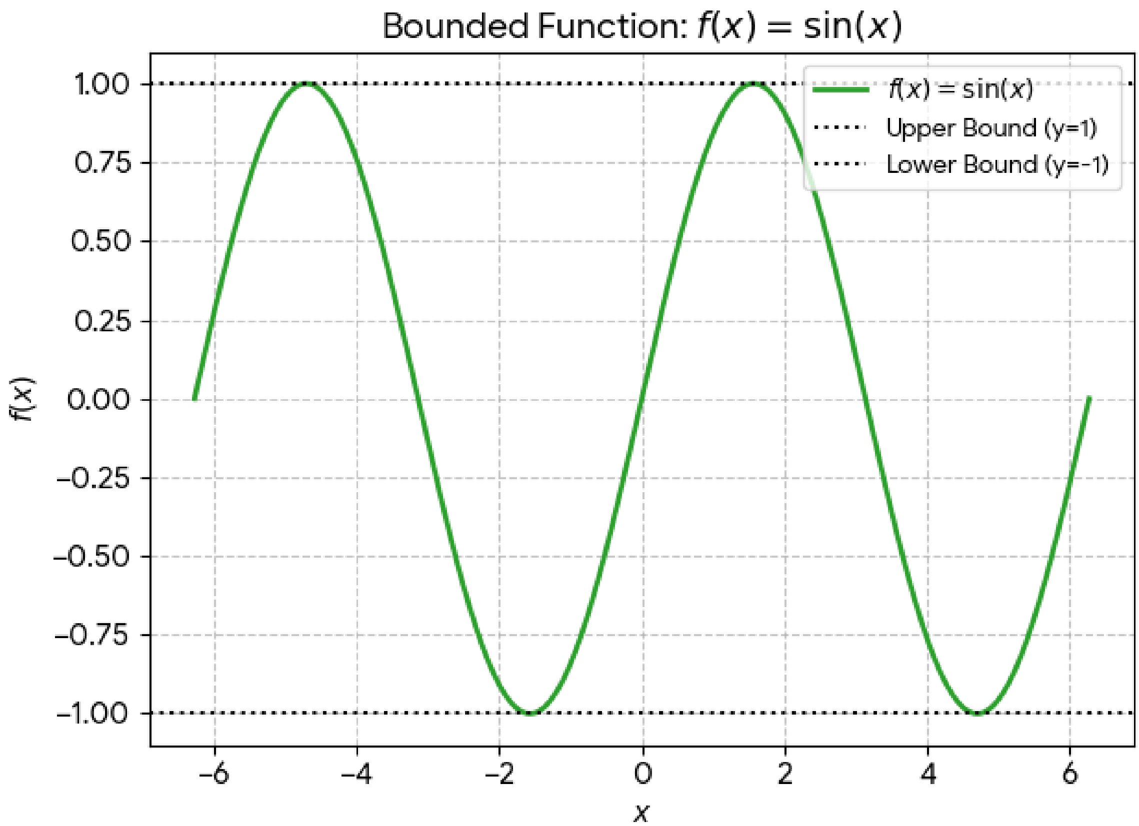
Continuity: A function is continuous if its graph can be drawn without lifting the pen. For example, f(x)=x2is continuous everywhere.

Monotonicity: A function is monotonic if it is always increasing or always decreasing. For instance, f(x)=exis always increasing.

Boundedness: A function is bounded if its values stay within a fixed range. For example, f(x)=sin xis bounded between -1 and 1.

Fill in the blank: A function whose values remain within a fixed range is said to be __________.

: Graphical examples of continuity, monotonicity, and boundedness



1.2.3 Symmetry and periodicity

Two further important properties are **symmetry** and **periodicity**.

An **even function** satisfies $f(-x)=f(x)$, showing symmetry about the y-axis. For example, $f(x)=x2$.

An **odd function** satisfies $f(-x)=-f(x)$, showing symmetry about the origin. For example, $f(x)=x3$.

A **periodic function** repeats its values at regular intervals. For instance, $f(x)=\sin x$has a period of 2π.

Fill in the blank: A function that satisfies $f(-x)=f(x)$is called an __________ function.

Box summarising definitions of even, odd, and periodic functions with examples.

Even, Odd, and Periodic Functions

Type	Mathematical Definition	Visual Symmetry	Example
Even	$f(-x) = f(x)$	Symmetric across the **y-axis** (Mirror image).	$f(x) = \cos(x)$ or $f(x) = x^2$
Odd	$f(-x) = -f(x)$	Symmetric about the **origin** (180° rotation).	$f(x) = \sin(x)$ or $f(x) = x^3$
Periodic	$f(x + P) = f(x)$	The graph repeats its shape every P units.	$f(x) = \tan(x)$ or square waves

Pilot Questions 1.2

A function expressed as a finite sum of powers of xwith constant coefficients is called a: A. Rational function B. Polynomial function C. Exponential function D. Logarithmic function (Linked to SLO 1.2)

A function that is always increasing or always decreasing is described as: A. Continuous B. Monotonic C. Bounded D. Periodic (Linked to SLO 1.3)

The function f(x)=sinxis: A. Unbounded B. Monotonic C. Bounded D. Discontinuous (Linked to SLO 1.3)

A function that satisfies f(-x)=f(x)is: A. Odd B. Even C. Periodic D. Rational (Linked to SLO 1.3)

The function f(x)=sinxhas a period of: A. B. 2πC. 3πD. 4π(Linked to SLO 1.3)

1.3 Operations on Real-Valued Functions

1.3.1 Basic operations on functions

Functions can be combined in various ways to create new functions. The most common **operations on functions** are addition, subtraction, multiplication, and division.

Addition: f+g)(x)=f(x)+g(x.

Subtraction: f-g)(x)=f(x)-g(x.

Multiplication: f·g)(x)=f(x)·g(x.

Division: $(f/g)(x)=f(x)g(x)$, provided $g(x)0$.

These operations allow us to build more complex functions from simpler ones.

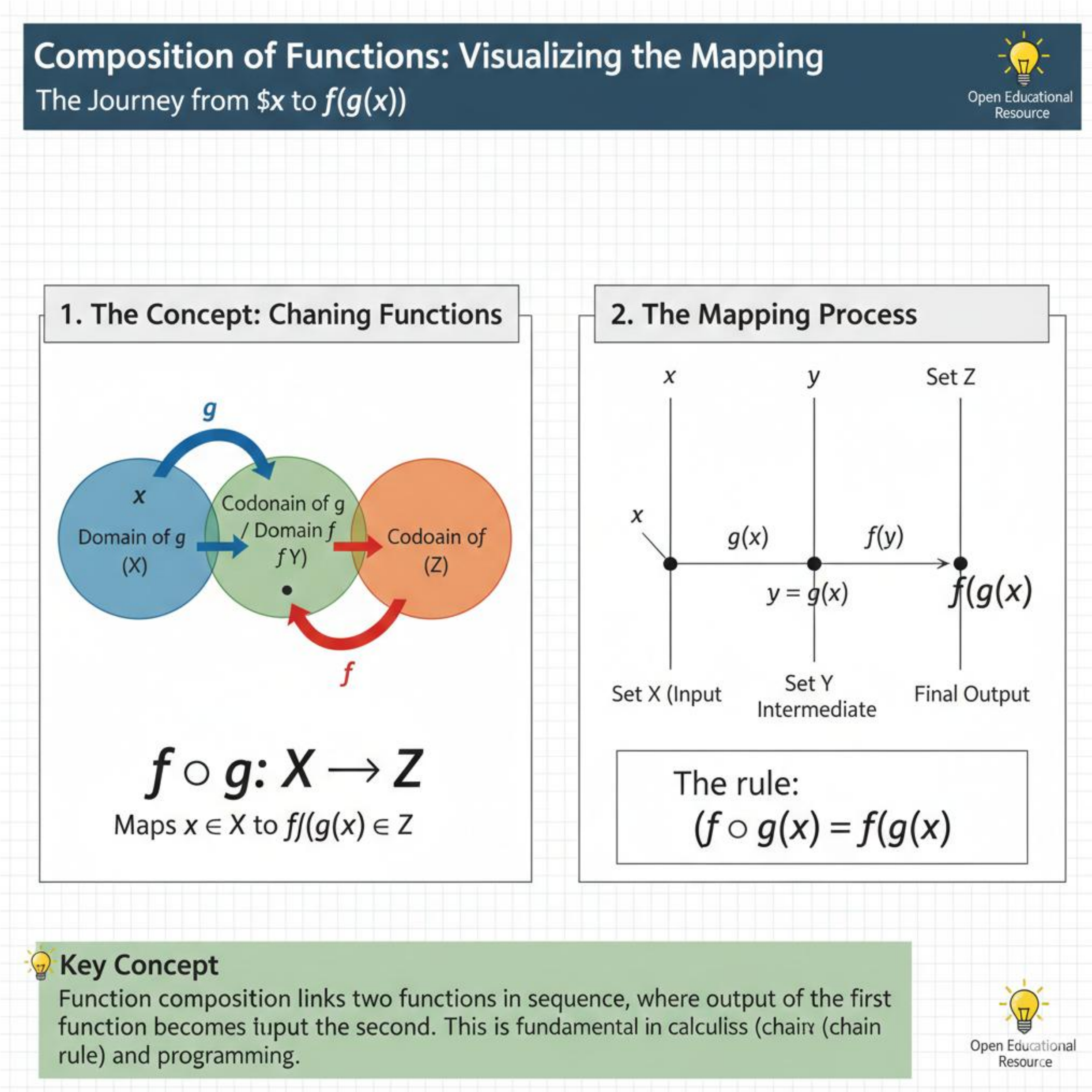
Fill in the blank: The operation $f+g)(x$is defined as __________.

[Insert figure here] Short description: Diagram showing addition and multiplication of two functions. Caption: Figure 1.4: Examples of basic operations on functions AI prompt: "Generate diagrams showing addition and multiplication of two functions f(x) and g(x)." Search string: "operations on functions addition multiplication OER diagrams"

1.3.2 Composition of functions

The **composition of functions** involves applying one function to the result of another. If $f$and $g$are functions, then the composition is written as $f\circ g)(x)=f(g(x)$. For example, if $f(x)=x2$and $g(x)=x+1$, then $(f\circ g)(x)=(x+1)2$.

Fill in the blank: The composition of functions is written as __________.



Composition of functions

1.3.3 Inverse functions

An **inverse function** reverses the effect of another function. If f(x)maps xto y, then the inverse function f-1(y)maps yback to x. A function has an inverse if it is one-to-one (bijective). For example, if f(x)=2x+3, then its inverse is f-1(y)=y-32.

Fill in the blank: A function that reverses the effect of another function is called an __________ function.

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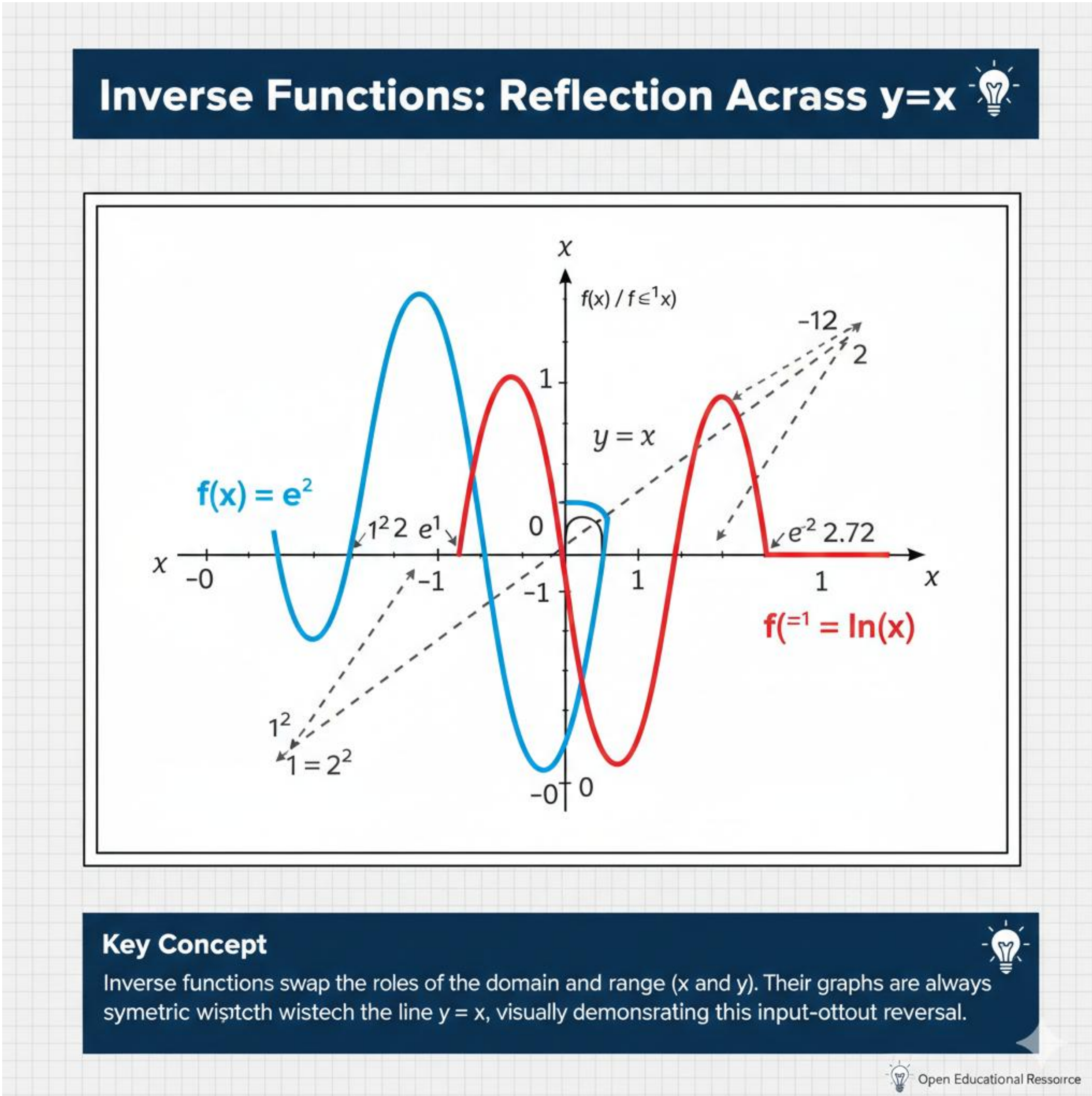
Graphical representation of a function and its inverse



Pilot Questions 1.3

The operation f+g)(xis defined as: A. f(x)·g(x)B. f(x)+g(x)C. f(x)-g(x)D. f(x)g(x)(Linked to SLO 1.4)

The composition of functions is written as: A. f+gB. f·gC. f∘gD. f/g(Linked to SLO 1.4)

If f(x)=x2and g(x)=x+1, then f∘g)(xis: A. x2+1B. (1)2C. x2+x+1D. x+1(Linked to SLO 1.4)

A function that reverses the effect of another function is called: A. Rational function B. Inverse function C. Polynomial function D. Periodic function (Linked to SLO 1.4)

The graph of a function and its inverse is symmetric about the line: A. y=0B. x=0C. y=xD. y=-x(Linked to SLO 1.4)

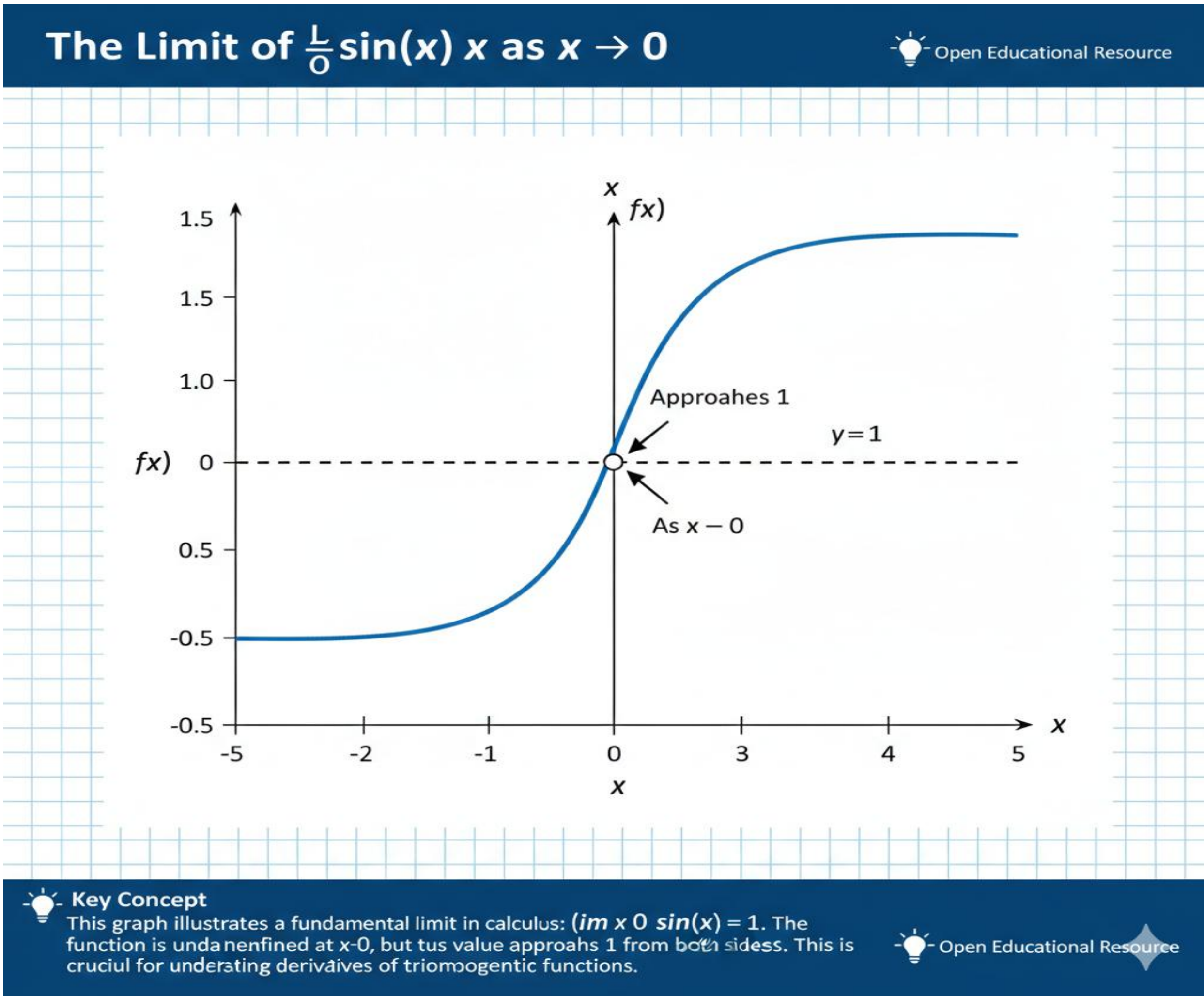
1.4 Limits and Continuity of Functions

1.4.1 Concept of limits

A **limit** describes the value that a function approaches as the input gets closer to a particular point. Limits help us understand the behaviour of functions near points where they may not be explicitly defined. For example, as xapproaches 0, the function sin x xapproaches 1.

Fill in the blank: The value that a function approaches as the input gets closer to a particular point is called a __________.

Illustration of the concept of a limit



1.4.2 Techniques for evaluating limits

There are several techniques for evaluating limits:

- **Direct substitution**: If the function is continuous at the point, simply substitute the value.
- **Factorisation**: Simplify expressions by factoring to remove indeterminate forms.
- **Rationalisation**: Multiply by a conjugate to simplify expressions involving roots.
- **Special limits**: Use known results such as $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Fill in the blank: The technique used to simplify expressions involving square roots when evaluating limits is called __________.

Table summarising direct substitution, factorisation, rationalisation, and special limits.

Summary of Limit Evaluation Techniques

Technique	When to Use	Procedure	Example
Direct Substitution	When $f(x)$ is continuous at c.	Plug the value c into the function.	$\lim_{x\to 2}(x^2+3) = 2^2+3=7$
Factoring	When substitution yields $0/0$ and the expression is a polynomial.	Factor terms and cancel the common factor causing the zero.	$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2)=4$
Rationalization	When substitution yields $0/0$ and contains square roots.	Multiply numerator and denominator by the **conjugate**.	$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$
L'Hôpital's Rule	For indeterminate forms $0/0$ or ∞/∞.	Differentiate the numerator and denominator separately.	$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1}=1$
Squeeze Theorem	When a function is "trapped" between two others with the same limit.	Find two functions $g(x)$ and $h(x)$ such that $g \le f \le h$.	$\lim_{x\to 0}x^2\sin(\frac{1}{x})=0$

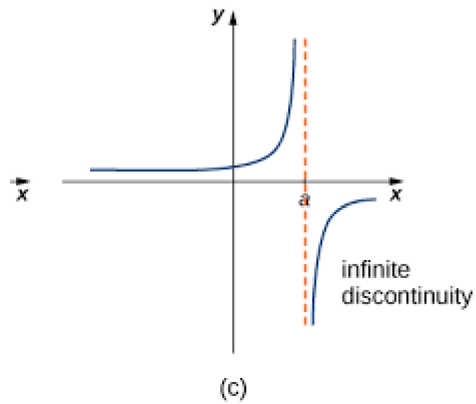
1.4.3 Continuity and discontinuity

A function is **continuous** at a point if the limit of the function as xapproaches that point equals the function's value at that point. If this condition fails, the function is **discontinuous**.

Types of discontinuities include:

- **Removable discontinuity**: A "hole" in the graph, often caused by a factor that can be cancelled.
- **Jump discontinuity**: A sudden change in function values.
- **Infinite discontinuity**: The function tends to infinity near the point.

Fill in the blank: A function is continuous at a point if the limit of the function as xapproaches that point equals the __________ at that point.



Types of discontinuities in functions

Pilot Questions 1.4

The value that a function approaches as the input gets closer to a particular point is called a: A. Range B. Limit C. Domain D. Codomain (Linked to SLO 1.5)

The technique used to simplify expressions involving square roots when evaluating limits is called: A. Factorisation B. Rationalisation C. Substitution D. Expansion (Linked to SLO 1.5)

A function is continuous at a point if: A. The limit equals the function's value at that point B. The function is bounded C. The function is monotonic D. The function is periodic (Linked to SLO 1.5)

A sudden change in function values is called a: A. Removable discontinuity B. Jump discontinuity C. Infinite discontinuity D. Rational discontinuity (Linked to SLO 1.5)

The function sin x xas x→0approaches: A. 0 B. 1 C. Infinity D. Undefined (Linked to SLO 1.5)

Series Summary

This Series has introduced you to the foundations of **real-valued functions of a real variable**, which are essential for calculus and analysis. We began by examining the concept of functions, including their definition, domain, codomain, and range, and how they can be represented graphically. We then explored the different types of functions such as polynomial, rational, trigonometric, exponential, and logarithmic, and studied their properties including continuity, monotonicity, boundedness, symmetry, and periodicity. Following that, we considered operations on functions, including addition, subtraction, multiplication, division, composition, and inverses. Finally, we discussed limits and continuity, which provide the gateway to differentiation and integration.

By completing this Series, you should now be able to **define** real-valued functions and their components (SLO 1.1), **explain** the main types of functions (SLO 1.2), **analyse** their properties (SLO 1.3), **demonstrate** operations on functions (SLO 1.4), and **evaluate** limits and continuity (SLO 1.5). These skills form a strong foundation for the more advanced topics that follow in this course.

Glossary of Terms

Codomain: The set of all potential output values that a function could produce.

Domain: The set of all possible input values for which a function is defined.

Even function: A function where f(-x)=f(x), showing symmetry about the y-axis.

Inverse function: A function that reverses the effect of another function, mapping outputs back to inputs.

Limit: The value that a function approaches as the input approaches a particular point.

Monotonic function: A function that is always increasing or always decreasing across its domain.

Odd function: A function where f(-x)=-f(x), showing symmetry about the origin.

Polynomial function: A function expressed as a finite sum of powers of xwith constant coefficients.

Range: The set of all actual output values produced by a function.

Rational function: A function defined as the ratio of two polynomials.

Symmetry: A property of functions where their graphs exhibit reflectional or rotational balance.

Concept Check Questions 1

Objective questions

A real-valued function assigns each element in the domain to exactly one __________ in the codomain. (Linked to SLO 1.1)

Which of the following is an example of a polynomial function? A. sin x B. ln x C. 3x2+2x+1D. x2+1x-2(Linked to SLO 1.2)

A function is monotonic if it is always __________ or always decreasing. (Linked to SLO 1.3)

The composition of functions is written as: A. f+gB. f·gC. f∘gD. f/g(Linked to SLO 1.4)

The graph of f(x)=x2is a __________ opening upwards. (Linked to SLO 1.5)

Short-answer questions

Define the domain of a function and give one example. (Linked to SLO 1.1)

State the difference between codomain and range. (Linked to SLO 1.1)

Explain what it means for a function to be bounded. (Linked to SLO 1.3)

Long-answer questions

Analyse the properties of polynomial and trigonometric functions, highlighting their similarities and differences. (Linked to SLO 1.2, SLO 1.3)

Basic response: Describe the general forms and properties of each.

Value-added response: Compare their behaviour in terms of continuity, boundedness, and periodicity, with examples.

Evaluate the importance of limits and continuity in preparing for calculus. (Linked to SLO 1.5)

Basic response: Explain how limits define the behaviour of functions at specific points.

Value-added response: Discuss how continuity and limits underpin differentiation and integration, with practical examples.

Answers to Pilot Questions 1

Real number in the codomain

Domain

Range

x,f(x)plotted in the Cartesian plane

Parabola opening upwards

Rational function

Monotonic

Bounded

Even function

2π

Denominator not equal to zero

f(g(x))

2 Mean Value Theorem and Taylor Series Expansion

Series outline

Part 2.1: The Mean Value Theorem

Statement of the theorem

Geometric interpretation

Applications in problem solving

Part 2.2: Taylor Series Expansion

Definition and general form of Taylor series

Maclaurin series as a special case

Examples of common expansions (e.g., ex, $\sin x$ **,** $\cos x$ **)**

Part 2.3: Error and Remainder Term in Taylor Series

Lagrange form of the remainder

Estimating approximation errors

Practical applications of error bounds

Introduction

The **Mean Value Theorem (MVT)** and **Taylor Series Expansion** are two central results in calculus that connect the behaviour of functions to powerful methods of approximation. The MVT provides a precise link between the average rate of change of a function over an interval and its instantaneous rate of change at some point within that interval. Taylor series, on the other hand, allow us to approximate complex functions using polynomials, making them easier to compute and analyse. Together, these concepts form the backbone of many applications in mathematics, science, and engineering.

The aim of this Series is to enable you to apply the Mean Value Theorem to problems involving rates of change, and to construct and use Taylor series expansions to approximate functions and estimate errors in calculations.

We shall commence this Series by examining the Mean Value Theorem, its formal statement, geometric interpretation, and applications. Next, we will explore Taylor series expansion, including the special case of Maclaurin series, with examples of common functions such as exponential and trigonometric forms. Finally, we will consider the error and remainder term in Taylor series, which provides a measure of accuracy in approximations and ensures confidence in practical applications.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 state the Mean Value Theorem clearly and interpret its geometric meaning,

SLO 2.2 apply the Mean Value Theorem to solve problems involving rates of change,

SLO 2.3 construct Taylor series expansions for functions about a given point,

SLO 2.4 derive Maclaurin series as a special case of Taylor series,

2.1 The Mean Value Theorem

2.1.1 Statement of the theorem

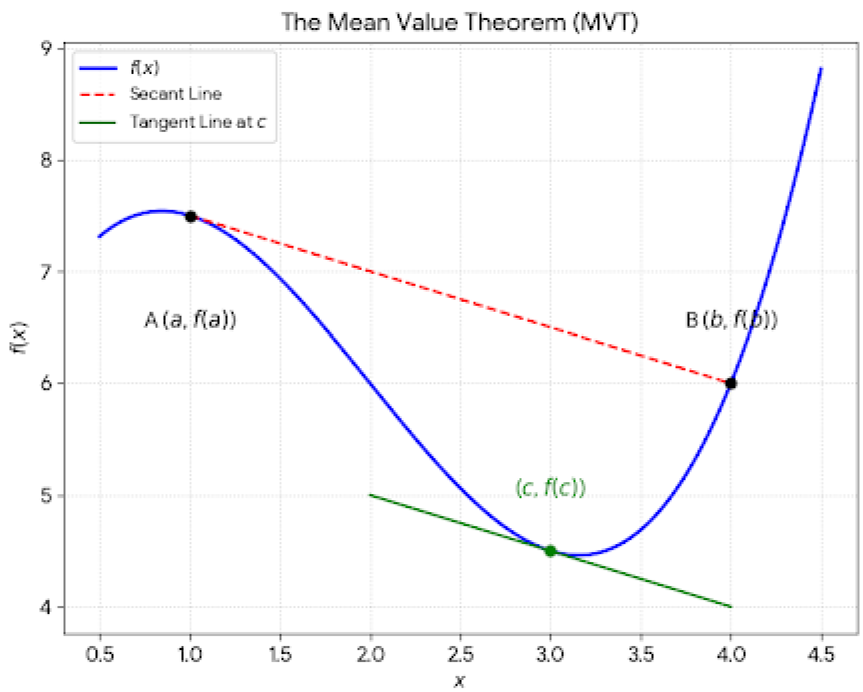
The **Mean Value Theorem (MVT)** is a fundamental result in calculus. It states that if a function f(x)is continuous on a closed interval band differentiable on the open interval b, then there exists at least one point c∈(a,b)such that

$$f'(c)=f(b)-f(a)b-a.$$

This means that the instantaneous rate of change of the function at some point cis equal to the average rate of change of the function over the interval b.

Fill in the blank: The Mean Value Theorem requires a function to be __________ on band differentiable on b.

Geometric illustration of the Mean Value Theorem



2.1.2 Geometric interpretation

Geometrically, the theorem tells us that for a smooth curve joining two points, there is at least one point where the tangent to the curve is parallel to the secant line connecting the two points. This interpretation helps us visualise the theorem and understand its significance in terms of slopes.

Fill in the blank: The tangent line at some point cis parallel to the __________ line joining points aand b.

2.1.3 Applications in problem solving

The Mean Value Theorem has several applications:

It provides a foundation for proving other results in calculus, such as Rolle's Theorem.

It is used to estimate changes in functions and to justify approximations.

It helps in analysing motion, showing that if an object travels a certain distance in a given time, there must be a moment when its instantaneous speed equals its average speed.

Fill in the blank: If an object travels a certain distance in a given time, there must be a moment when its instantaneous speed equals its __________ speed.

Box summarising applications of MVT in proving other theorems, estimating changes, and analysing motion.

Core Applications in Calculus

Application	Description	Significance
Monotonicity Test	Proves that if $f'(x) > 0$ on an interval, the function is increasing.	Allows us to use derivatives to sketch functions accurately.
Constant Function Theorem	If $f'(x) = 0$ for all x in an interval, then $f(x)$ is a constant.	Foundational for understanding antiderivatives and the "$+C$" in integration.
Error Estimation	Used to find the maximum possible error in linear approximations and Taylor polynomials.	Critical for numerical analysis to ensure computer simulations stay within safe bounds.
Uniqueness of Roots	Can be used to prove that a function has *at most* one root in a specific interval.	Used in root-finding algorithms to determine if a search is complete.

Pilot Questions 2.1

The Mean Value Theorem states that there exists a point csuch that: A. f(c)=f(b)-f(a)b-aB. f'(c)=f(b)-f(a)b-aC. f(c)=f(a)+f(b)D. f'(c)=f(a)+f(b)(Linked to SLO 2.1)

The Mean Value Theorem requires a function to be: A. Continuous on band differentiable on bB. Differentiable on band continuous on bC. Continuous everywhere only D. Differentiable everywhere only (Linked to SLO 2.1)

Geometrically, the tangent line at some point cis parallel to the: A. Axis B. Secant line C. Normal line D. Graph itself (Linked to SLO 2.1)

The Mean Value Theorem can be used to prove: A. Rolle's Theorem B. Fundamental Theorem of Algebra C. Binomial Theorem D. Pythagoras' Theorem (Linked to SLO 2.2)

In motion analysis, the Mean Value Theorem shows that instantaneous speed at some point equals: A. Maximum speed B. Minimum speed C. Average speed D. Constant speed (Linked to SLO 2.2)

2.2 Taylor Series Expansion

2.2.1 Definition and general form of Taylor series

A **Taylor series** is a way of expressing a function as an infinite sum of terms calculated from its derivatives at a single point. If a function f(x)is infinitely differentiable at a point a, then its Taylor series about ais given by:

$$f(x)=f(a)+f'(a)(x-a)+f''(a)2!(x-a)2+f3(a)3!(x-a)3+\cdots$$

This expansion allows us to approximate functions using polynomials, which are easier to compute and analyse.

Fill in the blank: A Taylor series expresses a function as an infinite sum of terms calculated from its __________ at a single point.

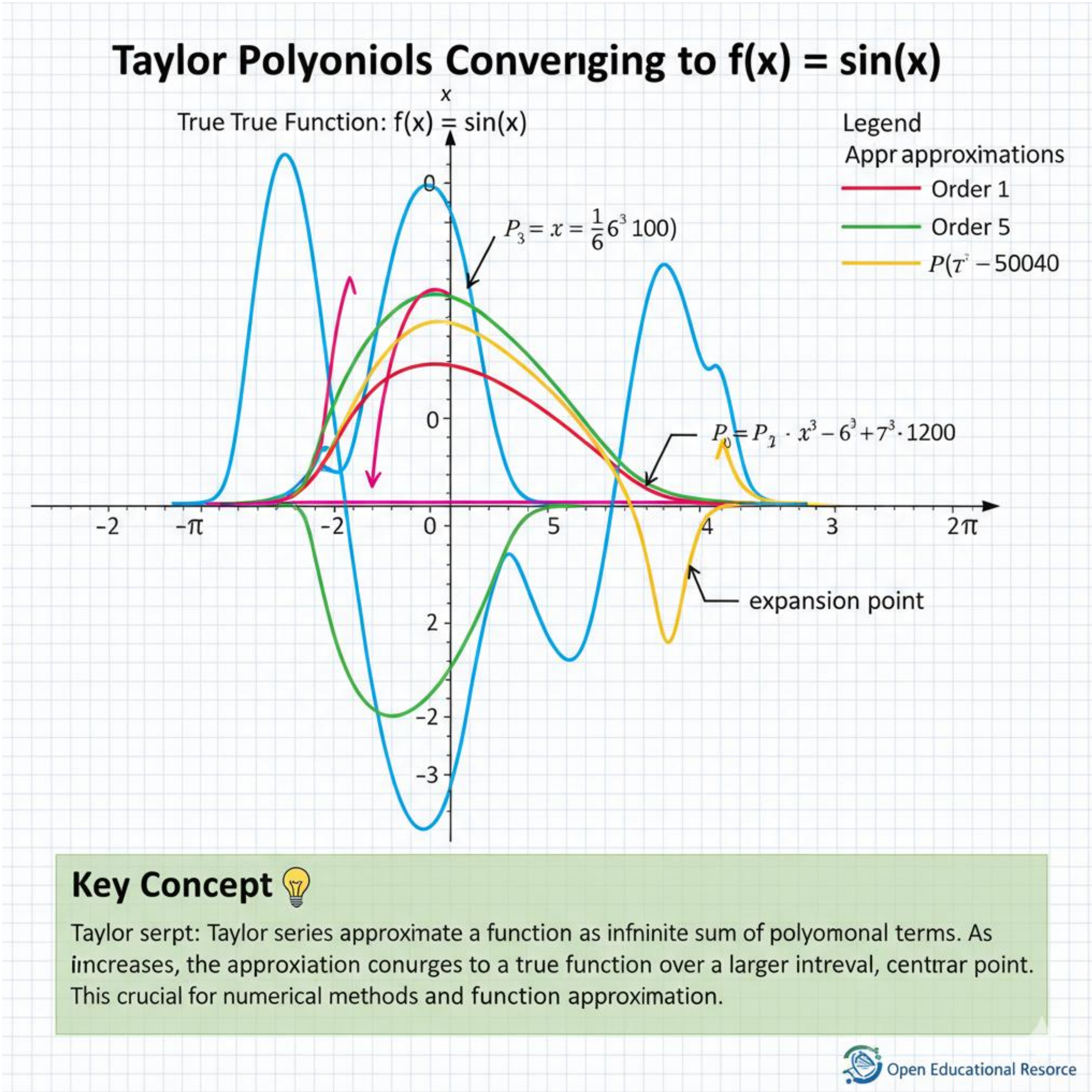


Illustration of Taylor polynomial approximations approaching a function

2.2.2 Maclaurin series as a special case

A **Maclaurin series** is a special case of the Taylor series where the expansion is taken about a=0. The general form is:

$$f(x)=f(0)+f'(0)x+f''(0)2!x2+f3(0)3!x3+\cdots$$

For example:

$ex=1+x+x22!+x33!+\cdots$

sin x=x-x33!+x55!-⋯

cos x=1-x22!+x44!-⋯

Fill in the blank: A Maclaurin series is a Taylor series expansion about the point __________.

Table listing Maclaurin series for ex, sin x , and cos x ”

Common Maclaurin Series

The table below shows the expansions for the three most fundamental functions in calculus and engineering.

Function	Maclaurin Series Expansion	Sigma Notation	Interval of Convergence
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$

2.2.3 Examples of common expansions

Taylor and Maclaurin series are widely used to approximate functions. For instance:

For small values of x, sin x≈x.

For small values of x, ex≈1+x.

For small values of x, cos x≈1-x22.

These approximations are useful in physics, engineering, and numerical analysis, where exact values are difficult to compute.

Fill in the blank: For small values of x, the approximation \sin x \approx __________ is often used.

Box summarising applications of Taylor and Maclaurin series in approximation, physics, and engineering

Real-World Utility Across Disciplines

Field	Application	Why it Matters
Numerical Analysis	**Function Approximation**	Computers don't "know" what $\sin(x)$ is; they calculate it using the first few terms of a Maclaurin series to a specific decimal precision.
Physics	**Small Angle Approximation**	In pendulum motion, we replace $\sin(\theta)$ with θ. This linearizes the differential equation, turning a messy problem into a simple harmonic oscillator.
Engineering	**Linearization**	Complex non-linear systems (like aircraft flight dynamics) are approximated by linear Taylor series at specific equilibrium points to design stable control systems.
Computational Finance	**Option Pricing**	Taylor series are used to calculate "The Greeks" (like Delta and Gamma), which measure how sensitive an option's price is to small changes in the underlying stock price.

Pilot Questions 2.2

A Taylor series expresses a function as: A. A finite sum of constants B. An infinite sum of terms from derivatives C. A ratio of polynomials D. A trigonometric identity (Linked to SLO 2.3)

A Maclaurin series is a Taylor series expansion about the point: A. a=1B. a=0C. a=-1D. a=π(Linked to SLO 2.4)

The Maclaurin series for exbegins with: A. 1+x+x22!B. x-x33!C. 1-x22!D. x2+x3(Linked to SLO 2.4)

For small values of x, sin x can be approximated by: A. 1+xB. xC. 1-x22D. x33!(Linked to SLO 2.3)

Taylor and Maclaurin series are particularly useful in: A. Geometry B. Physics and engineering C. Literature D. History (Linked to SLO 2.5)

2.3 Error and Remainder Term in Taylor Series

2.3.1 Lagrange form of the remainder

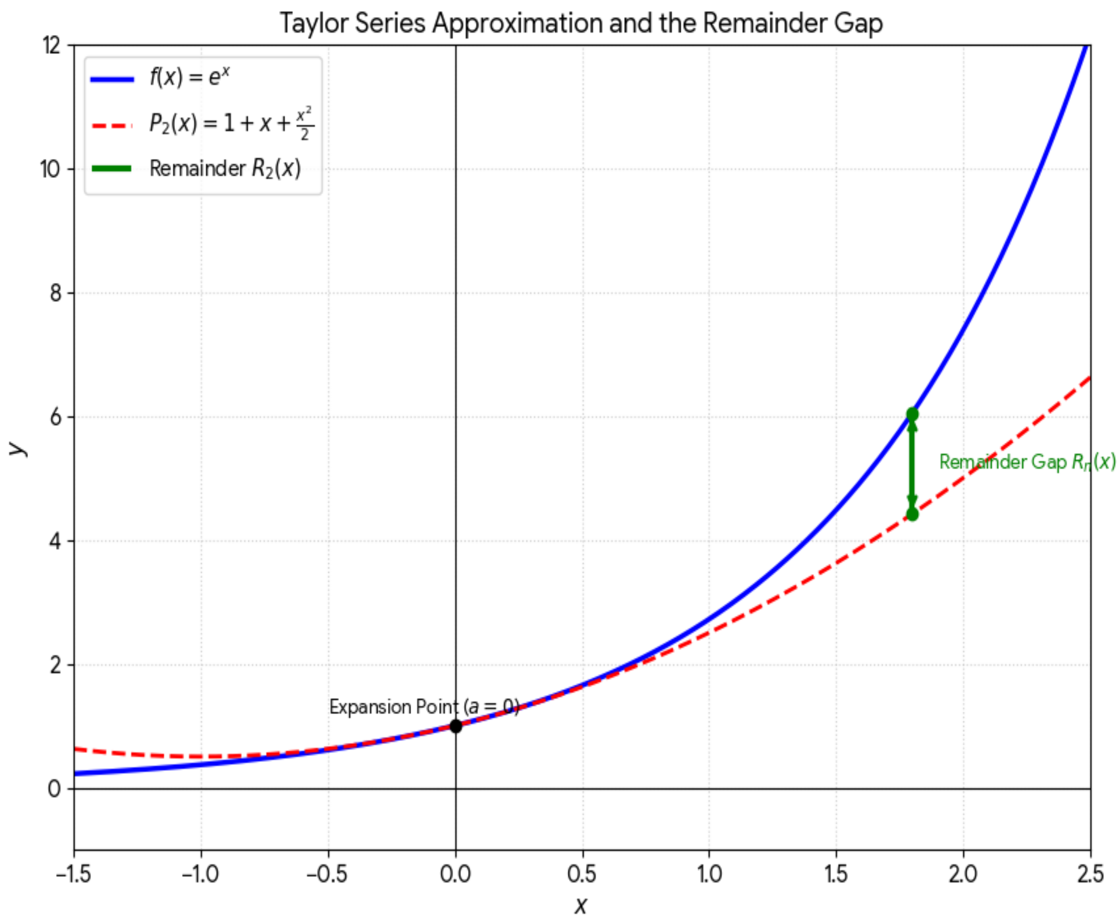
When we approximate a function using a Taylor polynomial, the difference between the actual function and the polynomial is called the **remainder term**. The **Lagrange form of the remainder** is given by:

$$Rn(x)=f1(c)(n+1)!(x\text{-}a)n+1$$

for some cbetween aand x. This expression tells us how far the Taylor polynomial of degree nis from the true value of the function.

Fill in the blank: The difference between the actual function and its Taylor polynomial approximation is called the __________ term.

Illustration of remainder term in Taylor series approximation



2.3.2 Estimating approximation errors

The remainder term allows us to **estimate errors** in approximations. By bounding the derivative f1(c), we can determine how accurate the polynomial approximation is. For example, when approximating sin x near 0 using its Maclaurin series, the error decreases rapidly as more terms are included.

Fill in the blank: The accuracy of a Taylor polynomial approximation depends on the size of the __________ term.

Table showing error estimates for approximations of ex, sin x , and cos x . AI prompt

Taylor Series Error Estimation

When we use a Taylor polynomial to approximate a function, we need to know how much we can trust the result. The **Lagrange Error Bound** (also known as the Taylor Remainder Theorem) provides a mathematical way to calculate the maximum possible error for a given number of terms.

Error Bounds for Common Maclaurin Series

The following table summarizes the error bound $|R_n(x)|$ for the three most common expansions centered at $a = 0$. In all cases, c is a value between 0 and x that maximizes the $(n+1)$-th derivative.

Function	n-th Degree Polynomial ($P_n(x)$)	Max Error Bound $|R_n(x)| \leq$	Notes on Error Behavior
e^x	$\sum_{i=0}^{n} \frac{x^i}{i!}$	$\frac{e^k}{(n+1)!}|x|^{n+1}$	$k = x$ if $x > 0$; $k = 0$ if $x < 0$. Error grows rapidly as x increases.
$\sin(x)$	$\sum_{i=0}^{n}(-1)^i \frac{x^{2i+1}}{(2i+1)!}$	$\frac{|x|^{n+1}}{(n+1)!}$	Derivative of $\sin(x)$ is always ≤ 1, making error calculation very simple.
$\cos(x)$	$\sum_{i=0}^{n}(-1)^i \frac{x^{2i}}{(2i)!}$	$\frac{|x|^{n+1}}{(n+1)!}$	Similar to sine, the error is bounded by the magnitude of the next term.

2.3.3 Practical applications of error bounds

Error bounds are crucial in practical applications. In numerical analysis, physics, and engineering, approximations are often used instead of exact values. Knowing the error bound ensures that results are reliable. For instance, when calculating trajectories in physics or designing circuits in engineering, Taylor approximations are used with confidence because the error can be controlled.

Fill in the blank: In practical applications, knowing the __________ ensures that approximations are reliable.

Box summarising applications of error bounds in numerical analysis, physics, and engineering.

Field	Application	Why the Error Bound is Critical
Numerical Analysis	**Algorithm Precision**	When a computer computes \sqrt{x} or \ln(x)$, the error bound determines how many terms the CPU must process to guarantee "double-precision" (15-17 decimal places).
Physics	**Perturbation Theory**	In quantum mechanics or celestial mechanics, we approximate complex equations. The error bound tells us if the "neglected" higher-order physics are significant enough to change the outcome.
Civil Engineering	**Structural Tolerance**	When calculating the deflection of a beam under a load, engineers use Taylor series to simplify beam equations. The error bound ensures the simplification doesn't underestimate the stress on the material.
Control Systems	**Sensor Calibration**	Sensors often produce non-linear signals. To "linearize" them for a flight controller, the error bound determines the "safe operating range" where the linear model remains accurate.

Pilot Questions 2.3

The difference between the actual function and its Taylor polynomial approximation is called the: A. Error term B. Remainder term C. Approximation term D. Polynomial term (Linked to SLO 2.5)

The Lagrange form of the remainder is expressed as: A. fn(c)n!(x-a)nB. f1(c)(n+1)!(x-a)n+1C. f1(c)(n-1)!(x-a)n-1D. f(c)(n+1)!(x-a)n+1(Linked to SLO 2.5)

The accuracy of a Taylor polynomial approximation depends on the size of the: A. Domain B. Range C. Remainder term D. Polynomial degree only (Linked to SLO 2.5)

In numerical analysis, error bounds are important because they: A. Eliminate the need for approximations B. Ensure approximations are reliable C. Make functions continuous D. Remove periodicity (Linked to SLO 2.5)

Series Summary

This Series has focused on two powerful tools in calculus: the **Mean Value Theorem (MVT)** and the **Taylor Series Expansion**. We began by stating and interpreting the MVT, exploring its geometric meaning and applications in problem solving, particularly in motion analysis. We then studied Taylor series, learning how functions can be expressed as infinite sums of derivatives, and considered the special case of Maclaurin series with examples such as ex, $\sin x$, and $\cos x$. Finally, we examined the error and remainder term in Taylor series, understanding how to estimate approximation errors and apply error bounds in practical contexts.

By completing this Series, you should now be able to **state** and **apply** the Mean Value Theorem (SLO 2.1, SLO 2.2), **construct** Taylor series expansions (SLO 2.3), **derive** Maclaurin series (SLO 2.4), and **evaluate** remainder terms and error bounds (SLO 2.5). These skills equip you to solve problems involving rates of change and approximations, preparing you for deeper applications in analysis and applied mathematics.

Glossary of Terms

Average rate of change: The change in function values divided by the change in input values over an interval.

Instantaneous rate of change: The derivative of a function at a specific point.

Maclaurin series: A Taylor series expansion about the point $a=0$.

Mean Value Theorem (MVT): A theorem stating that for a continuous and differentiable function, there exists a point where the instantaneous rate of change equals the average rate of change.

Remainder term: The difference between the actual function and its Taylor polynomial approximation.

Taylor series: An infinite sum of terms derived from a function's derivatives at a single point.

Concept Check Questions 2

Objective questions

The Mean Value Theorem requires a function to be __________ on band differentiable on b. (Linked to SLO 2.1)

Geometrically, the tangent line at some point cis parallel to the __________ line joining points aand b. (Linked to SLO 2.1)

A Maclaurin series is a Taylor series expansion about the point: A. a=0B. a=1C. a=-1D. a=π(Linked to SLO 2.4)

The remainder term in Taylor series is used to estimate: A. Symmetry B. Error in approximation C. Continuity D. Periodicity (Linked to SLO 2.5)

For small values of x, cos x can be approximated by: A. 1-x22B. xC. 1+xD. x33!(Linked to SLO 2.3)

Short-answer questions

State the Mean Value Theorem in words. (Linked to SLO 2.1)

Explain the geometric interpretation of the Mean Value Theorem. (Linked to SLO 2.1)

Write the first three terms of the Maclaurin series for sin x . (Linked to SLO 2.4)

Long-answer questions

Construct the Taylor series expansion of exabout a=0and discuss its convergence. (Linked to SLO 2.3)

Basic response: Write the expansion and explain that it converges for all real x.

Value-added response: Discuss why the series converges everywhere and its importance in analysis and applications.

Evaluate the importance of the remainder term in Taylor series, with examples of how error bounds are applied in practice. (Linked to SLO 2.5)

Basic response: Explain that the remainder term measures the difference between the function and its polynomial approximation.

Value-added response: Provide examples from physics or engineering where error bounds ensure reliable approximations.

Answers to Pilot Questions 2

Part 2.1

B

A

B

A

C

Part 2.2

B

B

A

B

B

Part 2.3

B

B

C

B

B

3 Real-Valued Functions of Two and Three Variables

Series outline

Part 3.1: Concept of Functions of Two and Three Variables

Definition and examples

Domains and ranges in higher dimensions

Graphical representation (surfaces and level curves)

Part 3.2: Limits and Continuity of Multivariable Functions

Definition of limits in two and three variables

Continuity and discontinuity in higher dimensions

Illustrative examples

Part 3.3: Partial Derivatives

Definition and notation

Geometric interpretation

Examples and applications

Part 3.4: Differentiability and the Gradient Vector

Differentiability of multivariable functions

Gradient vector and its significance

Applications in optimisation and directional derivatives

Introduction

Functions of two and three variables extend the ideas you have already studied with single-variable functions into higher dimensions. Instead of describing how one quantity depends on another, these functions describe how an output depends on two or more inputs. Such functions are essential in modelling real-world phenomena, from temperature distributions across a surface to pressure variations in a fluid. Understanding them prepares you for advanced topics in multivariable calculus, optimisation, and applied mathematics.

The aim of this Series is to introduce you to the fundamental concepts, properties, and techniques for working with real-valued functions of two and three variables, equipping you with the skills to analyse, differentiate, and apply them in problem solving.

We shall commence this Series by defining functions of two and three variables, exploring their domains, ranges, and graphical representations such as surfaces and level curves. Next, we will study limits and continuity in higher dimensions, with examples to illustrate how these concepts extend beyond single-variable functions. Following that, we will examine partial derivatives, their notation, and geometric interpretation. Finally, we will conclude with differentiability and the gradient vector, highlighting their applications in optimisation and directional derivatives.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 3.1 define real-valued functions of two and three variables and identify their domains and ranges,

SLO 3.2 illustrate functions of two and three variables using graphical representations such as surfaces and level curves,

SLO 3.3 evaluate limits and assess continuity of multivariable functions,

SLO 3.4 compute partial derivatives of functions of two and three variables and explain their geometric meaning,

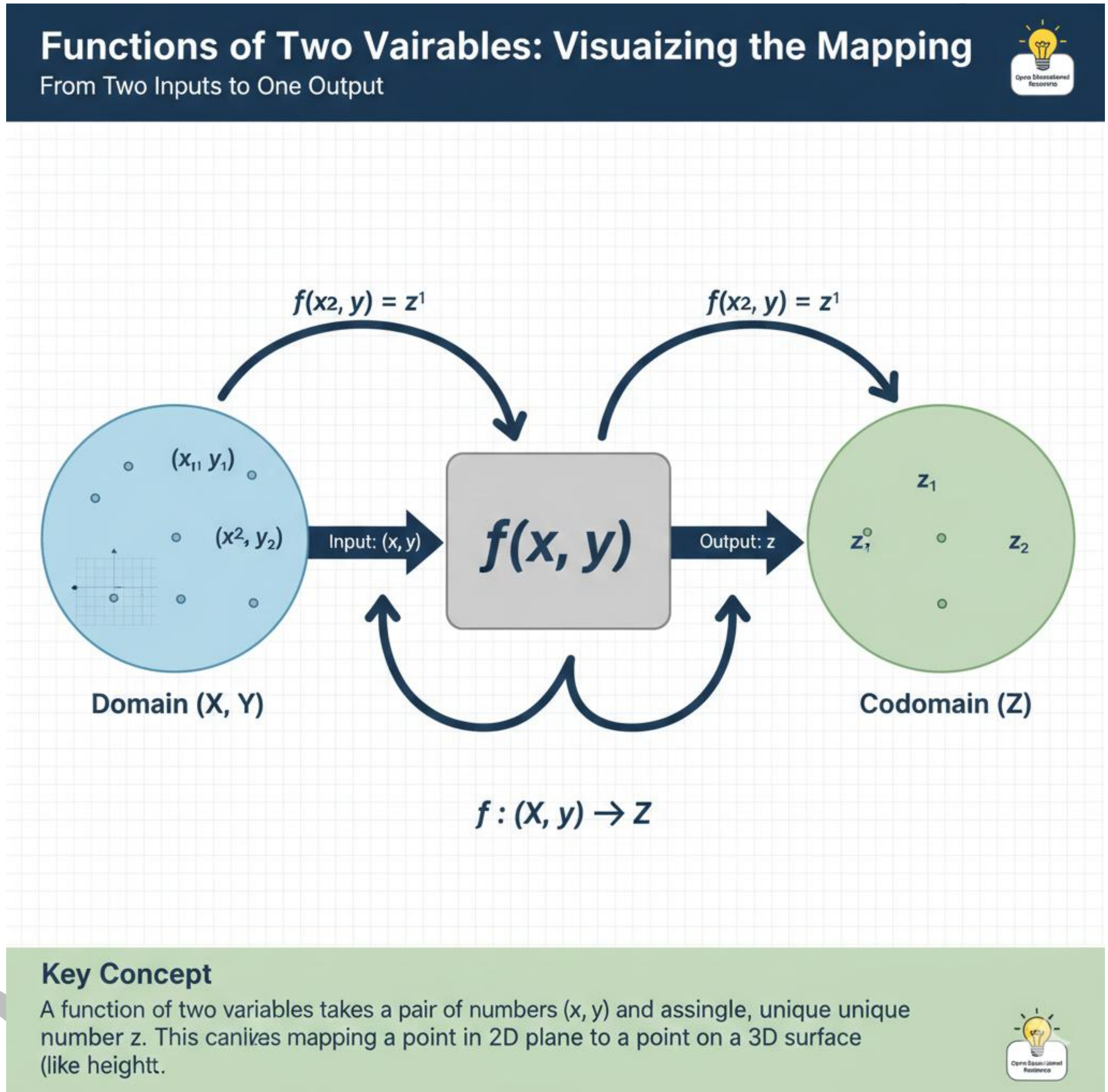
SLO 3.5 analyse differentiability of multivariable functions and apply the gradient vector in optimisation and directional derivatives.

3.1 Concept of Functions of Two and Three Variables

3.1.1 Definition and examples

A function of two variables is a rule that assigns each ordered pair yin a domain to a single real number z. Similarly, a function of three variables assigns each ordered triple zto a single real number w. For example, the function f(x,y)=x2+y2maps pairs of real numbers to their squared sum, while g(x,y,z)=x+y+zmaps triples to their sum.

Fill in the blank: A function of two variables assigns each ordered pair yto a single __________ number.



Mapping of a function of two variables

3.1.2 Domains and ranges in higher dimensions

The domain of a multivariable function is the set of all input pairs or triples for which the function is defined. The range is the set of all possible outputs. For example, for f(x,y)=x2+y2, the domain is all real pairs y, while the range is all non-negative real numbers.

Fill in the blank: The set of all possible outputs of a multivariable function is called the __________.

Table showing examples of domains and ranges for functions of two and three variables.

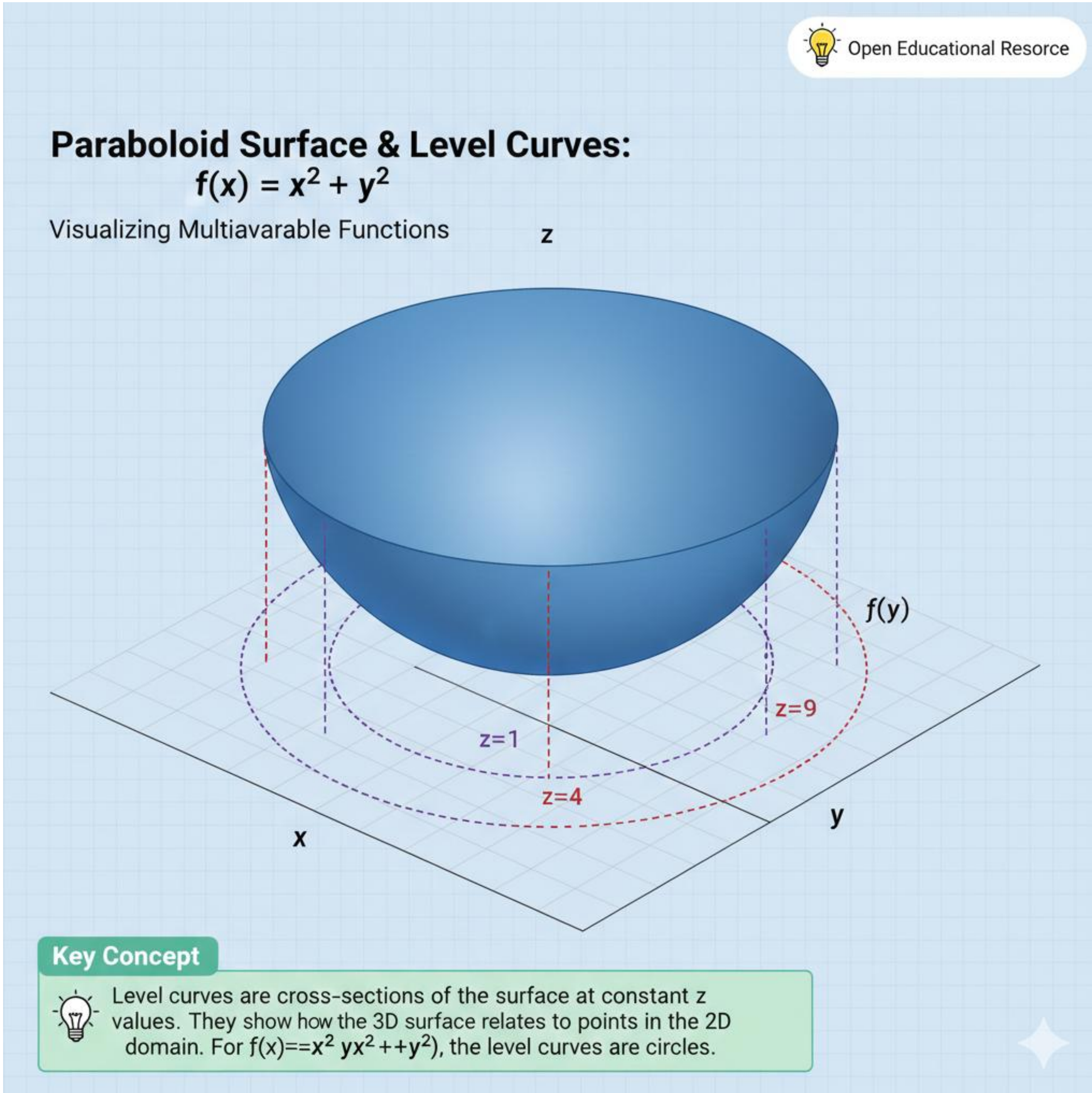
Examples of Domains and Ranges

Function Type	Example Equation	Domain (Input Space)	Range (Output Values)
Two Variables (Polynomial)	$f(x,y) = x^2 + y^2$	All points in $\mathbb{R}^2$ (the entire xy-plane)	$[0, \infty)$
Two Variables (Square Root)	$f(x,y) = \sqrt{9 - x^2 - y^2}$	Disk defined by $x^2 + y^2 \leq 9$ (Radius ≤ 3)	$[0, 3]$
Two Variables (Logarithmic)	$f(x,y) = \ln(x + y)$	Half-plane where $y > -x$	$(-\infty, \infty)$
Three Variables (Distance)	$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$	All points in $\mathbb{R}^3$ (3D space)	$[0, \infty)$
Three Variables (Rational)	$f(x,y,z) = \frac{1}{x^2+y^2+z^2}$	All of $\mathbb{R}^3$ except the origin $(0,0,0)$	$(0, \infty)$

3.1.3 Graphical representation (surfaces and level curves)

Functions of two variables can be represented graphically as surfaces in three-dimensional space. For example, the function f(x,y)=x2+y2produces a paraboloid surface. Another useful representation is through level curves, which are curves in the plane where the function takes constant values. For three-variable functions, we use level surfaces, which are surfaces in space where the function is constant.

Fill in the blank: Curves in the plane where a function of two variables takes constant values are called __________ curves.



Surface and level curves of f(x,y)=x2+y2

Questions 3.1

A function of two variables assigns each ordered pair yto a single: A. Complex number B. Real number C. Arbitrary set D. Vector (Linked to SLO 3.1)

The set of all possible outputs of a multivariable function is called the: A. Domain B. Codomain C. Range D. Level curve (Linked to SLO 3.1)

The function $f(x,y)=x2+y2$produces a surface called a: A. Sphere B. Paraboloid C. Cylinder D. Plane (Linked to SLO 3.2)

Curves in the plane where a function of two variables takes constant values are called: A. Level curves B. Gradient curves C. Tangent curves D. Domain curves (Linked to SLO 3.2)

For the function $f(x,y,z)=x+y+z$, the domain is: A. All real triples zB. Only positive triples C. Only integer triples D. Undefined (Linked to SLO 3.1)

3.2 Limits and Continuity of Multivariable Functions

3.2.1 Definition of limits in two and three variables

A limit of a multivariable function describes the value that the function approaches as the input variables get closer to a particular point. For functions of two variables, we write:

$$\lim x,y)\to(a,bf(x,y)=L$$

if the values of $f(x,y)$get arbitrarily close to Las yapproaches b. Similarly, for three variables:

$$\lim x,y,z)\to(a,b,cf(x,y,z)=L.$$

Fill in the blank: The limit of a multivariable function describes the value the function approaches as the input variables get closer to a particular __________.

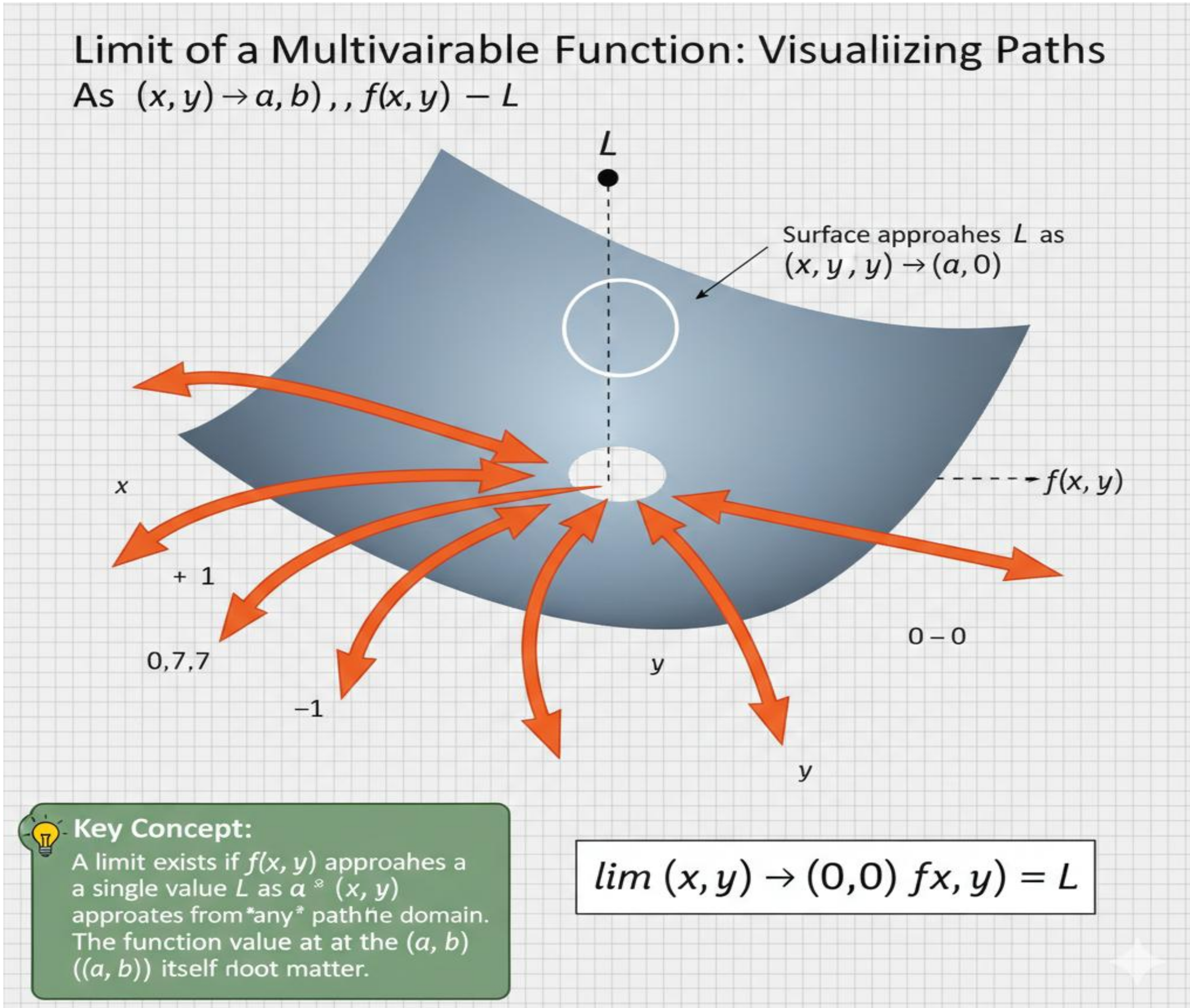


Illustration of a limit in two variables

3.2.2 Continuity and discontinuity in higher dimensions

A function of two or three variables is continuous at a point if the limit of the function at that point equals the function's value there. If this condition fails, the function is discontinuous.

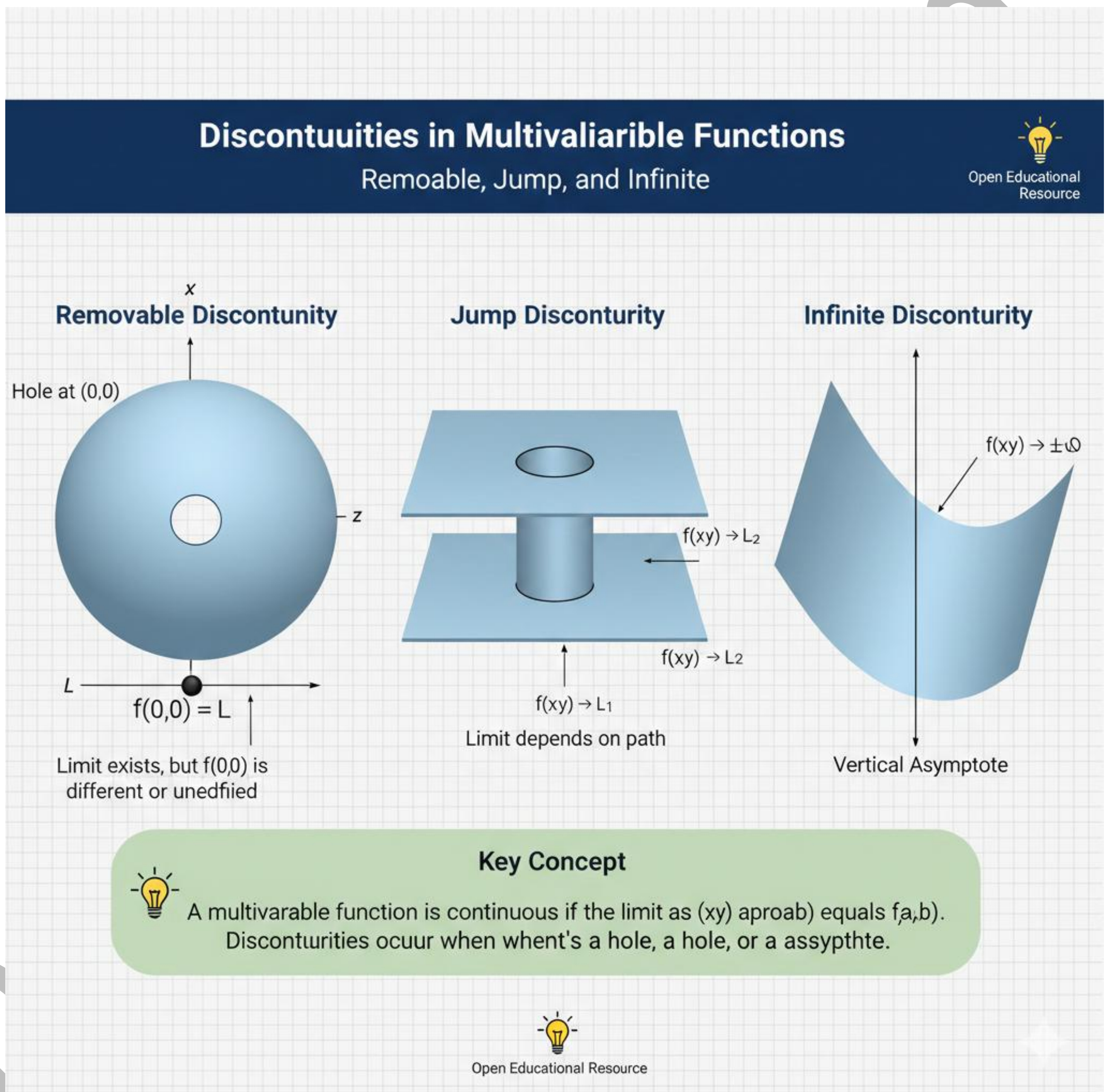
Types of discontinuities include:

Removable discontinuity: A "hole" in the surface that can be filled.

Jump discontinuity: A sudden change in values across a boundary.

Infinite discontinuity: The function tends to infinity near a point.

Fill in the blank: A function is continuous at a point if the limit at that point equals the __________ at that point.



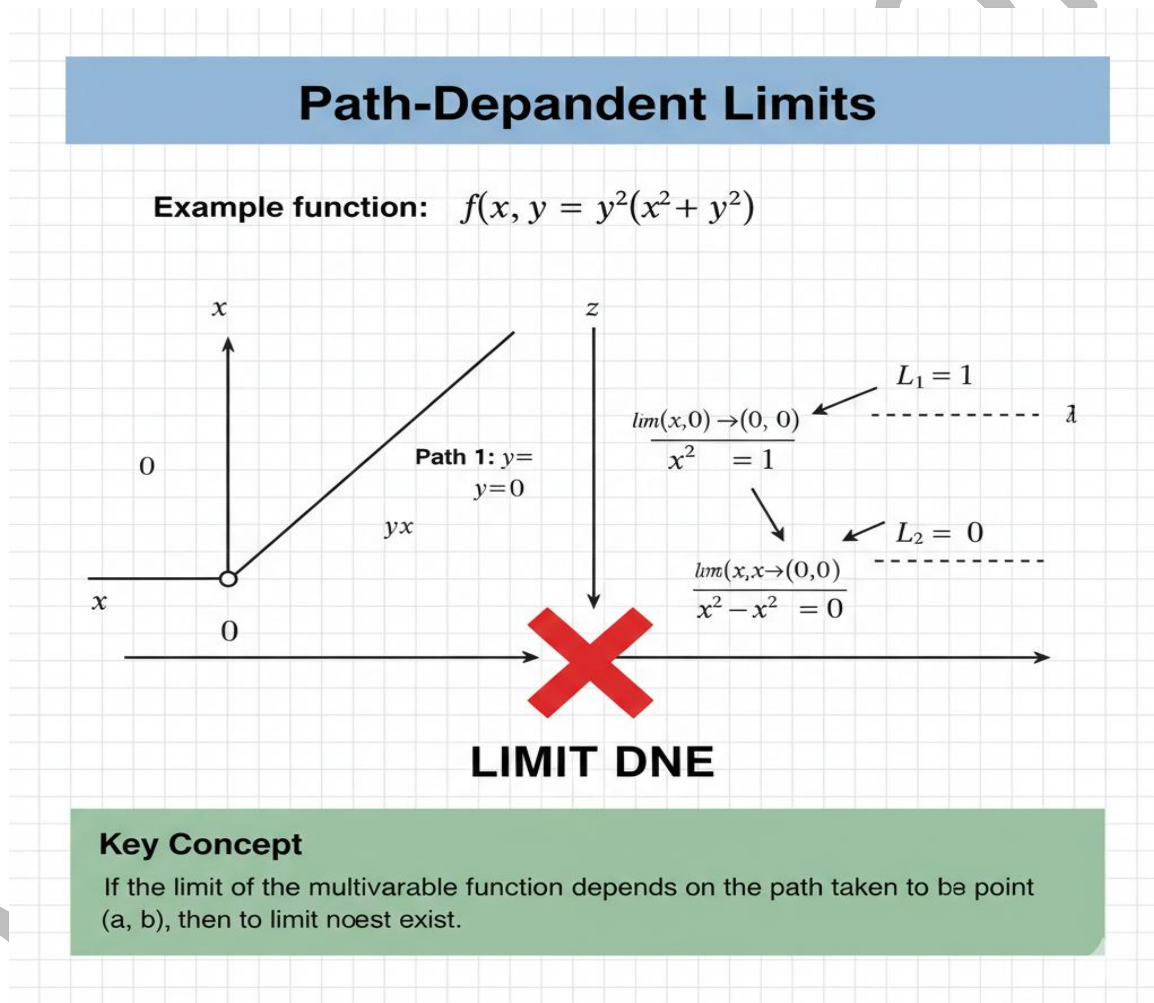
Types of discontinuities in multivariable functions

3.2.3 Illustrative examples

Consider the function f(x,y)=x2-y2x2+y2. As yapproaches 0, the limit depends on the path taken. Along y=0, the limit is 1, but along x=0, the limit is -1. Since the limit depends on the path, the function has no limit at 0. This illustrates how limits in multiple variables can be more complex than in single-variable functions.

Fill in the blank: If the limit of a multivariable function depends on the path taken, then the limit __________ at that point.

Box explaining the function f(x,y)=x2-y2x2+y2and its path-dependent limits.



Pilot Questions 3.2

The limit of a multivariable function describes the value the function approaches as the input variables get closer to a particular: A. Range B. Domain C. Point D. Curve (Linked to SLO 3.3)

A function of two variables is continuous at a point if: A. The limit equals the function's value at that point B. The function is bounded C. The function is monotonic D. The function is periodic (Linked to SLO 3.3)

A sudden change in values across a boundary is called a: A. Removable discontinuity B. Jump discontinuity C. Infinite discontinuity D. Rational discontinuity (Linked to SLO 3.3)

If the limit of a multivariable function depends on the path taken, then the limit: A. Exists uniquely B. Does not exist C. Is continuous D. Is bounded (Linked to SLO 3.3)

For the function f(x,y)=x2-y2x2+y2, the limit at 0: A. Exists and equals 0 B. Exists and equals 1 C. Exists and equals -1 D. Does not exist (Linked to SLO 3.3)

3.3 Partial Derivatives

3.3.1 Definition and notation

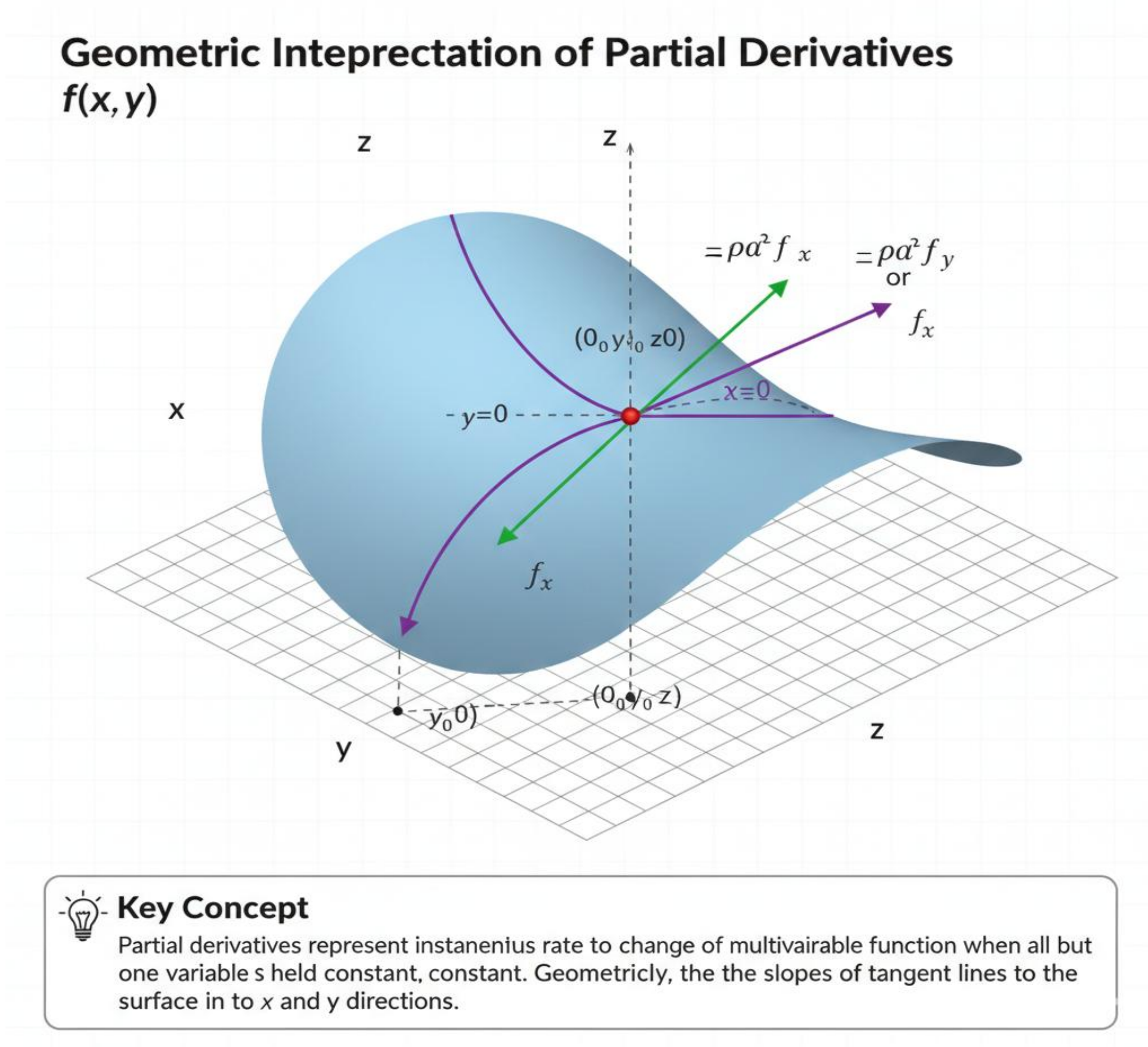
A partial derivative measures how a multivariable function changes when one variable is varied while the others are held constant. For a function f(x,y), the partial derivative with respect to xis denoted by:

$$fx=\lim h\to 0f(x+h,y)-f(x,y)h$$

Similarly, the partial derivative with respect to yis:

$$fy=\lim h\to 0f(x,y+h)-f(x,y)h.$$

Fill in the blank: A partial derivative measures how a function changes when one variable is varied while the others are __________.



Geometric meaning of partial derivatives

3.3.2 Geometric interpretation

Geometrically, partial derivatives represent the slope of the tangent line to the surface in the direction of one variable. For example, fxgives the slope of the surface in the x-direction, while fygives the slope in the y-direction.

Fill in the blank: The partial derivative fxgives the slope of the surface in the __________ direction.

3.3.3 Examples and applications

For f(x,y)=x2y+y3:

$$fx=2xy,fy=x2+3y2.$$

Applications include:

Physics: Rate of change of temperature with respect to spatial coordinates.

Economics: Sensitivity of cost functions to changes in input variables.

Engineering: Stress and strain analysis in materials.

Fill in the blank: In physics, partial derivatives can represent the rate of change of __________ with respect to spatial coordinates.

Box summarising applications in physics, economics, and engineering.

Multidisciplinary Impact

Field	Application	Example Scenario
Physics	**Thermodynamics**	Determining how the pressure (P) of a gas changes with temperature (T) while the volume (V) remains fixed: $(\frac{\partial P}{\partial T})_V$.
Economics	**Marginal Productivity**	Calculating how much additional output a firm produces by adding one more unit of labor while keeping capital investment constant.
Engineering	**Heat Transfer**	Using the Heat Equation to find how temperature varies at a specific point over time (t) and across space (x, y, z).
Data Science	**Gradient Descent**	Updating weights in a neural network by finding the partial derivative of the error function with respect to each individual weight.

Pilot Questions 3.3

A partial derivative measures how a function changes when: A. All variables change simultaneously B. One variable changes while others are constant C. No variable changes D. The function is periodic (Linked to SLO 3.4)

The partial derivative fxgives the slope of the surface in the: A. y-direction B. x-direction C. z-direction D. Diagonal direction (Linked to SLO 3.4)

For f(x,y)=x2y+y3, fyis: A. 2xyB. x2+3y2C. x2yD. y2(Linked to SLO 3.4)

In physics, partial derivatives can represent the rate of change of: A. Temperature B. Velocity C. Pressure D. All of the above (Linked to SLO 3.4)

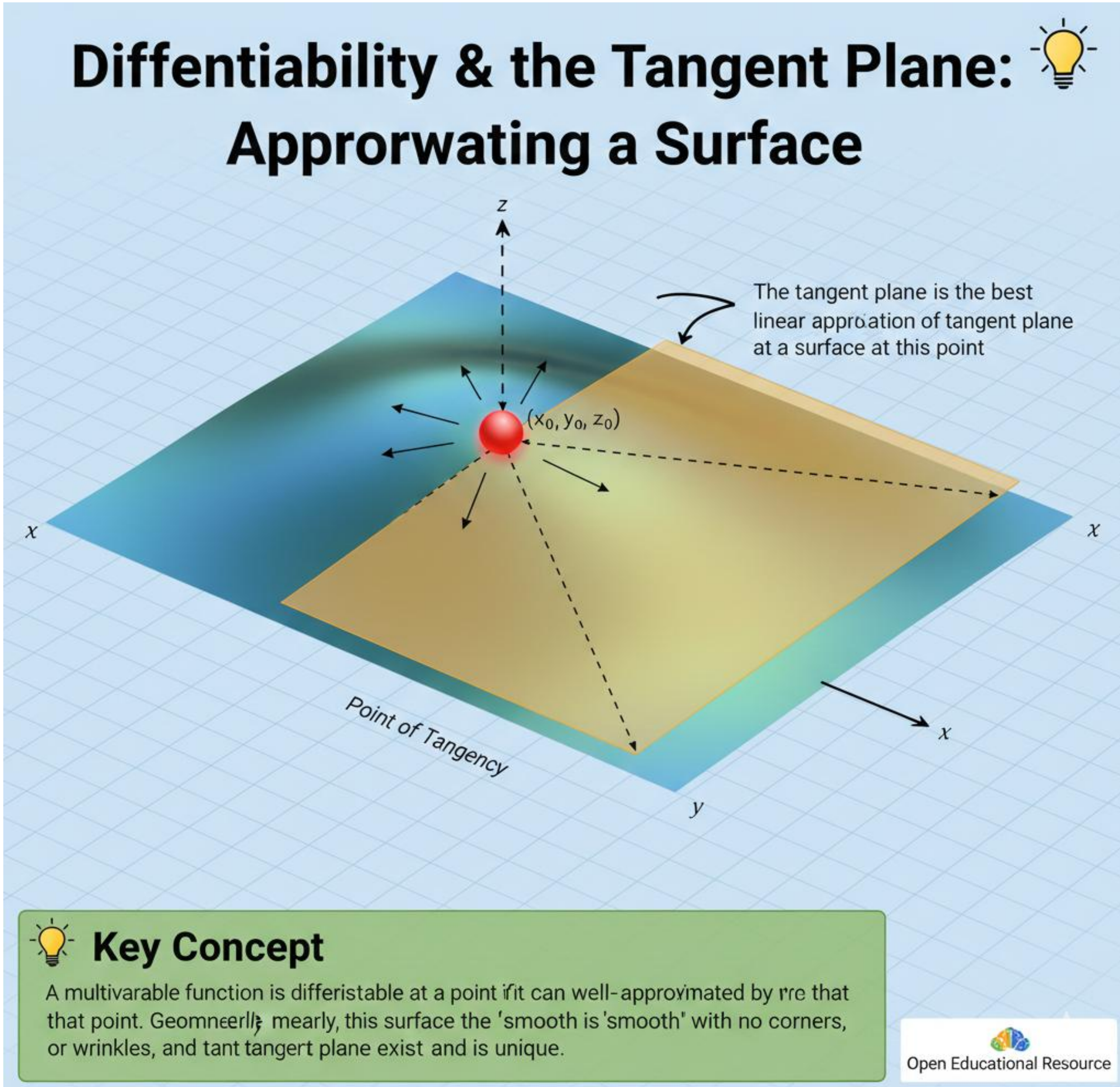
In economics, partial derivatives are used to measure: A. Symmetry B. Sensitivity of cost functions C. Periodicity D. Continuity (Linked to SLO 3.4)

3.4 Differentiability and the Gradient Vector

3.4.1 Differentiability of multivariable functions

A function of two or three variables is differentiable at a point if it can be well-approximated near that point by a linear function. Differentiability implies continuity, but the reverse is not always true. Formally, differentiability means that small changes in the input variables produce predictable, linear changes in the output.

Fill in the blank: Differentiability of a multivariable function implies __________, but the reverse is not always true.



Tangent plane illustrating differentiability of a function of two variables

3.4.2 Gradient vector and its significance

The gradient vector of a function f(x,y)is defined as:

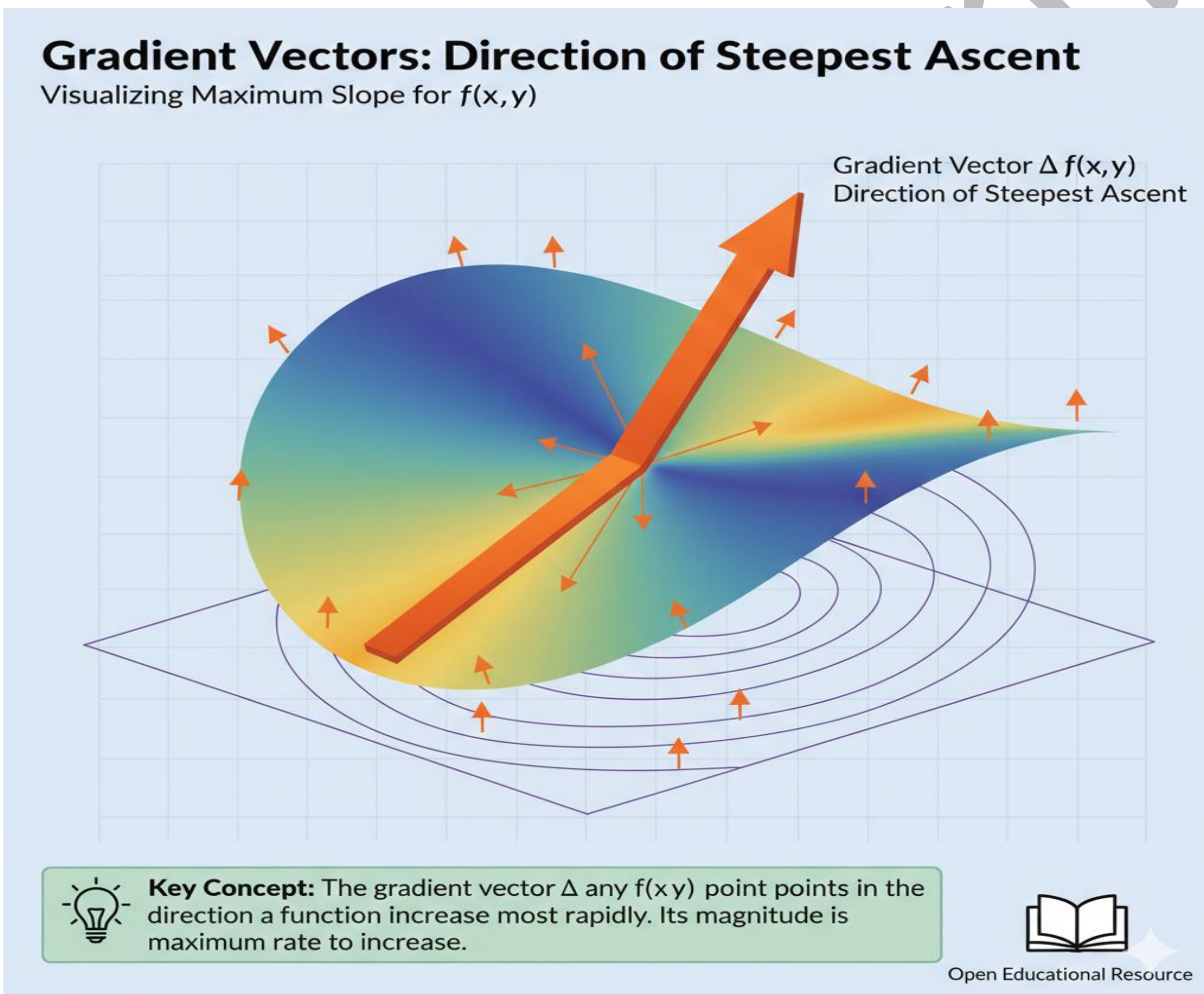
$$f(x,y)=fy.$$

For three variables:

$$f(x,y,z)=fz.$$

The gradient points in the direction of the greatest rate of increase of the function, and its magnitude gives the rate of increase in that direction.

Fill in the blank: The gradient vector points in the direction of the greatest rate of __________ of the function.



Gradient vector pointing in direction of steepest ascent

3.4.3 Applications in optimisation and directional derivatives

Optimisation: The gradient is used to find maxima and minima of functions. For example, in economics, it helps identify optimal production levels.

Directional derivatives: The gradient can be used to compute the rate of change of a function in any direction. The directional derivative of fin the direction of a unit vector uis given by:

$$Duf(x,y)=f(x,y)\cdot u.$$

Fill in the blank: The directional derivative of a function in the direction of a unit vector uis given by f·u, which is the __________ of the gradient and the vector.

Box summarising applications of gradient vector in optimisation and directional derivatives.

Core Applications in Science and Industry

Application	Field	Mechanism
Gradient Descent	**Machine Learning**	Algorithms minimize "loss functions" by moving in the opposite direction of the gradient ($-\nabla f$) to find the lowest error state.
Terrain Analysis	**Geology / GIS**	Directional derivatives calculate the steepness of a trail in any specific compass direction, not just North-South or East-West.
Fluid Flow	**Meteorology**	The pressure gradient force explains how air moves from high to low pressure, which is the primary driver of wind speeds and directions.
Optimal Design	**Engineering**	Gradients are used to find the "steepest ascent" to maximize the strength-to-weight ratio of aerospace components.

Pilot Questions 3.4

Differentiability of a multivariable function implies: A. Continuity B. Periodicity C. Symmetry D. Boundedness (Linked to SLO 3.5)

The gradient vector of f(x,y)is: A. f(x),f(y)B. fyC. yD. f′(x),f′(y)(Linked to SLO 3.5)

The gradient vector points in the direction of: A. Least rate of increase B. Greatest rate of increase C. Constant rate of change D. Zero change (Linked to SLO 3.5)

The directional derivative of fin the direction of a unit vector uis given by: A. f·uB. f+uC. f-uD. f×u(Linked to SLO 3.5)

In optimisation problems, the gradient is used to find: A. Symmetry B. Maxima and minima C. Periodicity D. Continuity (Linked to SLO 3.5)

Series Summary

This Series introduced you to the foundations of real-valued functions of two and three variables, extending the concepts from single-variable calculus into higher dimensions. We began by defining functions of two and three variables, exploring their domains, ranges, and graphical representations such as surfaces and level curves. Next, we studied limits and continuity in higher dimensions, noting how path-dependent limits can complicate existence. We then examined partial derivatives, their notation, geometric meaning, and applications in physics, economics, and engineering. Finally, we discussed differentiability and the gradient vector, highlighting its role in optimisation and directional derivatives.

By completing this Series, you should now be able to define multivariable functions and their domains/ranges (SLO 3.1), illustrate them graphically (SLO 3.2), evaluate limits and continuity (SLO 3.3), compute partial derivatives (SLO 3.4), and apply the gradient vector in optimisation and directional derivatives (SLO 3.5).

Glossary of Terms

Domain (multivariable): The set of all input pairs or triples for which a function is defined.

Range (multivariable): The set of all possible outputs of a multivariable function.

Level curve: A curve in the plane where a function of two variables takes constant values.

Level surface: A surface in space where a function of three variables takes constant values.

Partial derivative: The derivative of a multivariable function with respect to one variable, holding others constant.

Gradient vector: A vector of partial derivatives pointing in the direction of greatest increase of the function.

Directional derivative: The rate of change of a function in the direction of a given vector.

Differentiability: The property of a function being well-approximated by a linear function near a point.

Concept Check Questions 3

Objective questions

A function of two variables assigns each ordered pair yto a single __________. (Linked to SLO 3.1)

Curves in the plane where a function of two variables takes constant values are called __________. (Linked to SLO 3.2)

A function is continuous at a point if the limit at that point equals the __________ at that point. (Linked to SLO 3.3)

The partial derivative fxgives the slope of the surface in the __________ direction. (Linked to SLO 3.4)

The gradient vector points in the direction of the greatest rate of __________ of the function. (Linked to SLO 3.5)

Short-answer questions

Define a level curve and give an example. (Linked to SLO 3.2)

Explain what it means for a limit of a multivariable function to be path-dependent. (Linked to SLO 3.3)

Compute fxand fyfor f(x,y)=x2y+y3. (Linked to SLO 3.4)

Long-answer questions

Analyse the difference between continuity and differentiability in multivariable functions. (Linked to SLO 3.3, SLO 3.5)

Basic response: Describe continuity and differentiability separately.

Value-added response: Discuss why differentiability implies continuity but not vice versa, with examples.

Discuss the role of the gradient vector in optimisation problems and directional derivatives. (Linked to SLO 3.5)

Basic response: Explain that the gradient points in the direction of steepest ascent.

Value-added response: Provide examples from economics or physics where the gradient is used to find maxima/minima or rates of change in specific directions.

Answers to Pilot Questions 3

Part 3.1

B

C

B

A

A

Part 3.2

C

A

B

B

D

Part 3.3

B

B

B

D

B

Part 3.4

A

B

B

A

B

4 Differentiation of Functions of Two and Three Variables

Series outline

Part 4.1: Higher-Order Partial Derivatives

- Definition and notation
- Mixed partial derivatives
- Clairaut's theorem on equality of mixed partials

Part 4.2: The Chain Rule in Multivariable Calculus

- Chain rule for functions of two variables
- Chain rule for functions of three variables
- Applications in composite functions

Part 4.3: Directional Derivatives and Gradient Vector

- Definition of directional derivatives
- Relationship with the gradient vector
- Applications in physics and optimisation

Part 4.4: Tangent Planes and Linear Approximations

- Equation of a tangent plane to a surface
- Linear approximation of multivariable functions
- Practical examples and applications

Introduction

Differentiation of functions of two and three variables extends the familiar rules of single-variable calculus into higher dimensions. Instead of dealing with curves, we now work with surfaces and hypersurfaces, where rates of change can be measured in multiple directions. This makes differentiation a powerful tool for analysing complex systems in mathematics, physics, engineering, and economics.

The aim of this Series is to equip you with the techniques and interpretations of differentiation in multivariable calculus. You will learn how to compute higher-order partial derivatives, apply the chain rule to composite functions, evaluate directional derivatives, and use the gradient vector to understand rates of change. Finally, you will explore tangent planes and linear approximations, which provide geometric and practical insights into differentiability.

We shall commence this Series by studying higher-order partial derivatives, including mixed partials and Clairaut's theorem. Next, we will examine the chain rule in multivariable calculus, showing how composite functions are differentiated. Following that, we will explore directional derivatives and the gradient vector, highlighting their applications in optimisation and physics. Finally, we will conclude with tangent planes and linear approximations, which connect algebraic differentiation with geometric intuition.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 4.1 compute higher-order partial derivatives, including mixed partials, and apply Clairaut's theorem,

SLO 4.2 apply the chain rule to differentiate composite functions of two and three variables,

SLO 4.3 evaluate directional derivatives and explain their relationship with the gradient vector,

SLO 4.4 construct equations of tangent planes and use linear approximations for multivariable functions,

SLO 4.5 apply differentiation techniques of multivariable functions to solve practical problems in mathematics, physics, and engineering.

4.1 Higher-Order Partial Derivatives

4.1.1 Definition and notation

A higher-order partial derivative is obtained by differentiating a function of several variables more than once. For example, for a function f(x,y):

First-order partial derivatives:

$$fx,fy$$

Second-order partial derivatives:

2fx2,2fy2,2fxy,2fyx

Fill in the blank: A higher-order partial derivative is obtained by differentiating a function of several variables more than __________.

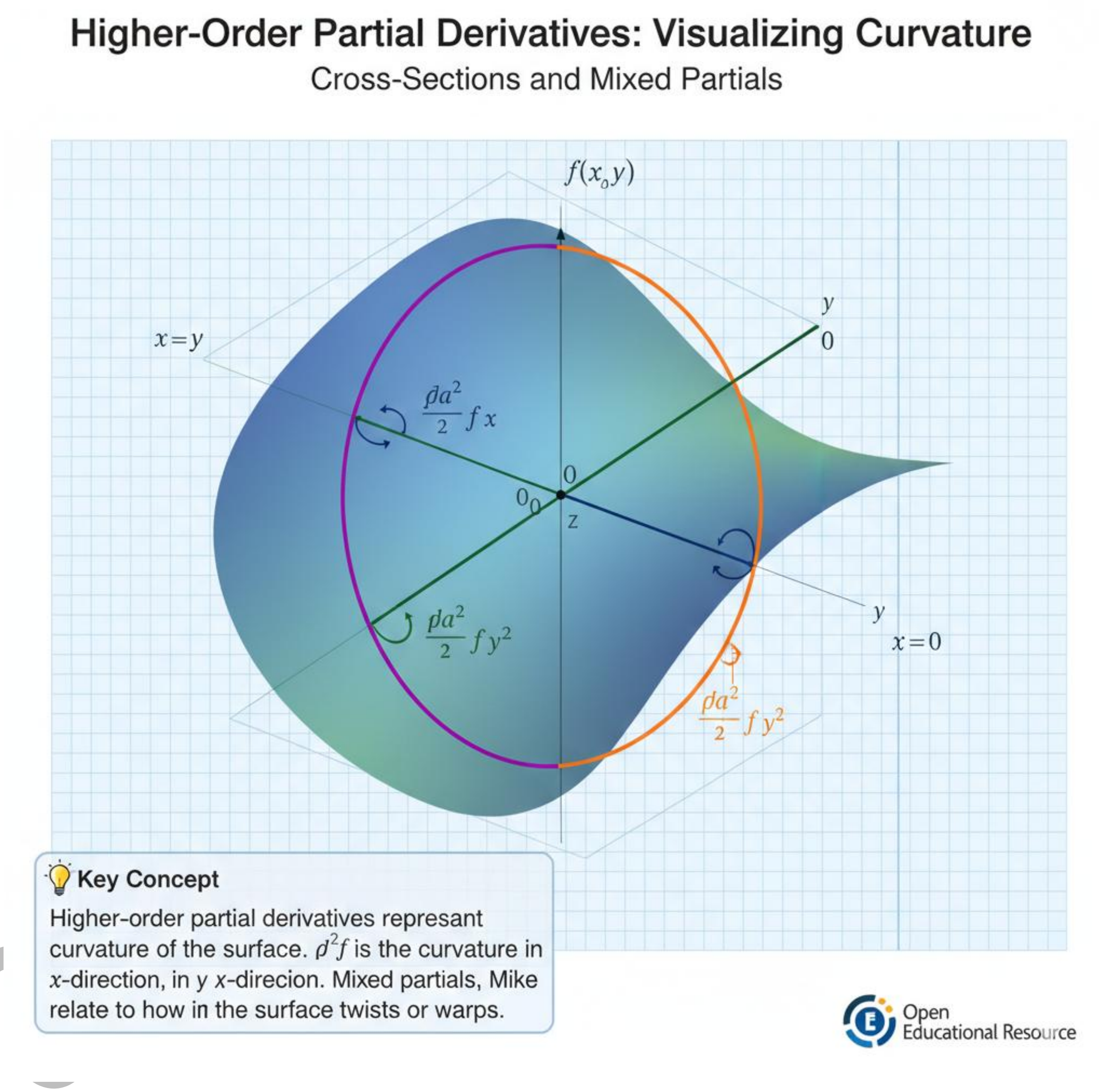


Illustration of higher-order partial derivatives

4.1.2 Mixed partial derivatives

Mixed partial derivatives involve differentiating with respect to different variables in succession. For example:

2fxy=xfy.

Fill in the blank: A mixed partial derivative involves differentiating with respect to __________ variables in succession.

4.1.3 Clairaut's theorem on equality of mixed partials

Clairaut's theorem states that if the mixed partial derivatives of a function are continuous near a point, then they are equal:

2fxy=2fyx.

This result simplifies calculations and ensures consistency in multivariable differentiation.

Fill in the blank: Clairaut's theorem states that if mixed partial derivatives are __________ near a point, then they are equal.

Box summarising Clairaut's theorem and its importance in multivariable calculus

The Core Definition

If a function $f(x, y)$ is defined on a region D in the xy-plane, and the mixed partial derivatives f_{xy} and f_{yx} are both **continuous** on D, then:

$$f_{xy} = f_{yx}$$

In Leibniz notation, this is expressed as:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Why It Matters

Benefit	Application
Computational Ease	If one order of differentiation is significantly easier than the other (e.g., differentiating $\ln(x)$ before e^y), you can switch the order without changing the result.
Verification	In physics and engineering, equality of mixed partials is used to verify that a multivariable function is "well-behaved" and physically realistic.
Exact Equations	In thermodynamics and differential equations, Clairaut's Theorem is the basis for the **Exactness Test**. If $M_y = N_x$, it proves a conservative vector field exists.
Hessian Symmetry	The **Hessian Matrix** (used in optimization and second derivative tests) is symmetric because of this theorem, which simplifies the math for finding local extrema.

Pilot Questions 4.1

A higher-order partial derivative is obtained by differentiating a function: A. Once B. Twice or more C. Not at all D. Only with respect to one variable (Linked to SLO 4.1)

A mixed partial derivative involves differentiating with respect to: A. The same variable repeatedly B. Different variables in succession C. No variables D. Only one variable (Linked to SLO 4.1)

Clairaut's theorem states that if mixed partial derivatives are continuous near a point, then they are: A. Unequal B. Equal C. Undefined D. Zero (Linked to SLO 4.1)

For f(x,y)=x2y, 2fxyis: A. 2xB. x2C. 2yD. 0(Linked to SLO 4.1)

Higher-order partial derivatives are useful because they: A. Simplify notation only B. Provide information about curvature and behaviour of functions C. Eliminate continuity requirements D. Replace limits entirely (Linked to SLO 4.1)

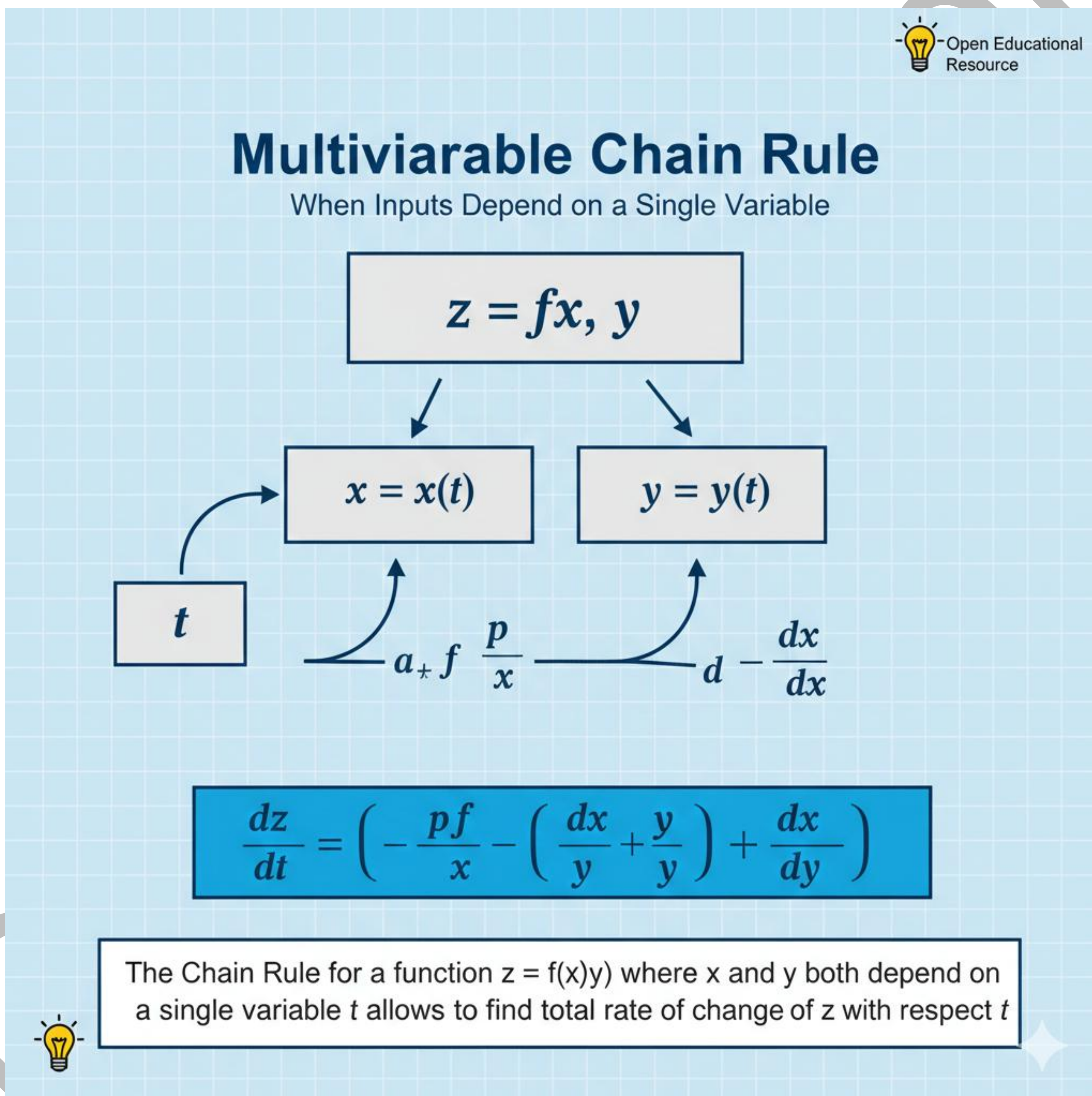
4.2.1 Chain rule for functions of two variables

The chain rule allows us to differentiate composite functions. Suppose z=f(x,y), where both xand ydepend on another variable t. Then:

dzdt=fxdxdt+fydydt.

This formula shows how changes in taffect zthrough both xand y.

Fill in the blank: The chain rule allows us to differentiate __________ functions.



Chain rule for functions of two variables

4.2.2 Chain rule for functions of three variables

If w=f(x,y,z), where x,y,zdepend on another variable t, then:

dwdt=fxdxdt+fydydt+fzdzdt.

This generalises the chain rule to three variables.

Fill in the blank: For three variables, the chain rule involves partial derivatives with respect to x, y, and __________.

4.2.3 Applications in composite functions

The chain rule is widely used in:

- Physics: Relating rates of change in motion when variables depend on time.
- Economics: Differentiating cost functions where inputs depend on other parameters.
- Engineering: Analysing systems where multiple variables depend on a single parameter.

Example: If z=x2+y2, with x=t2and y=sin t, then:

dzdt=2x·dxdt+2y·dydt=2t2·(2t)+2sin t·cos t.

Fill in the blank: In physics, the chain rule is often used to relate rates of change in __________ when variables depend on time.

Box summarising applications of the chain rule in physics, economics, and engineering.

Applications of the Multivariable Chain Rule

The multivariable **Chain Rule** is the mathematical tool used to track how a complex system changes when its underlying components are moving or evolving. It allows us to calculate a "total rate of change" by summing up the indirect influences of multiple variables.

Critical Uses Across Disciplines

Field	Application	Concept in Action
Physics	**Thermodynamics**	Calculating how the internal energy (U) of a system changes over time as both pressure (P) and volume (V) change: $\frac{dU}{dt} = \frac{\partial U}{\partial P}\frac{dP}{dt} + \frac{\partial U}{\partial V}\frac{dV}{dt}$.
Economics	**Cost Sensitivity**	Analyzing how a company's total profit changes over time as labor costs and raw material prices fluctuate simultaneously due to market trends.
Engineering	**Fluid Dynamics**	Tracking the temperature change of a fluid particle as it moves through a pipe (the "Material Derivative"), accounting for both local temperature changes and the particle's motion.

Pilot Questions 4.2

The chain rule allows us to differentiate: A. Composite functions B. Linear functions only C. Constant functions D. Rational functions only (Linked to SLO 4.2)

For z=f(x,y)with xand ydepending on t, the chain rule is: A. dzdt=f(x,y)B. dzdt=fxdxdt+fydydtC. dzdt=dxdt+dydtD. dzdt=ft(Linked to SLO 4.2)

For three variables, the chain rule involves partial derivatives with respect to: A. x,y,zB. x,yonly C. tonly D. None of the above (Linked to SLO 4.2)

In physics, the chain rule is often used to relate rates of change in: A. Geometry B. Motion C. Literature D. History (Linked to SLO 4.2)

If z=x2+y2, with x=t2and y=sin t, then dzdtis: A. 2t2+cos tB. 4t3+2sin tcos tC. t2+sin tD. 2t+cos t(Linked to SLO 4.2)

4.3 Directional Derivatives and Gradient Vector

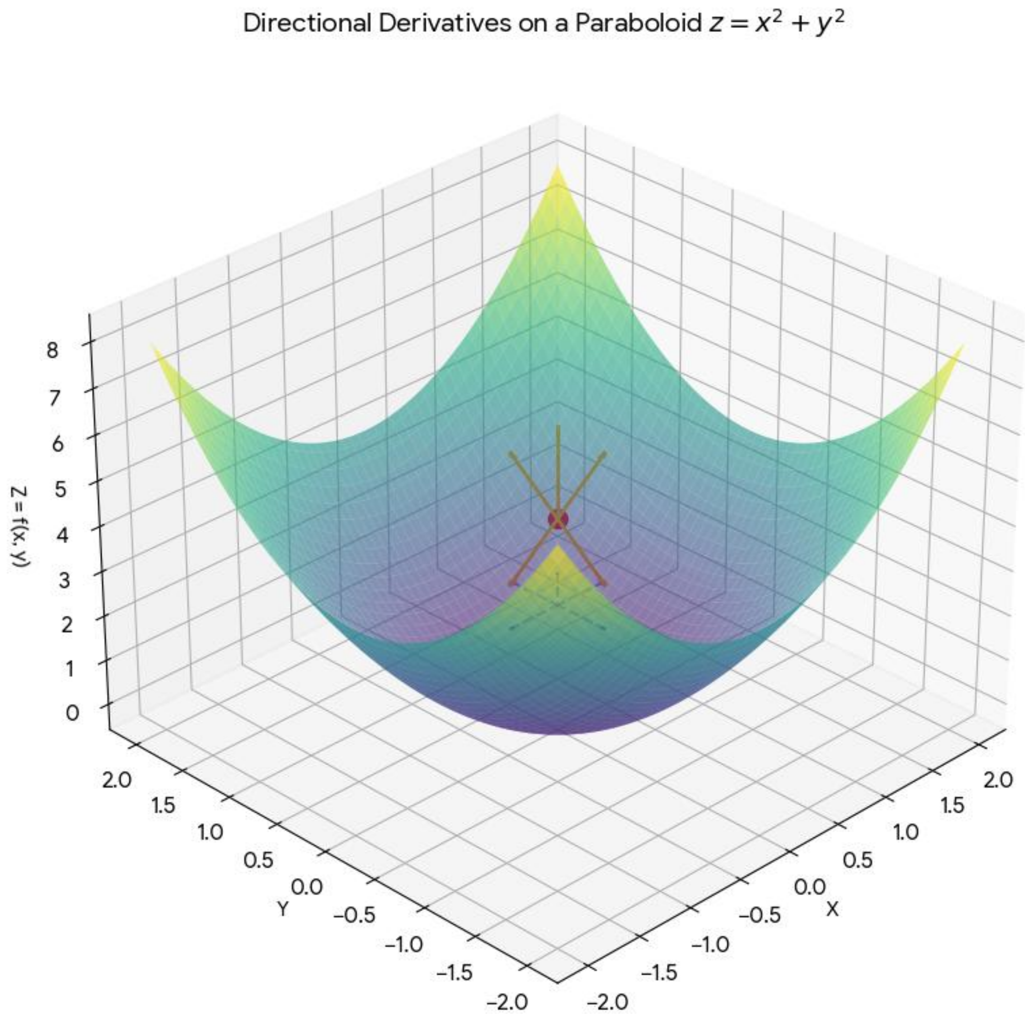
4.3.1 Definition of directional derivatives

A directional derivative measures the rate of change of a function in the direction of a given vector. For a function f(x,y), the directional derivative in the direction of a unit vector u=(u1,u2)is:

$$Duf(x,y)=f(x,y)\cdot u.$$

This formula shows how the gradient interacts with the chosen direction to give the rate of change.

Fill in the blank: A directional derivative measures the rate of change of a function in the direction of a given __________.



Directional derivatives along different directions

4.3.2 Relationship with the gradient vector

The gradient vector f(x,y)points in the direction of the greatest rate of increase of the function. The directional derivative is the dot product of the gradient and the chosen direction vector.

If upoints in the same direction as the gradient, the directional derivative is maximised.

If upoints opposite to the gradient, the directional derivative is minimised (negative).

If uis perpendicular to the gradient, the directional derivative is zero.

Fill in the blank: The directional derivative is maximised when the direction vector points in the same direction as the __________.

4.3.3 Applications in physics and optimisation

Physics: Directional derivatives describe how temperature, pressure, or other physical quantities change in specific directions.

Optimisation: The gradient is used to find maxima and minima of functions, with directional derivatives helping to analyse behaviour along chosen paths.

Engineering: Used in stress analysis, where changes in force or displacement are studied along specific directions.

Example: If f(x,y)=x2+y2, then f=(2x,2y). At point 1, the gradient is 2. In the direction of u=12(1,1), the directional derivative is:

$$Duf(1,1)=(2,2)\cdot 12=42=22.$$

Fill in the blank: In optimisation, the gradient is used to find __________ and __________ of functions.

Box summarising applications in physics, optimisation, and engineering.

Applications: Gradient Vector & Directional Derivatives

The gradient vector ∇f and directional derivatives $D_{\mathbf{u}}f$ are the "GPS" of multivariable calculus. While the gradient tells you the direction of the steepest path, the directional derivative calculates the exact slope of the path you choose to take.

Critical Roles in Science and Industry

Field	Application	Concept in Action
Optimization	**Gradient Descent**	Used in **Machine Learning** to train neural networks. The algorithm calculates the gradient of an "error surface" and moves in the opposite direction ($-\nabla f$) to find the minimum error.
Physics	**Potential Fields**	In electromagnetics, the electric field $\mathbf{E}$ is the negative gradient of the electric potential V ($\mathbf{E} = -\nabla V$). Electrons move in the direction of steepest potential change.
Civil Engineering	**Topographic Grading**	When building roads on hills, directional derivatives are used to ensure the slope in the direction of the road never exceeds a grade where cars lose traction.
Heat Transfer	**Thermal Flux**	Heat flows in the direction of the negative temperature gradient. Directional derivatives calculate the rate of cooling in a specific direction from a heat source.

Pilot Questions 4.3

A directional derivative measures the rate of change of a function in the direction of a given: A. Point B. Vector C. Plane D. Gradient only (Linked to SLO 4.3)

The directional derivative of f(x,y)in the direction of uis given by: A. f·uB. f+uC. f-uD. f×u(Linked to SLO 4.3)

The directional derivative is maximised when the direction vector points in the same direction as the: A. Gradient B. Domain C. Range D. Tangent plane (Linked to SLO 4.3)

If the direction vector is perpendicular to the gradient, the directional derivative is: A. Positive B. Negative C. Zero D. Undefined (Linked to SLO 4.3)

In optimisation, the gradient is used to find: A. Symmetry and continuity B. Maxima and minima C. Periodicity and boundedness D. Tangent planes only (Linked to SLO 4.3)

4.4 Tangent Planes and Linear Approximations

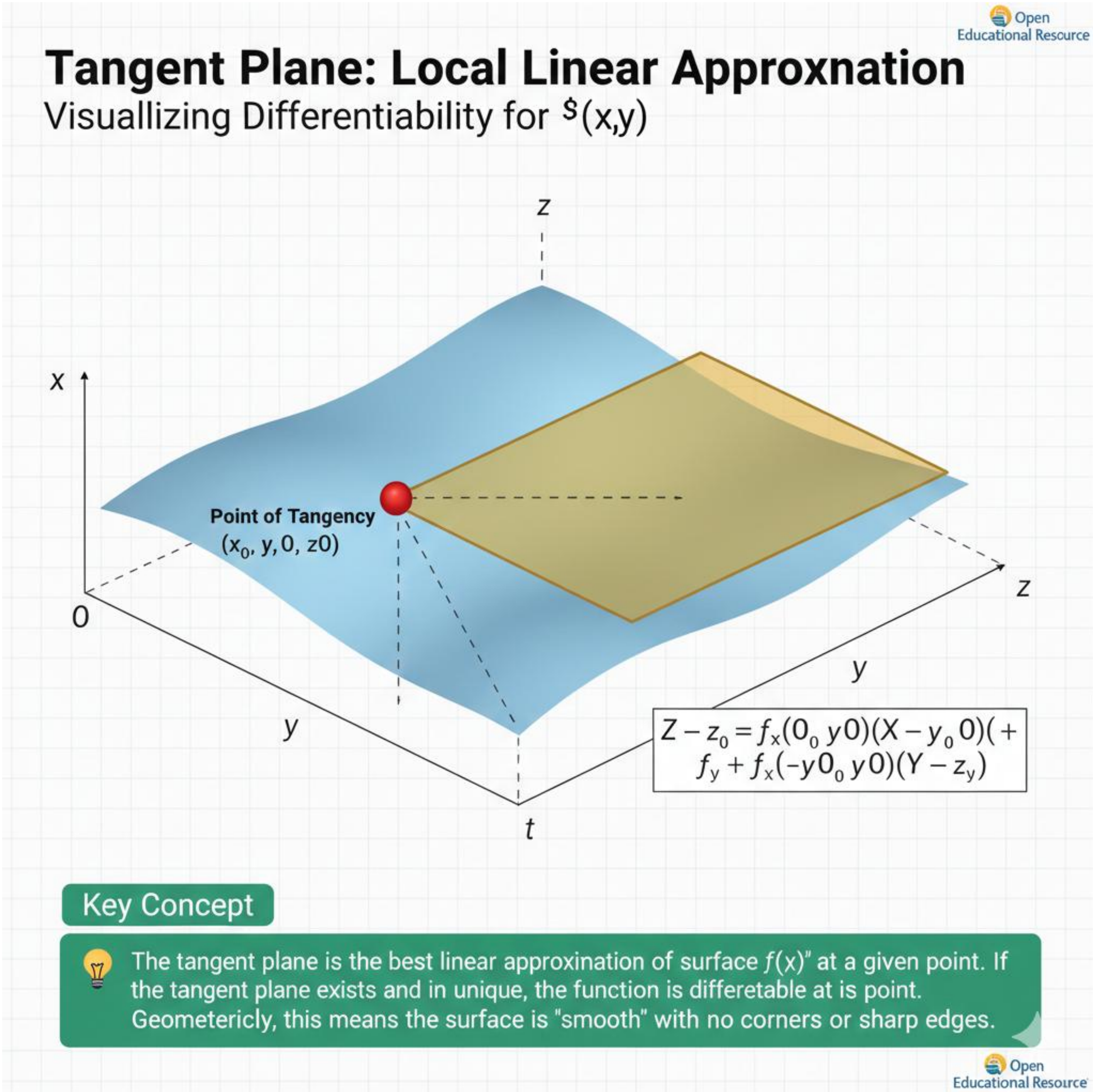
4.4.1 Equation of a tangent plane to a surface

For a function z=f(x,y), the tangent plane at point z0is given by:

$$z-z0=fx(x0,y0)(x-x0)+fy(x0,y0)(y-y0).$$

This plane provides a local linear approximation of the surface near the point.

Fill in the blank: The tangent plane provides a local __________ approximation of the surface near a point.



Tangent plane to a surface at a point

4.4.2 Linear approximation of multivariable functions

The tangent plane leads to the linear approximation formula:

$$f(x,y) \approx f(x0,y0)+fx(x0,y0)(x-x0)+fy(x0,y0)(y-y0).$$

This approximation is useful for estimating function values near a known point.

Fill in the blank: Linear approximation is used to estimate function values near a known __________.

4.4.3 Practical examples and applications

Physics: Approximating potential energy surfaces near equilibrium points.

Economics: Estimating cost functions near given production levels.

Engineering: Approximating stress-strain relationships near specific operating conditions.

Example: For $f(x,y)=x2+y2$at point 1:

$$f(1,1)=2, fx=2x, fy=2y.$$

At 1, the tangent plane is:

$$z-2=2(x-1)+2(y-1).$$

Fill in the blank: In economics, tangent planes and linear approximations can be used to estimate __________ functions near given production levels.

Box summarising applications in physics, economics, and engineering

Applications of Tangent Planes & Linear Approximations

In multivariable calculus, the **tangent plane** serves as the "best linear approximation" of a surface $f(x,y)$ near a specific point. By replacing a complex, curved surface with a flat plane, we can simplify difficult calculations into manageable linear equations.

Critical Utility Across Fields

Field	Application	Why We Use Approximations
Engineering	**Error Propagation**	Engineers use differentials (dz) from the tangent plane to estimate how small errors in measuring x and y will impact the final result (z), such as the volume of a manufactured part.
Physics	**Small Deviation Modeling**	Many physics laws (like the pendulum or spring equations) are non-linear. By using a tangent plane, physicists can model behavior near an equilibrium point where the system acts linearly.
Economics	**Marginal Analysis**	The tangent plane represents the total effect of marginal changes. It helps economists estimate how a simultaneous change in labor and capital will affect total production output.
Computer Graphics	**Surface Shading**	To render 3D objects realistically, software uses the tangent plane (and its normal vector) to calculate how light reflects off a specific point on a curved surface.

Pilot Questions 4.4

The tangent plane provides a local __________ approximation of the surface near a point. A. Linear B. Quadratic C. Cubic D. Constant (Linked to SLO 4.4)

The equation of the tangent plane to z=f(x,y)at z0is: A. z=f(x,y)B. z-z0=fx(x0,y0)(x-x0)+fy(x0,y0)(y-y0)C. z=x+yD. z=f(x0,y0)(Linked to SLO 4.4)

Linear approximation is used to estimate function values near a known: A. Gradient B. Point C. Plane D. Vector (Linked to SLO 4.4)

In physics, tangent planes are used to approximate: A. Potential energy surfaces near equilibrium B. Velocity functions C. Historical data D. Literature trends (Linked to SLO 4.5)

In economics, tangent planes and linear approximations can be used to estimate: A. Cost functions near production levels B. Symmetry of graphs C. Continuity of functions D. Periodicity of demand (Linked to SLO 4.5)

Series 4 Summary

This Series focused on the differentiation of functions of two and three variables, extending the concepts of partial derivatives into more advanced techniques. We began with higher-order partial derivatives, including mixed partials and Clairaut's theorem. Next, we studied the chain rule in multivariable calculus, which allows us to differentiate composite functions. We then explored directional derivatives and the gradient vector, highlighting their relationship and applications in optimisation and physics. Finally, we examined tangent planes and linear approximations, which provide geometric and practical tools for estimating function values near given points.

By completing this Series, you should now be able to compute higher-order partial derivatives (SLO 4.1), apply the chain rule (SLO 4.2), evaluate directional derivatives and gradients (SLO 4.3), and construct tangent planes and linear approximations (SLO 4.4).

Glossary of Terms

Higher-order partial derivative: A derivative obtained by differentiating a function of several variables more than once.

Mixed partial derivative: A derivative taken with respect to different variables in succession.

Clairaut's theorem: States that if mixed partial derivatives are continuous near a point, they are equal.

Chain rule: A formula for differentiating composite functions in multivariable calculus.

Directional derivative: The rate of change of a function in the direction of a given vector.

Gradient vector: A vector of partial derivatives pointing in the direction of greatest increase of a function.

Tangent plane: A plane that locally approximates a surface at a point.

Linear approximation: An estimate of function values near a known point using the tangent plane.

Concept Check Questions 4

Objective questions

A higher-order partial derivative is obtained by differentiating a function more than __________. (Linked to SLO 4.1)

Clairaut's theorem states that if mixed partial derivatives are __________ near a point, then they are equal. (Linked to SLO 4.1)

The chain rule allows us to differentiate __________ functions. (Linked to SLO 4.2)

A directional derivative measures the rate of change of a function in the direction of a given __________. (Linked to SLO 4.3)

The tangent plane provides a local __________ approximation of the surface near a point. (Linked to SLO 4.4)

Short-answer questions

Define a mixed partial derivative and give an example. (Linked to SLO 4.1)

State the chain rule for a function z=f(x,y)where xand ydepend on t. (Linked to SLO 4.2)

Explain the relationship between the gradient vector and directional derivatives. (Linked to SLO 4.3)

Long-answer questions

Discuss Clairaut's theorem and its importance in simplifying multivariable differentiation. (Linked to SLO 4.1)

Basic response: State the theorem and its condition.

Value-added response: Provide examples showing equality of mixed partials and explain why continuity is essential.

Analyse the role of tangent planes and linear approximations in applied contexts such as physics or economics. (Linked to SLO 4.4, SLO 4.5)

Basic response: Explain tangent planes as local approximations.

Value-added response: Provide real-world examples of how tangent planes are used to estimate values in practice.

Answers to Pilot Questions 4

Part 4.1

B

B

B

A

B

Part 4.2

A

B

A

B

B

Part 4.3

B

A

A

C

B

Part 4.4

A

B

B

A

A

5 Maxima and Minima of Functions of Two Variables

Series outline

Part 5.1: Critical Points of Functions of Two Variables

Definition of critical points

Finding critical points using partial derivatives

Part 5.2: Second Derivative Test for Two Variables

Hessian determinant

Classification of critical points (local maxima, local minima, saddle points)

Part 5.3: Constrained Optimisation and Lagrange Multipliers

Optimisation with constraints

Method of Lagrange multipliers

Applications in economics and engineering

Part 5.4: Practical Applications of Maxima and Minima

Real-world problems in physics, economics, and engineering

Case studies and examples

Introduction

Optimisation in higher dimensions is one of the most powerful applications of multivariable calculus. In this Series, we focus on the maxima and minima of functions of two variables, which extend the familiar ideas of single-variable optimisation into surfaces and multidimensional contexts. These concepts are essential in mathematics, physics, economics, and engineering, where systems often depend on more than one variable.

The aim of this Series is to equip you with the tools to identify and classify critical points, apply second derivative tests, and solve constrained optimisation problems. You will learn how to use partial derivatives to locate critical points, apply the Hessian determinant to classify them, and employ the method of Lagrange multipliers to handle constraints. Finally, you will explore practical applications of maxima and minima in real-world problems.

We shall commence this Series by studying critical points of functions of two variables, where partial derivatives vanish or fail to exist. Next, we will examine the second derivative test, using the Hessian determinant to classify critical points as local maxima, minima, or saddle points. Following that, we will explore constrained optimisation and Lagrange multipliers, a powerful method for solving problems with restrictions. Finally, we will conclude with practical applications of maxima and minima, showing how these techniques are used in physics, economics, and engineering.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 5.1** identify and compute critical points of functions of two variables using partial derivatives,
- **SLO 5.2** apply the second derivative test with the Hessian determinant to classify critical points as local maxima, minima, or saddle points,
- **SLO 5.3** solve constrained optimisation problems using the method of Lagrange multipliers,
- **SLO 5.4** analyse and apply maxima and minima techniques to practical problems in physics, economics, and engineering.

5.1 Critical Points of Functions of Two Variables

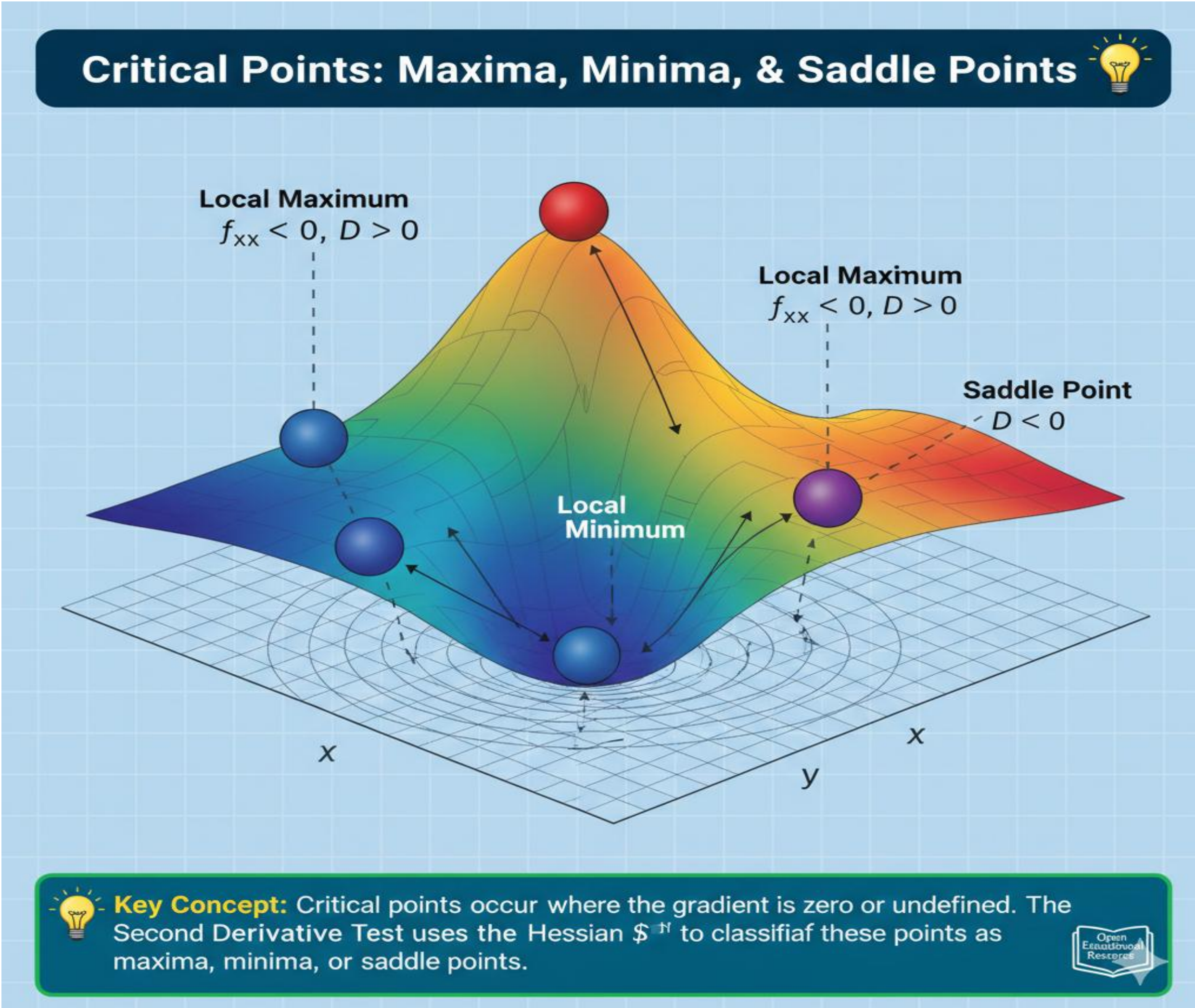
5.1.1 Definition of critical points

A **critical point** of a function f(x,y)is a point y0where the first-order partial derivatives vanish or fail to exist:

$$fx(x0,y0)=0,fy(x0,y0)=0.$$

Critical points are candidates for local maxima, local minima, or saddle points.

Fill in the blank: A critical point occurs when the first-order partial derivatives of a function __________ or fail to exist.



Critical points of a function of two variables

5.1.2 Finding critical points using partial derivatives

To find critical points:

Compute the first-order partial derivatives fxand fy.

Solve the system of equations:

$$fx=0,fy=0.$$

The solutions are the critical points.

Example: For f(x,y)=x2+y2:

$$f_x=2x, f_y=2y.$$

Setting both equal to zero gives the critical point 0.

Fill in the blank: To find critical points, we solve the system of equations formed by setting the first-order partial derivatives equal to __________.

5.1.3 Types of critical points (overview)

Critical points can be classified into:

- **Local maximum**: Function value is greater than nearby points.
- **Local minimum**: Function value is smaller than nearby points.
- **Saddle point**: Function value is higher in some directions and lower in others.

Fill in the blank: A saddle point is a critical point where the function is higher in some directions and __________ in others.

Box summarising local maxima, local minima, and saddle points.

Types of Critical Points in Multivariable Functions

A **critical point** (a, b) occurs where the gradient is zero ($\nabla f = \mathbf{0}$) or where the partial derivatives are undefined. Once a critical point is found, we use the **Second Derivative Test** to classify it.

Classification Guide

To classify a point, we calculate the **Hessian determinant** (or Discriminant), D:

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

Type of Point	Mathematical Condition	Geometric Visualization
Local Maximum	$D > 0$ and $f_{xx} < 0$	The surface curves downward in all directions, like the top of a dome.
Local Minimum	$D > 0$ and $f_{xx} > 0$	The surface curves upward in all directions, like the bottom of a bowl.
Saddle Point	$D < 0$	The surface curves up in one direction and down in another, like a mountain pass or a physical horse saddle.
Inconclusive	$D = 0$	The test fails; the point could be any of the above or a "flat" spot. Further analysis is needed.

Pilot Questions 5.1

A critical point occurs when the first-order partial derivatives: A. Are positive B. Are negative C. Vanish or fail to exist D. Are undefined everywhere (Linked to SLO 5.1)

To find critical points, we solve the system of equations formed by setting the first-order partial derivatives equal to: A. 1 B. 0 C. Infinity D. Undefined (Linked to SLO 5.1)

For f(x,y)=x2+y2, the critical point is: A. 1B. 0C. 2D. None (Linked to SLO 5.1)

A saddle point is a critical point where the function is higher in some directions and __________ in others. A. Equal B. Lower C. Undefined D. Constant (Linked to SLO 5.1)

Critical points are candidates for: A. Local maxima, minima, or saddle points B. Only maxima C. Only minima D. Only saddle points (Linked to SLO 5.1)

5.2 Second Derivative Test for Two Variables

5.2.1 Hessian determinant

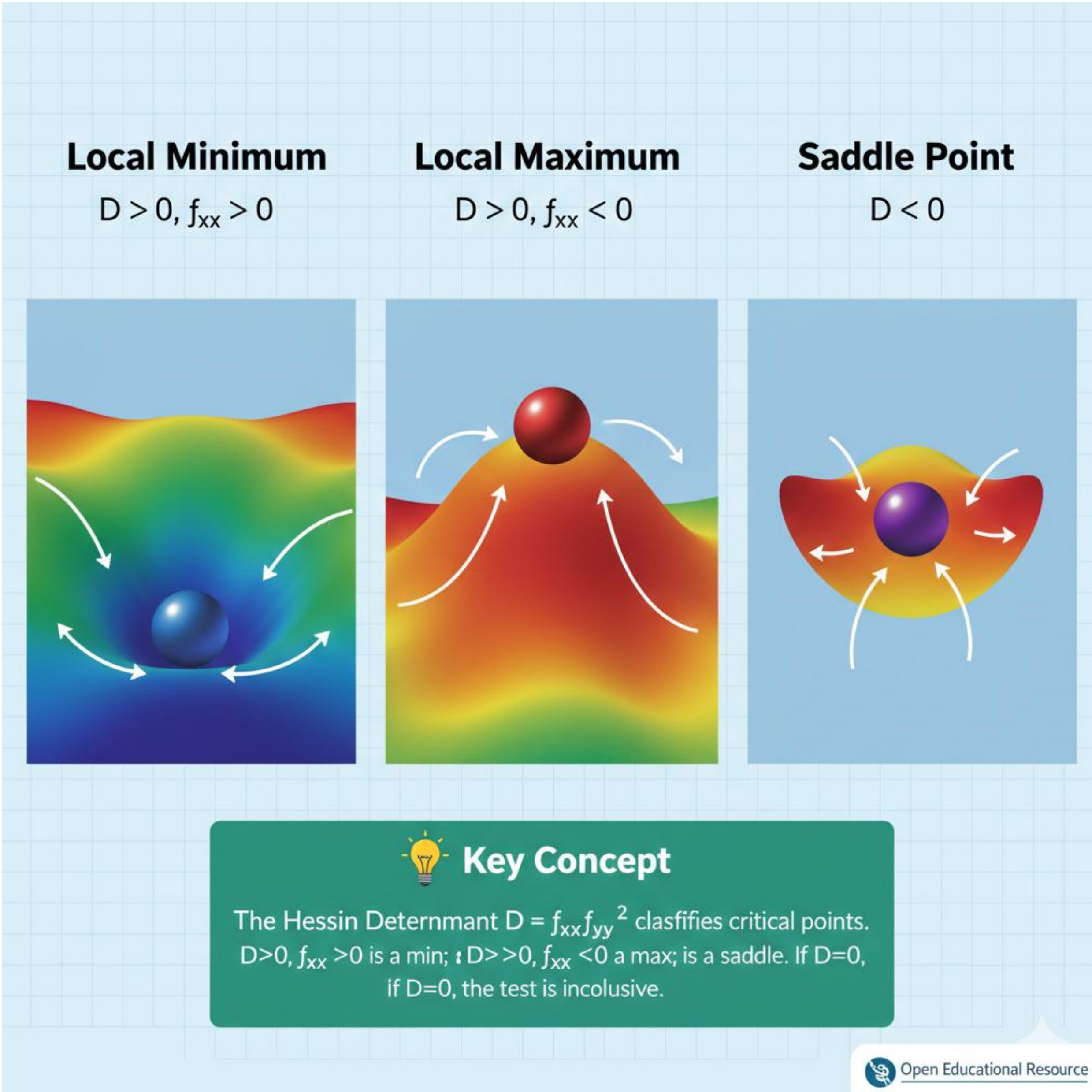
To classify critical points, we use the **Hessian determinant**. For a function f(x,y), let:

$$fxx=2fx2,fyy=2fy2,fxy=2fxy.$$

The Hessian determinant at a point y0is:

$$D=fxx(x0,y0)\cdot fyy(x0,y0)-fxy(x0,y0)2.$$

Fill in the blank: The Hessian determinant is defined as $D=fxxfyy-(fxy)2$at a __________ point.



Hessian determinant used to classify critical points

5.2.2 Classification of critical points

The second derivative test states:

If D>0and fxx>0, then y0is a **local minimum**.

If D>0and fxx<0, then y0is a **local maximum**.

If D<0, then y0is a **saddle point**.

If D=0, the test is inconclusive.

Fill in the blank: If D>0and fxx<0, the critical point is a local __________.

5.2.3 Example

For f(x,y)=x2+y2:

$$fxx=2, fyy=2, fxy=0.$$

At 0:

$$D=(2)(2)-(0)2=4>0, fxx=2>0.$$

Thus, 0is a **local minimum**.

Fill in the blank: For f(x,y)=x2+y2, the critical point at 0is classified as a local __________.

Box summarising conditions for local maxima, minima, and saddle points using the Hessian determinant.

The Second Derivative Test for Multivariable Functions

The **Second Derivative Test** is the definitive tool for classifying critical points of a function $f(x, y)$. After finding a point (a, b) where the gradient $\nabla f = \mathbf{0}$, this test uses the concavity and "twist" of the surface to determine its shape.

The Test Metric: The Hessian Determinant (D)

First, calculate the second-order partial derivatives and find the discriminant D at the critical point (a, b):

$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

Classification Conditions

Result	Mathematical Requirements	Visual Description
Local Minimum	$D > 0$ and $f_{xx} > 0$	The surface opens upward in all directions (like a bowl).
Local Maximum	$D > 0$ and $f_{xx} < 0$	The surface opens downward in all directions (like a mountain peak).
Saddle Point	$D < 0$	The surface opens up in one direction and down in another (like a mountain pass).
Inconclusive	$D = 0$	The test provides no information. The point could be a max, min, or saddle.

Pilot Questions 5.2

The Hessian determinant is defined as: A. fxx+fyyB. fxxfyy-(fxy)2C. fxx-fyyD. fxy2(Linked to SLO 5.2)

If D>0and fxx>0, the critical point is a: A. Local maximum B. Local minimum C. Saddle point D. Inconclusive (Linked to SLO 5.2)

If D>0and fxx<0, the critical point is a: A. Local maximum B. Local minimum C. Saddle point D. Inconclusive (Linked to SLO 5.2)

If D<0, the critical point is a: A. Local maximum B. Local minimum C. Saddle point D. Inconclusive (Linked to SLO 5.2)

For f(x,y)=x2+y2, the critical point at 0is classified as a: A. Local maximum B. Local minimum C. Saddle point D. Inconclusive (Linked to SLO 5.2)

5.3 Constrained Optimisation and Lagrange Multipliers

5.3.1 Optimisation with constraints

In many real-world problems, optimisation must be performed under certain restrictions. For example, maximising profit subject to resource limits or minimising cost subject to production requirements. These are called **constrained** optimisation problems.

Fill in the blank: Constrained optimisation problems involve finding maxima or minima subject to __________.

5.3.2 Method of Lagrange multipliers

The **Lagrange multiplier method** is a powerful technique for solving constrained optimisation problems.

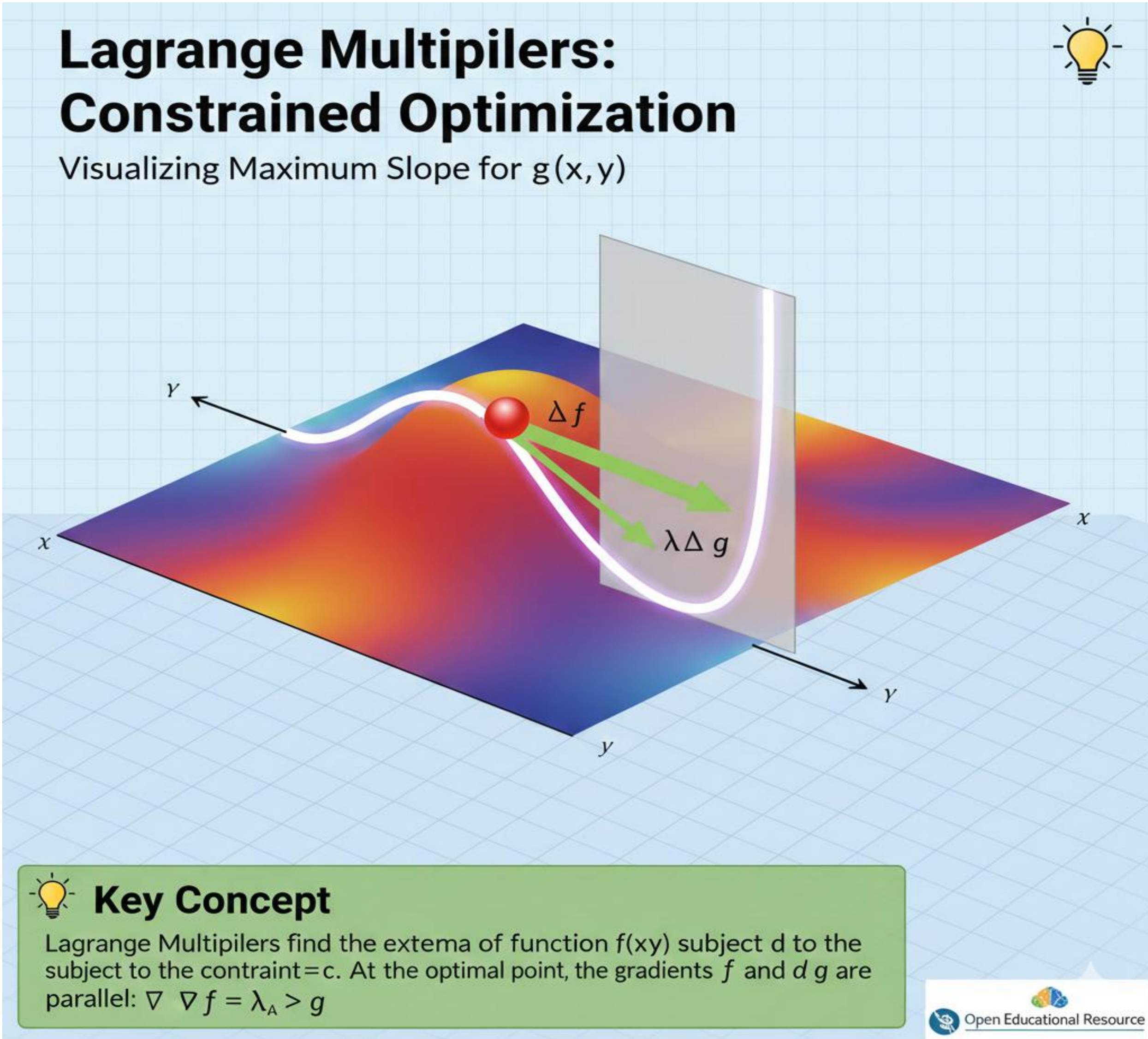
Suppose we want to optimise f(x,y)subject to the constraint g(x,y)=c. We introduce a new variable (the Lagrange multiplier) and form the **Lagrangian function**:

$$L(x,y,\lambda)=f(x,y)+\lambda(c-g(x,y)).$$

The critical points are found by solving:

$$Lx=0,Ly=0,L=0.$$

Fill in the blank: The Lagrangian function introduces a new variable called the __________ multiplier.



Lagrange multipliers method for constrained optimisation

5.3.3 Applications in economics and engineering

Economics: Maximising utility subject to budget constraints.

Engineering: Minimising energy subject to design restrictions.

Physics: Optimising systems subject to conservation laws.

Example: Maximise f(x,y)=xysubject to x+y=10.

Lagrangian: L(x,y,λ)=xy+λ(10-x-y).

Equations:

$$L_x = y - \lambda = 0, L_y = x - \lambda = 0, L = 10 - x - y = 0.$$

Solution: $x = y = 5$, maximum value $f(5,5) = 25$.

Fill in the blank: In economics, Lagrange multipliers are often used to maximise ___________ subject to budget constraints.

Box summarising applications in economics, engineering, and physics. **Pilot**

Applications of Lagrange Multipliers

The Method of **Lagrange Multipliers** is the gold standard for **constrained optimization**. It allows us to find the maximum or minimum values of a function $f(x, y, z)$ when we are restricted by one or more equations $g(x, y, z) = k$.

The core mathematical insight is that at the optimal point, the gradient of the function (∇f) and the gradient of the constraint (∇g) must be **parallel**:

$$\nabla f = \lambda \nabla g$$

Where λ (Lambda) is the Lagrange Multiplier.

Field	Application	The Objective vs. The Constraint
Economics	**Utility Maximization**	**Objective: Maximize a consumer's satisfaction (utility).** **Constraint: A fixed budget or income.**
Engineering	**Structural Integrity**	**Objective: Minimize the weight of a bridge or aircraft wing.**

Field	Application	The Objective vs. The Constraint
		Constraint: It must support a specific maximum load without failing.
Physics	**Least Action Principle**	**Objective: Determine the path of a particle by minimizing "Action."** **Constraint: The particle must stay on a specific surface or track.**
Manufacturing	**Cost Minimization**	**Objective: Minimize production costs.** **Constraint: Produce exactly 10,000 units of a product.**

Questions 5.3

Constrained optimisation problems involve finding maxima or minima subject to: A. No conditions B. Constraints C. Random choices D. Undefined values (Linked to SLO 5.3)

The Lagrangian function is defined as: A. f(x,y)+g(x,y)B. f(x,y)+λ(c-g(x,y))C. f(x,y)-g(x,y)D. f(x,y)+c(Linked to SLO 5.3)

The new variable introduced in the Lagrangian method is called the: A. Gradient B. Hessian C. Lagrange multiplier D. Constraint variable (Linked to SLO 5.3)

In economics, Lagrange multipliers are often used to maximise: A. Utility subject to budget constraints B. Velocity subject to time constraints C. Literature subject to grammar constraints D. History subject to cultural constraints (Linked to SLO 5.3)

For the problem maximise $f(x,y)=xy$subject to $x+y=10$, the solution is: A. $x=10,y=0$B. $x=0,y=10$C. $x=y=5$D. None of the above (Linked to SLO 5.3)

5.4 Practical Applications of Maxima and Minima

5.4.1 Applications in physics

Equilibrium problems: Potential energy functions often reach minima at equilibrium points.

Optics: Path of light minimises travel time (Fermat's principle).

Thermodynamics: Systems tend toward states of minimum free energy.

Fill in the blank: In physics, equilibrium points often correspond to __________ of potential energy functions.

5.4.2 Applications in economics

Profit maximisation: Firms maximise profit subject to cost and resource constraints.

Utility maximisation: Consumers maximise utility subject to budget constraints.

Cost minimisation: Firms minimise production costs subject to output requirements.

Fill in the blank: In economics, consumers maximise __________ subject to budget constraints.

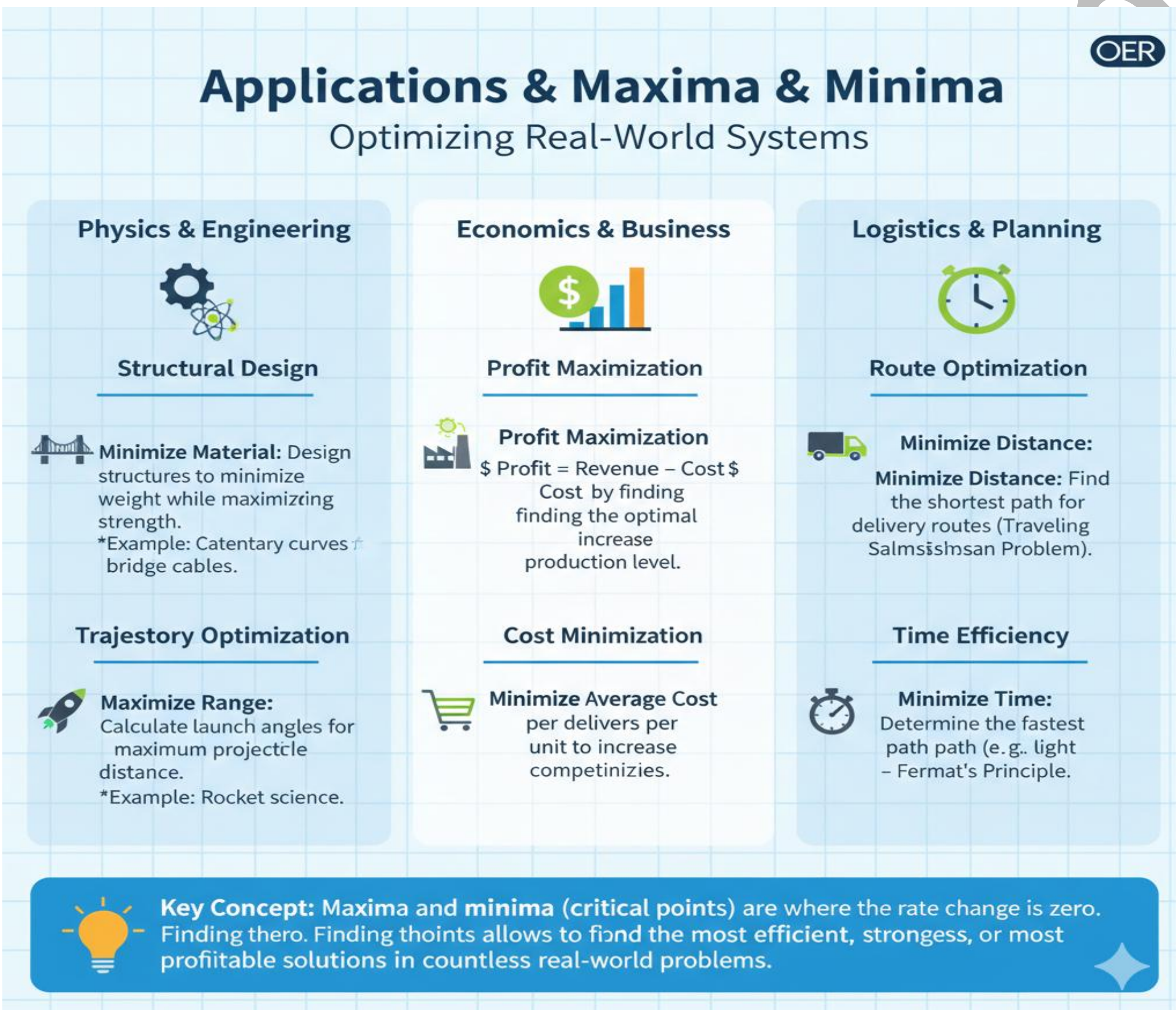
5.4.3 Applications in engineering

Structural design: Minimising stress or weight subject to safety constraints.

Control systems: Optimising performance subject to stability conditions.

Manufacturing: Minimising waste while maximising efficiency.

Fill in the blank: In engineering, optimisation is often used to minimise __________ or maximise efficiency.



Applications of maxima and minima

5.4.4 Case study example

Economics case study: A company produces two goods, xand y. Profit function:

$$P(x,y)=40x+30y-(x2+y2).$$

Critical points are found by setting partial derivatives to zero:

$$P_x = 40 - 2x = 0 \Rightarrow x = 20,$$

$$P_y = 30 - 2y = 0 \Rightarrow y = 15.$$

Thus, maximum profit occurs at 15.

Fill in the blank: In the case study, maximum profit occurs when $x=20$and __________.

Pilot Questions 5.4

In physics, equilibrium points often correspond to: A. Maxima of potential energy B. Minima of potential energy C. Saddle points of potential energy D. Undefined values (Linked to SLO 5.4)

In economics, consumers maximise __________ subject to budget constraints. A. Utility B. Cost C. Stress D. Energy (Linked to SLO 5.4)

In engineering, optimisation is often used to minimise: A. Stress or weight B. Utility C. Profit D. History (Linked to SLO 5.4)

Fermat's principle in optics states that light follows the path that: A. Maximises distance B. Minimises travel time C. Maximises cost D. Minimises stress (Linked to SLO 5.4)

In the case study, maximum profit occurs when $x=20$and: A. $y=10$B. $y=15$C. $y=20$D. $y=30$(Linked to SLO 5.4)

Series 5 Summary

This Series explored the maxima and minima of functions of two variables, extending optimisation techniques into multivariable contexts. We began with critical points, defined by vanishing partial derivatives, and learned how to identify them. Next, we studied the second derivative test, using the Hessian determinant to classify critical points as local maxima, minima, or saddle points. We then examined constrained optimisation and Lagrange multipliers, a powerful method for solving problems with restrictions. Finally, we applied these concepts to practical problems in physics, economics, and engineering, including equilibrium analysis, profit maximisation, and structural design.

By completing this Series, you should now be able to identify critical points (SLO 5.1), classify them using the second derivative test (SLO 5.2), solve constrained optimisation problems with Lagrange multipliers (SLO 5.3), and apply maxima and minima techniques to real-world problems (SLO 5.4).

Glossary of Terms

Critical point: A point where the first-order partial derivatives vanish or fail to exist.

Hessian determinant: A determinant used in the second derivative test to classify critical points.

Local maximum: A point where the function value is greater than nearby values.

Local minimum: A point where the function value is smaller than nearby values.

Saddle point: A critical point where the function is higher in some directions and lower in others.

Constrained optimisation: Optimisation performed subject to restrictions.

Lagrange multiplier: A variable introduced in the Lagrangian function to handle constraints.

Lagrangian function: A function combining the objective and constraint equations for optimisation.

Concept Check Questions 5

Objective questions

A critical point occurs when the first-order partial derivatives __________ or fail to exist. (Linked to SLO 5.1)

The Hessian determinant is defined as D=fxxfyy-(fxy)2at a __________ point. (Linked to SLO 5.2)

If D>0and fxx<0, the critical point is a local __________. (Linked to SLO 5.2)

The Lagrangian function introduces a new variable called the __________ multiplier. (Linked to SLO 5.3)

In physics, equilibrium points often correspond to __________ of potential energy functions. (Linked to SLO 5.4)

Short-answer questions

Define a saddle point and explain its significance. (Linked to SLO 5.2)

State the method of Lagrange multipliers for a function f(x,y)subject to g(x,y)=c. (Linked to SLO 5.3)

Give one example of constrained optimisation in economics. (Linked to SLO 5.4)

Long-answer questions

Discuss the second derivative test for functions of two variables. (Linked to SLO 5.2)

- **Basic response**: State the conditions for maxima, minima, and saddle points.
- **Value-added response**: Provide examples and explain why the test may be inconclusive when D=0.

Analyse the role of Lagrange multipliers in solving constrained optimisation problems, with applications in economics or engineering. (Linked to SLO 5.3, SLO 5.4)

- **Basic response**: Explain the method and its steps.
- **Value-added response**: Provide a real-world example of profit maximisation or structural optimisation.

Answers to Pilot Questions 5

Part 5.1

C

B

B

B

A

Part 5.2

B

B

A

C

B

Part 5.3

B

B

C

A

C

Part 5.4

B

A

A

B

B