La-Net

MTH101

ELEMENTARY MATHEMATICS I

Series 1: Foundations of Set Theory and Venn Diagrams

Series outline

Introduction

In everyday life, we often group objects, ideas, or people into categories, whether it is arranging books on a shelf, sorting clothes by colour, or classifying students by their courses. These groupings are examples of what mathematicians call **sets**, and they form the foundation of many areas of mathematics. Understanding sets and how they interact is essential because they provide the language and structure for more advanced topics such as algebra, probability, and logic. This Series will therefore introduce you to the basic principles of set theory and the use of Venn diagrams, which are powerful tools for visualizing relationships between sets.

The aim of this Series is to equip you with the ability to define sets clearly, perform operations on them, and represent these operations using Venn diagrams. By the end of the Series, you should be able to apply these concepts to solve mathematical problems and recognize their relevance in real-world contexts.

We shall commence this Series by exploring the basic concepts of sets, including their definitions, methods of description, and types. Next, we will examine subsets and set operations such as union, intersection, and complements. Following that, we will move on to Venn diagrams, where you will learn how to represent and interpret sets visually, including two-set and three-set diagrams. Finally, we will wrap up by considering practical applications of set theory, showing how these ideas can be used in everyday situations and in fields such as probability and logic.

Series Learning Outcomes

Upon completing this Series, you should be able to:

2.1 define the concept of a set and distinguish between different types of sets,

2.2 explain the idea of subsets and demonstrate how to perform basic set operations such as union, intersection, and complements,

2.3 illustrate relationships between sets using Venn diagrams and interpret two-set and three-set representations,

2.4 apply set theory and Venn diagrams to solve practical problems in everyday contexts and in mathematics.

1.1 Basic Concepts of Sets

1.1.1 Definition of a set

A **set** is a well-defined collection of distinct objects, called elements, which are grouped together as a single entity. The term "well-defined" means that there should be no ambiguity about whether an object belongs to the set or not. For example, the set of vowels in the English alphabet can be written as {a, e, i, o, u}.

Sets are usually denoted by capital letters such as A, B, or C, while their elements are listed inside curly brackets. If an element belongs to a set, we say it is a member of that set.

Fill in the blank: A ______ is a well-defined collection of distinct objects regarded as a single entity.

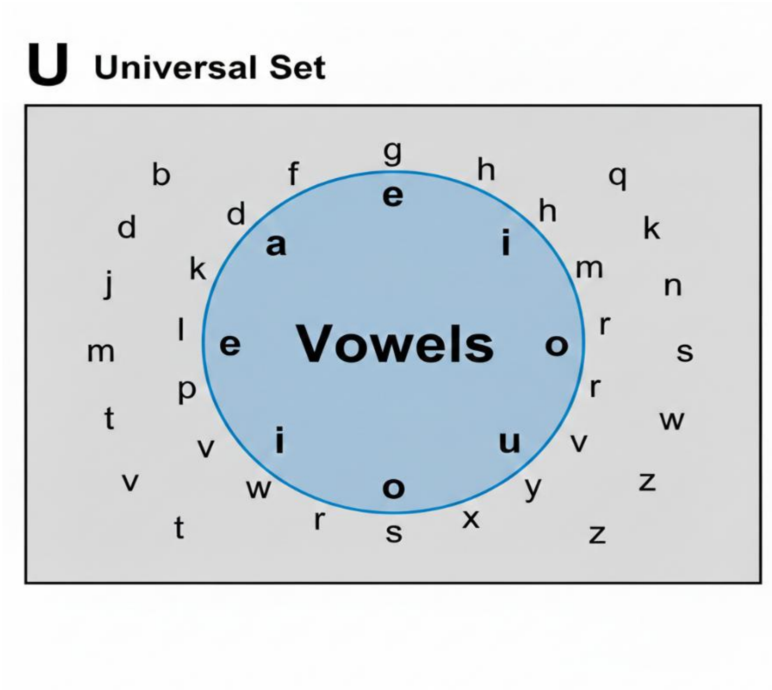


Figure 1.1 diagram of set notation vowels

1.1.2 Methods of describing sets

There are two common methods of describing sets:

Roster method: This involves listing all the elements of the set explicitly inside curly brackets. For example, the set of even numbers less than 10 can be written as {2, 4, 6, 8}.

Set-builder notation: This involves describing the property that characterises the elements of the set. For example, the same set of even numbers less than 10 can be written as {x | x is an even number less than 10}.

Fill in the blank: The ______ method lists all elements explicitly, while the ______ notation describes them by a property.

Box 1.1 Methods of describing sets Content:

Roster method: Listing elements explicitly.

Set-builder notation:

Table 1.1: set theory roster vs set-builder notation

Feature	Roster (Tabular) Method	Set-Builder Notation
Definition	Lists every single element within braces.	Describes the properties elements must satisfy.
Best Used For	Small, finite sets.	Large, infinite, or complex sets.
Visual Style	{1, 2, 3, 4, 5}	{x | x is a positive integer < 6}

1.1.3 Types of sets

Sets can be classified into different types based on their properties:

Finite set: A set with a countable number of elements. Example: {1, 2, 3, 4}.

Infinite set: A set with an uncountable or unlimited number of elements. Example: {1, 2, 3, …}.

Empty set (null set): A set with no elements, denoted by {} or Ø.

Universal set: A set that contains all possible elements under consideration in a particular context.

Fill in the blank: A set with no elements is called the ______ set.

Table 1.2 Types of sets with examples

Type of set	**Definition**	**Example**
Finite set	Countable number of elements	{1, 2, 3, 4}
Infinite set	Uncountable or unlimited number of elements	{1, 2, 3, …}
Empty set	No elements	{} or Ø
Universal set	All elements under consideration	U = {all letters of alphabet}

Pilot questions 1.1

Which of the following best defines a set? a) A random group of objects b) A well-defined collection of distinct objects c) A list of repeated items d) A group with no rules

Which method of describing sets involves listing all elements explicitly? a) Set-builder notation b) Roster method c) Universal method d) Infinite method

What symbol is commonly used to denote the empty set? a) {} or Ø b) U c) ∩ d) ∪

Which of the following is an example of an infinite set? a) {1, 2, 3, 4} b) {a, e, i, o, u} c) {1, 2, 3, …} d) {}

The set of all letters in the English alphabet is an example of which type of set? a) Finite set b) Infinite set c) Universal set d) Empty set

1.2 Subsets and Set Operations

1.2.1 Subsets and proper subsets

A **subset** is a set in which all its elements are contained within another set. If every element of set A is also an element of set B, then A is said to be a subset of B, written as $A \subseteq B$. A **proper subset** is a subset that is strictly smaller than the set it belongs to, meaning $A \subset B$ if A is contained in B but $A \neq B$.

Fill in the blank: If every element of set A is also in set B, then A is a ______ of

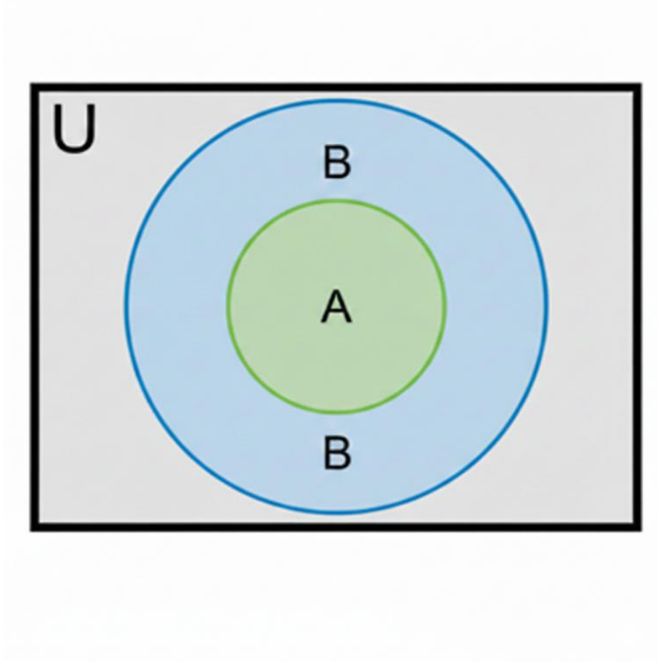


Figure 1.2 Venn diagram showing set A inside set B to illustrate subset

1.2.2 Union of sets

The union of two sets is the set containing all elements that belong to either of the sets. It is denoted by $A \cup B$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Fill in the blank: The union of sets A and B is written as ______.

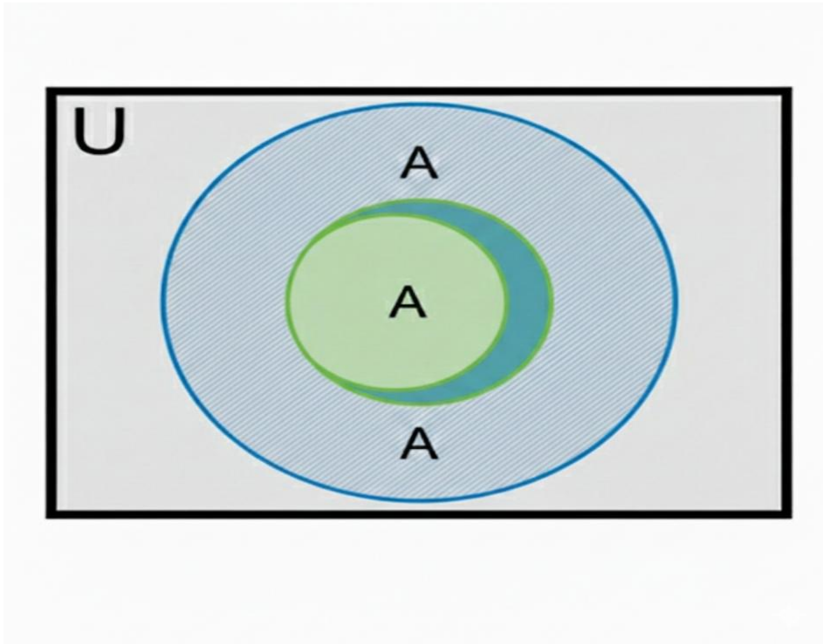


Figure 1.3 Union of two sets Venn diagram”

1.2.3 Intersection of sets

The intersection of two sets is the set containing all elements that are common to both sets. It is denoted by $A \cap B$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Fill in the blank: The intersection of sets A and B is written as ______.

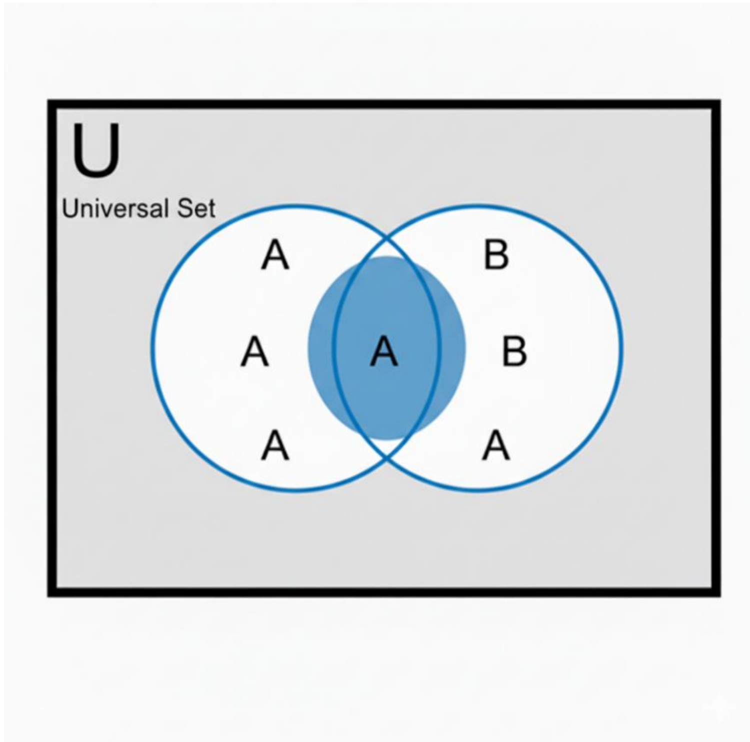


Figure 1.4 Intersection of two sets Venn diagram"

1.2.4 Complements of sets

The **complement** of a set A, written as A', is the set of all elements in the universal set U that are not in A. For example, if U = {1, 2, 3, 4, 5} and A = {1, 2}, then A' = {3, 4, 5}.

Fill in the blank: The complement of set A is written as ______.

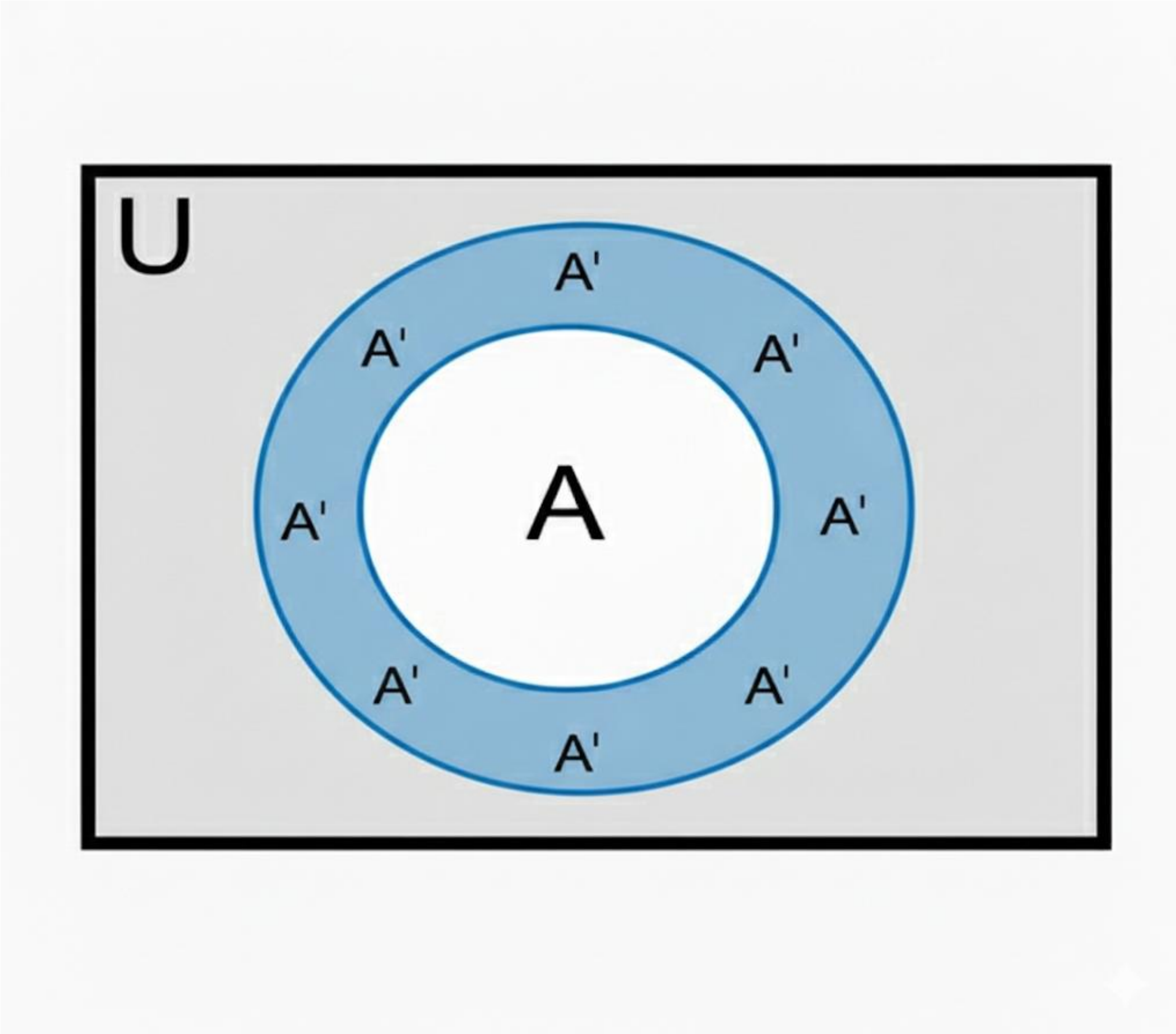


Figure 1.5 Complement of a **Venn diagram"**

Pilot questions 1.2

If A = {1, 2} and B = {1, 2, 3}, which statement is correct? a) A ⊂ B b) B ⊂ A c) A ∪ B = {1, 2} d) A ∩ B = {1, 2, 3}

Which symbol represents the union of two sets? a) ∩ b) ∪ c) ⊂ d) ⊆

If A = {2, 4, 6} and B = {4, 6, 8}, what is A ∩ B? a) {2, 4, 6, 8} b) {4, 6} c) {2, 8} d) {}

The complement of set A refers to: a) Elements in A only b) Elements in both A and B c) Elements in U but not in A d) Elements in A ∪ B

Which of the following is a proper subset? a) $\{1, 2\} \subset \{1, 2, 3\}$ b) $\{1, 2, 3\} \subset \{1, 2, 3\}$ c) $\{1, 2, 3\} \subset \{1, 2\}$ d) $\{\} \subset \{\}$

1.3 Venn Diagrams and Applications

1.3.1 Representation of sets using Venn diagrams

A **Venn diagram** is a pictorial representation of sets using closed curves, usually circles, to show relationships among sets. Each circle represents a set, and the overlap between circles shows elements common to the sets. Venn diagrams are particularly useful for visualising operations such as union, intersection, and complement.

Fill in the blank: A ______ diagram uses circles to represent sets and their relationships.

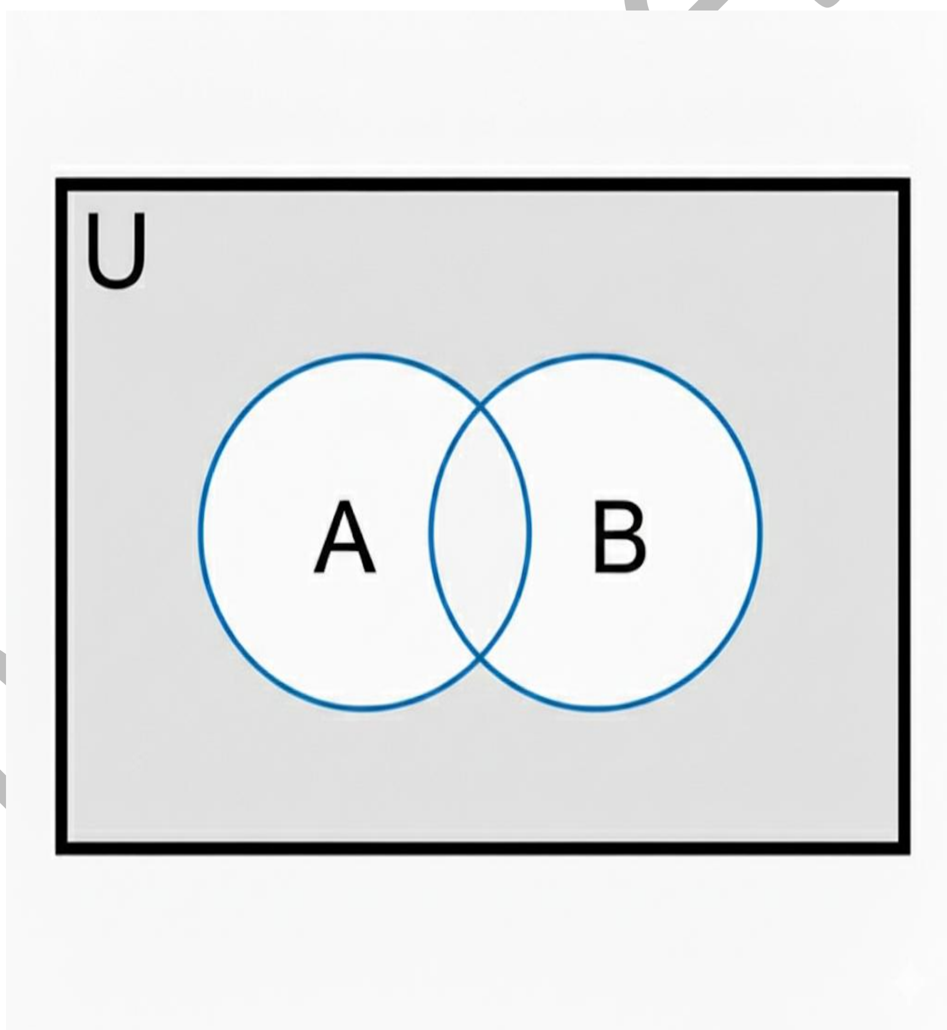


Figure 1.6 Basic Venn diagram representation of two

1.3.2 Two-set and three-set Venn diagrams

A two-set Venn diagram involves two circles, each representing a set. The overlapping region shows the intersection, while the non-overlapping regions show elements unique to each set.

A three-set Venn diagram involves three overlapping circles. It can represent more complex relationships, including intersections among all three sets, pairwise intersections, and unique elements of each set.

Fill in the blank: A ______ Venn diagram can represent intersections among three sets simultaneously.

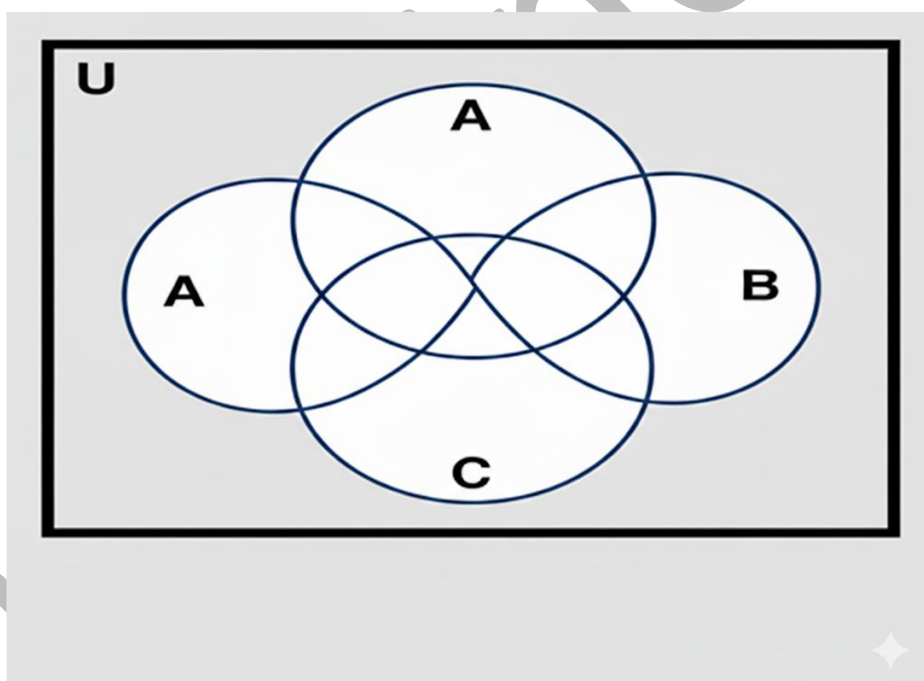


Figure 1.7 Basic Venn diagram of there sets

1.3.3 Problem-solving with Venn diagrams

Venn diagrams are not only visual aids but also problem-solving tools. They help in solving questions involving classification, counting, and logical reasoning. For example, if a survey shows that 40 students like mathematics, 30 like physics, and 20 like both, a Venn diagram can be used to determine how many students like only mathematics or only physics.

Fill in the blank: Venn diagrams are useful for solving problems involving ______ and logical reasoning.

Box 1.2 Example problem using Venn diagrams Content: A survey shows:

40 students like mathematics

30 students like physics

20 students like both subjects

Question: How many students like only mathematics? AI prompt suggestion: "Create a text box showing a Venn diagram word problem with numbers and sets." Search string: "OER Venn diagram word problem example"

Pilot questions 1.3

What does each circle in a Venn diagram represent? a) A number b) A set c) An equation d) A subset

In a two-set Venn diagram, what does the overlapping region represent? a) Union b) Complement c) Intersection d) Subset

How many circles are used in a three-set Venn diagram? a) One b) Two c) Three d) Four

Which of the following is a practical use of Venn diagrams? a) Solving quadratic equations b) Representing sets visually c) Drawing graphs of functions d) Calculating derivatives

If 40 students like mathematics, 30 like physics, and 20 like both, how many students like only mathematics? a) 20 b) 30 c) 40 d) 10

1.4 Practical Applications of Set Theory

1.4.1 Everyday examples of sets and subsets

Set theory is not confined to abstract mathematics, it appears in many everyday situations. For instance, when you classify fruits into categories such as citrus fruits {orange, lemon, lime} and tropical fruits {mango, pineapple, banana}, you are working with sets. Similarly, when you separate students into those who study mathematics and those who study physics, you are dealing with subsets of a larger group.

Fill in the blank: When you classify fruits into categories such as citrus or tropical, you are working with ______

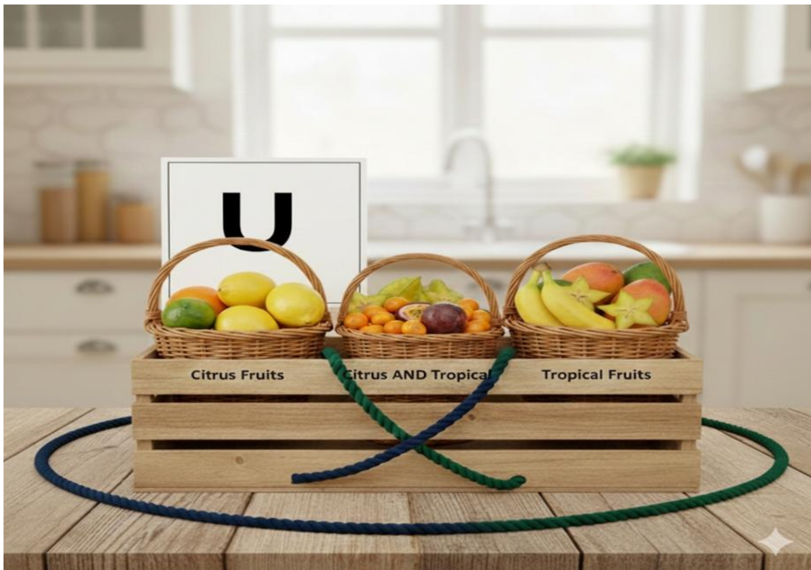


Plate 1.1 Everyday Classification of fruit as sets.

1.4.2 Use of Venn diagrams in probability and logic

Venn diagrams are widely used in probability and logic to illustrate relationships between events. For example, in probability, if event A represents students who passed mathematics and event B represents students who passed physics, then $A \cap B$ represents students who passed both subjects. In logic, Venn diagrams can be used to test the validity of arguments by showing how categories overlap or remain distinct.

Fill in the blank: In probability, the intersection of two events represents students who passed ______ subjects.

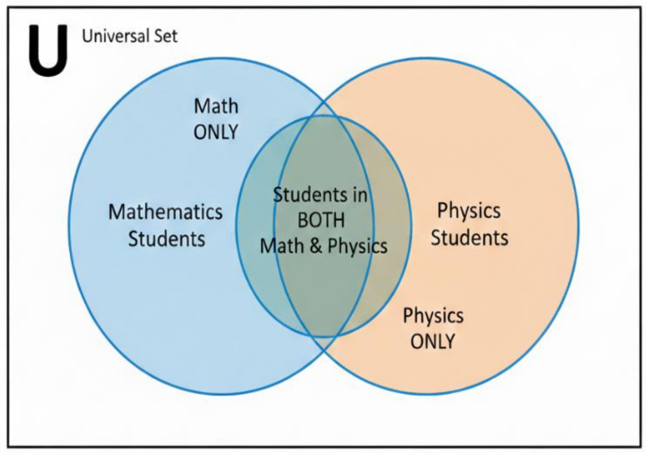


Figure 1.8 Venn diagram applied to probability Venn diagram

Pilot questions 1.4

Which of the following is an everyday example of sets? a) Sorting fruits into categories b) Solving quadratic equations c) Drawing graphs of functions d) Calculating derivatives

In probability, what does $A \cap B$ represent? a) Students who passed only mathematics b) Students who passed only physics c) Students who passed both subjects d) Students who failed both subjects

Which tool is commonly used in logic to test validity of arguments? a) Graphs b) Equations c) Venn diagrams d) Matrices

If citrus fruits are {orange, lemon, lime}, what type of set is this? a) Infinite set b) Finite set c) Empty set d) Universal set

Which of the following best describes the use of Venn diagrams in probability? a) To calculate derivatives b) To illustrate relationships between events c) To solve quadratic equations d) To classify numbers

Series Summary

This Series introduced you to the foundations of set theory and the use of Venn diagrams, which are essential tools for organising and visualising mathematical relationships. The purpose was to help you build confidence in defining sets, performing operations on them, and applying these concepts to practical situations in mathematics and everyday life.

We began by examining the basic concepts of sets, including their definitions, methods of description, and types. Then we explored subsets and set operations such as union, intersection, and complements. Following that, we studied Venn diagrams, learning how to represent sets visually and solve problems using two-set and three-set diagrams. Finally, we considered practical applications of set theory in everyday contexts and in areas such as probability and logic. By completing this Series, you should now be able to define sets and their types (SLO 2.1), explain subsets and operations (SLO 2.2), illustrate relationships using Venn diagrams (SLO 2.3), and apply these concepts to solve real-world problems (SLO 2.4).

Glossary of Terms

Complement: The set of all elements in the universal set that are not in a given set.

Element: An individual object or member contained within a set.

Empty set (null set): A set with no elements, denoted by {} or Ø.

Finite set: A set with a countable number of elements.

Infinite set: A set with an uncountable or unlimited number of elements.

Intersection: The set containing elements common to two or more sets, denoted by ∩.

Proper subset: A subset that contains fewer elements than the set it belongs to.

Roster method: A way of describing a set by listing all its elements explicitly.

Set: A well-defined collection of distinct objects regarded as a single entity.

Set-builder notation: A way of describing a set by stating the property that its elements satisfy.

Subset: A set whose elements are all contained within another set.

Union: The set containing all elements from two or more sets, denoted by ∪.

Universal set: The set that contains all possible elements under consideration in a particular context.

Venn diagram: A pictorial representation of sets using circles to show relationships among them.

Concept Check Questions [Series 1]

Objective questions

Which notation is used to represent the union of two sets? (Linked to SLO 2.2) a) ∩ b) ∪ c) ⊂ d) Ø

What does the shaded overlap in a two-set Venn diagram represent? (Linked to SLO 2.3) a) Union b) Intersection c) Complement d) Subset

Which of the following is an example of a finite set? (Linked to SLO 2.1) a) {1, 2, 3, …} b) {} c) {a, e, i, o, u} d) U = {all integers}

Short-answer questions

Define a proper subset and give one example. (Linked to SLO 2.2)

Explain the difference between the roster method and set-builder notation. (Linked to SLO 2.1)

Long-answer questions

Illustrate with examples how Venn diagrams can be used to solve classification problems involving three sets. (Linked to SLO 2.3)

Analyse how set theory can be applied in probability to determine outcomes of combined events. (Linked to SLO 2.4)

Answers to Concept Check Questions [Series 1]

Objective questions

b) ∪

b) Intersection

c) {a, e, i, o, u}

Short-answer questions

A proper subset is a subset that contains fewer elements than the set it belongs to. Example: {1, 2} ⊂ {1, 2, 3}.

The roster method lists all elements explicitly, for example {2, 4, 6}, while set-builder notation describes elements by a property, for example {x | x is an even number less than 10}.

Long-answer questions

Basic response: A three-set Venn diagram uses three overlapping circles to represent sets A, B, and C. The overlaps show intersections among all three sets, pairwise intersections, and unique elements. For example, in a survey of students who like mathematics, physics, and chemistry, the diagram can show how many students like one, two, or all three subjects. *Value-added response*: Beyond classification, three-set Venn diagrams can be used in logic to test argument validity or in computer science to represent database queries involving multiple conditions.

Basic response: In probability, set theory helps to represent events. For example, if A is the event of passing mathematics and B is the event of passing physics, then $A \cap B$ represents students who passed both subjects. *Value-added response*: In advanced probability, set theory underpins concepts such as mutually exclusive events, conditional probability, and sample spaces. Venn diagrams can be extended to illustrate these relationships, making them useful in statistics and decision-making.

Answers to Pilot Questions [Series 1]

Part 1.1

b) A well-defined collection of distinct objects

b) Roster method

a) {} or Ø

c) {1, 2, 3, …}

c) Universal set

Part 1.2

a) $A \subset B$

b) ∪

b) {4, 6}

c) Elements in U but not in A

a) $\{1, 2\} \subset \{1, 2, 3\}$

Part 1.3

b) A set

c) Intersection

c) Three

b) Representing sets visually

a) 20

Part 1.4

a) Sorting fruits into categories

c) Students who passed both subjects

c) Venn diagrams

b) Finite set

b) To illustrate relationships between events

Series 2: Number Systems, Sequences, and Induction

2.1 Number Systems

2.1.1 Real Numbers

2.1.2 Integers, Rational, and Irrational Numbers

2.1.3 Properties of Number Systems

2.2 Sequences and Series

2.2.1 Definition and Notation of Sequences

2.2.2 Types of Sequences

2.2.3 Series and Summation Notation:

2.3 Mathematical Induction

2.3.1 Principle of Mathematical Induction

2.3.2 Steps in Proving Statements

2.3.3 Applications of Induction in Number Theory

Introduction

In our daily routines, we instinctively organize the world around us. Whether you are sorting laundry by color, organizing a digital music library by genre, or a library is categorizing books by subject, you are engaging in the fundamental act of grouping. In mathematics, these collections are known as sets. They serve as the essential building blocks for nearly every branch of mathematics, providing a rigorous language and structure for complex fields like algebra, probability, and logic.

The purpose of this series is to introduce you to the formal principles of set theory and the use of Venn diagrams. These diagrams are indispensable visual tools that allow us to see how different groups overlap, remain distinct, or nest within one another. By mastering these concepts, you will gain a powerful framework for organizing information and solving problems both in the classroom and in real-world scenarios.

We will begin by defining what constitutes a set and exploring the different ways to describe them. From there, we will investigate how sets interact through operations like union, intersection, and complementation. We will then translate these concepts into visual representations using two-set and three-set Venn diagrams. Finally, we will apply these tools to practical situations, demonstrating their utility in logic and decision-making.

Series Learning Outcomes

Upon completing this Series, you should be able to:

2.1 Define the concept of a set and distinguish between different types of sets (finite, infinite, empty, and universal).

2.2 Explain the idea of subsets and demonstrate how to perform basic set operations such as union, intersection, and complements.

2.3 Illustrate relationships between sets using Venn diagrams and interpret two-set and three-set representations.

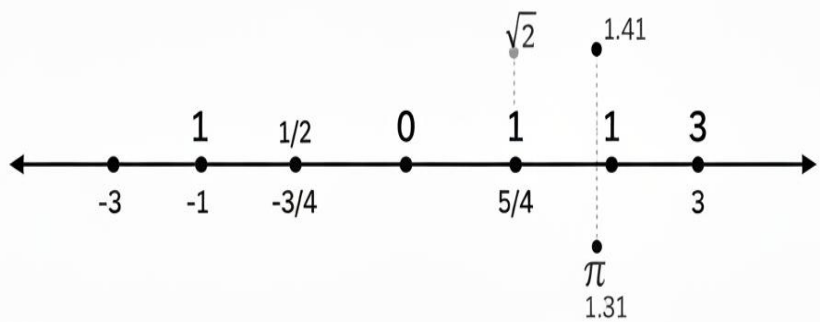
2.4 Apply set theory and Venn diagrams to solve practical problems in everyday contexts and in mathematics.

2.1 Number Systems

2.1.1 Real numbers

Real numbers are all numbers that can be represented on the number line. They include both rational numbers (fractions and integers) and irrational numbers (numbers that cannot be expressed as fractions, such as $\sqrt{2}$ or π). Real numbers are fundamental because they form the basis of most calculations in algebra and trigonometry.

Fill in the blank: Numbers such as $\sqrt{2}$ and π are examples of ______ numbers.



Representation of real numbers on a number line real numbers diagram"

2.1.2 Integers, rational and irrational numbers

Integers are whole numbers that include positive numbers, negative numbers, and zero. For example, {..., -3, -2, -1, 0, 1, 2, 3, ...}.

Rational numbers are numbers that can be expressed as a fraction p/q, where p and q are integers and q ≠ 0. Examples include 1/2, -3/4, and 5.

Irrational numbers are numbers that cannot be expressed as a fraction. Their decimal expansions are non-terminating and non-repeating. Examples include $\sqrt{2}$ and π.

Fill in the blank: A number with a non-terminating, non-repeating decimal expansion is called an ______ number.

Table 2.1 Classification of numbers

Type of number	Definition	Examples
Integers	Whole numbers (positive, negative, zero)	-3, 0, 5
Rational numbers	Numbers expressed as p/q, $q \neq 0$	1/2, -3/4, 5
Irrational numbers	Non-terminating, non-repeating decimals	$\sqrt{2}$, π

2.1.3 Properties of number systems

Number systems have important properties that govern operations:

Closure: The result of adding, subtracting, multiplying, or dividing (except division by zero) two real numbers is also a real number.

Commutativity: The order of addition or multiplication does not affect the result, e.g., $a + b = b + a$.

Associativity: Grouping of numbers does not affect the result, e.g., $(a + b) + c = a + (b + c)$.

Distributivity: Multiplication distributes over addition, e.g., $a(b + c) = ab + ac$.

Fill in the blank: The property that states $a + b = b + a$ is called ______.

[Insert box here] Caption: Box 2.1 Properties of number systems Content:

Closure

Commutativity

Associativity

Distributivity

Property	***Definition***	***Mathematical Example***
Closure	***Performing an operation on two real numbers results in another real number.***	***If $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$***
Commutativity	***Changing the order of the numbers does not change the result.***	***$a + b = b + a$*** ***$a \times b = b \times a$***
Associativity	***Changing the grouping of the numbers does not change the result.***	***$(a + b) + c = a + (b + c)$*** ***$(a \times b) \times c = a \times (b \times c)$***
Distributivity	***Multiplying a number by a sum is the same as multiplying each addend separately.***	***$a(b + c) = ab + ac$***

Table 2.2 text box summarising closure, commutativity, associativity, and distributivity with examples

Pilot questions 2.1

Which of the following numbers is irrational? a) 3/4 b) √2 c) -5 d) 0

Which property states that (a + b) + c = a + (b + c)? a) Closure b) Commutativity c) Associativity d) Distributivity

Which of the following is a rational number? a) π b) √2 c) 1/3 d) √3

What is the set of integers? a) {1, 2, 3, …} b) {..., -3, -2, -1, 0, 1, 2, 3, ...} c) {0, 1, 2, 3, …} d) {1, 3, 5, …}

Which property states that a(b + c) = ab + ac? a) Closure b) Commutativity c) Associativity d) Distributivity

2.2 Sequences and Series

2.2.1 Definition and notation of sequences

A **sequence** is an ordered list of numbers arranged according to a specific rule. Each number in the sequence is called a **term**, and the position of the term is important. Sequences are often denoted as $\{a_1, a_2, a_3, \ldots\}$, where a_n represents the nth term.

Fill in the blank: Each number in a sequence is called a ______.

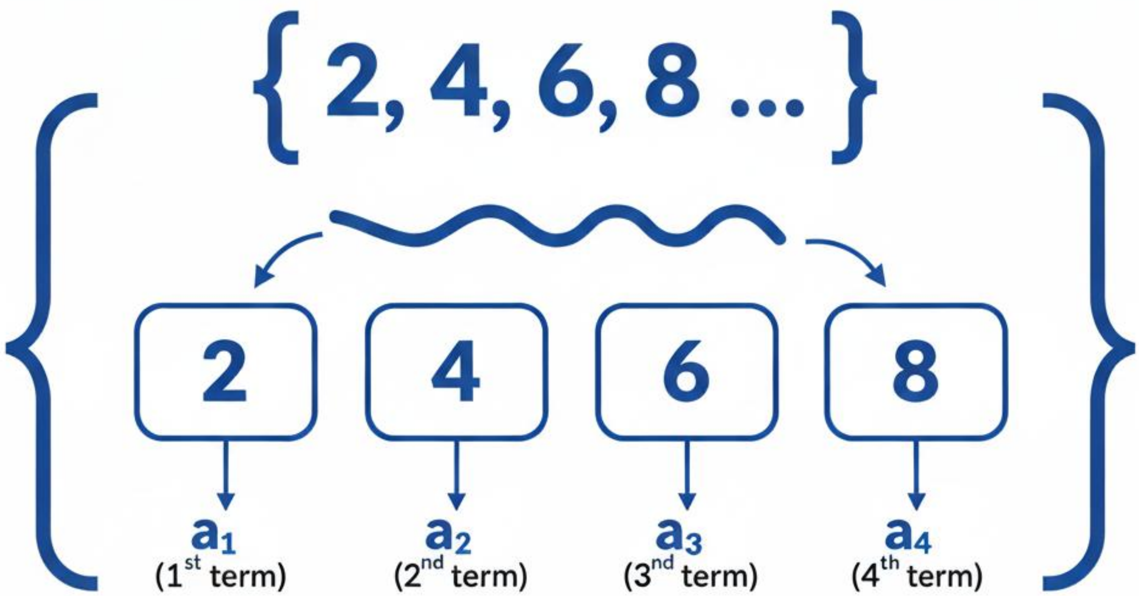


Figure 2.2 Representation of a arithmetic sequence terms"

2.2.2 Types of sequences (arithmetic, geometric)

An **arithmetic sequence** is a sequence in which each term after the first is obtained by adding a fixed number, called the **common difference**. For example, {3, 6, 9, 12, ...} has a common difference of 3.

A **geometric sequence** is a sequence in which each term after the first is obtained by multiplying by a fixed number, called the **common ratio**. For example, {2, 4, 8, 16, …} has a common ratio of 2.

Fill in the blank: In an arithmetic sequence, the fixed number added to each term is called the ______.

Type of sequence	**Rule**	**Example**	**Key feature**
Arithmetic	Add a fixed number	{3, 6, 9, 12, …}	Common difference
Geometric	Multiply by a fixed number	{2, 4, 8, 16, …}	Common ratio

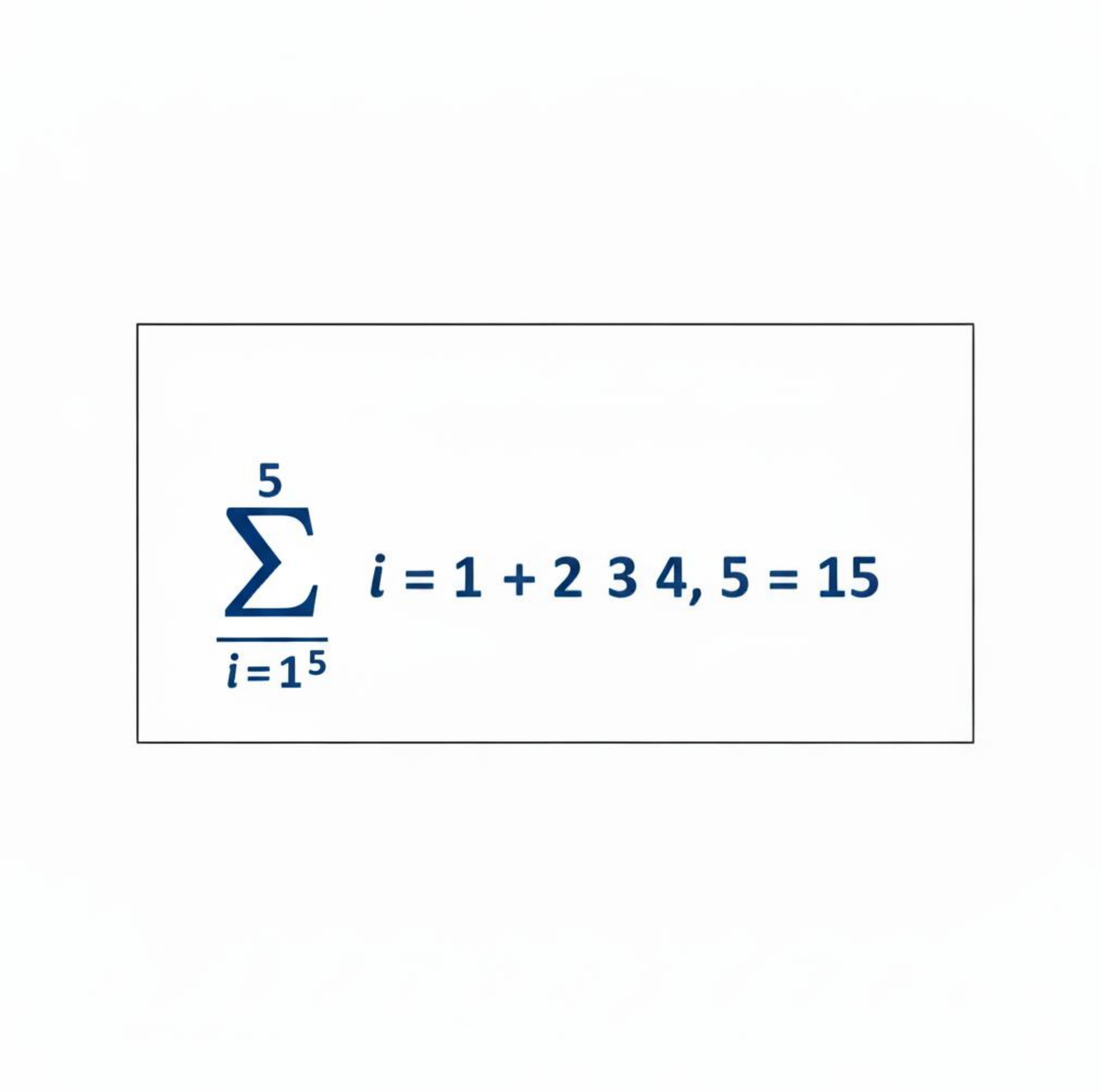
Table 2.3 comparing arithmetic and geometric sequences

2.2.3 Series and summation notation

A **series** is the sum of the terms of a sequence. For example, the series corresponding to the sequence {2, 4, 6} is 2 + 4 + 6 = 12.

The **summation notation** uses the Greek letter sigma (Σ) to represent the sum of terms. For example, Σ from i = 1 to 4 of i = 1 + 2 + 3 + 4 = 10.

Fill in the blank: The Greek letter ______ is used to represent summation notation.



Example of summation notation

Pilot questions 2.2

What is each number in a sequence called? a) Ratio b) Term c) Difference d) Series

Which of the following is an arithmetic sequence? a) {2, 4, 8, 16} b) {3, 6, 9, 12} c) {1, 2, 4, 8} d) {5, 10, 20, 40}

In a geometric sequence, the fixed number multiplied to each term is called: a) Common difference b) Common ratio c) Term d) Series

What does Σ represent in mathematics? a) Product b) Difference c) Summation d) Ratio

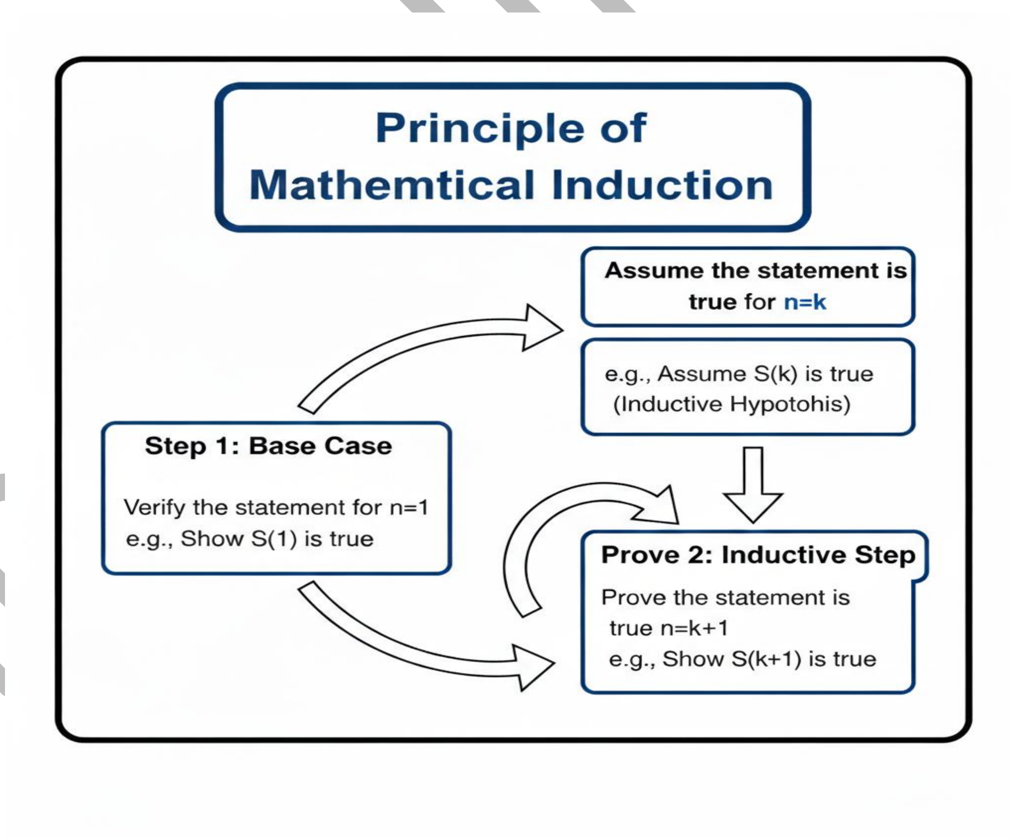
If the sequence is {2, 4, 6}, what is the corresponding series? a) 2 × 4 × 6 b) 2 + 4 + 6 c) 6 + 4 + 2 d) 2 − 4 − 6

2.3 Mathematical Induction

2.3.1 Principle of mathematical induction

The **principle of mathematical induction** is a method of proving statements that are true for all natural numbers. It works by showing that if a statement holds for the first case (usually n = 1), and if assuming it holds for n = k implies it also holds for n = k + 1, then the statement is true for all natural numbers.

Fill in the blank: Mathematical induction proves statements by showing they hold for the base case and the ______ step.



A flow diagram showing principle of mathematical induction

2.3.2 Steps in proving statements

To prove a statement using induction, follow these steps:

Base case: Verify that the statement is true for the first natural number (usually n = 1).

Inductive hypothesis: Assume the statement is true for n = k.

Inductive step: Show that if the statement is true for n = k, then it must also be true for n = k + 1.

Fill in the blank: The assumption that the statement holds for n = k is called the _______ hypothesis.

[Insert box here] Caption: Box 2.3 Steps in mathematical induction Content:

Step 1: Base case

Step 2: Inductive hypothesis

Step 3: Inductive step

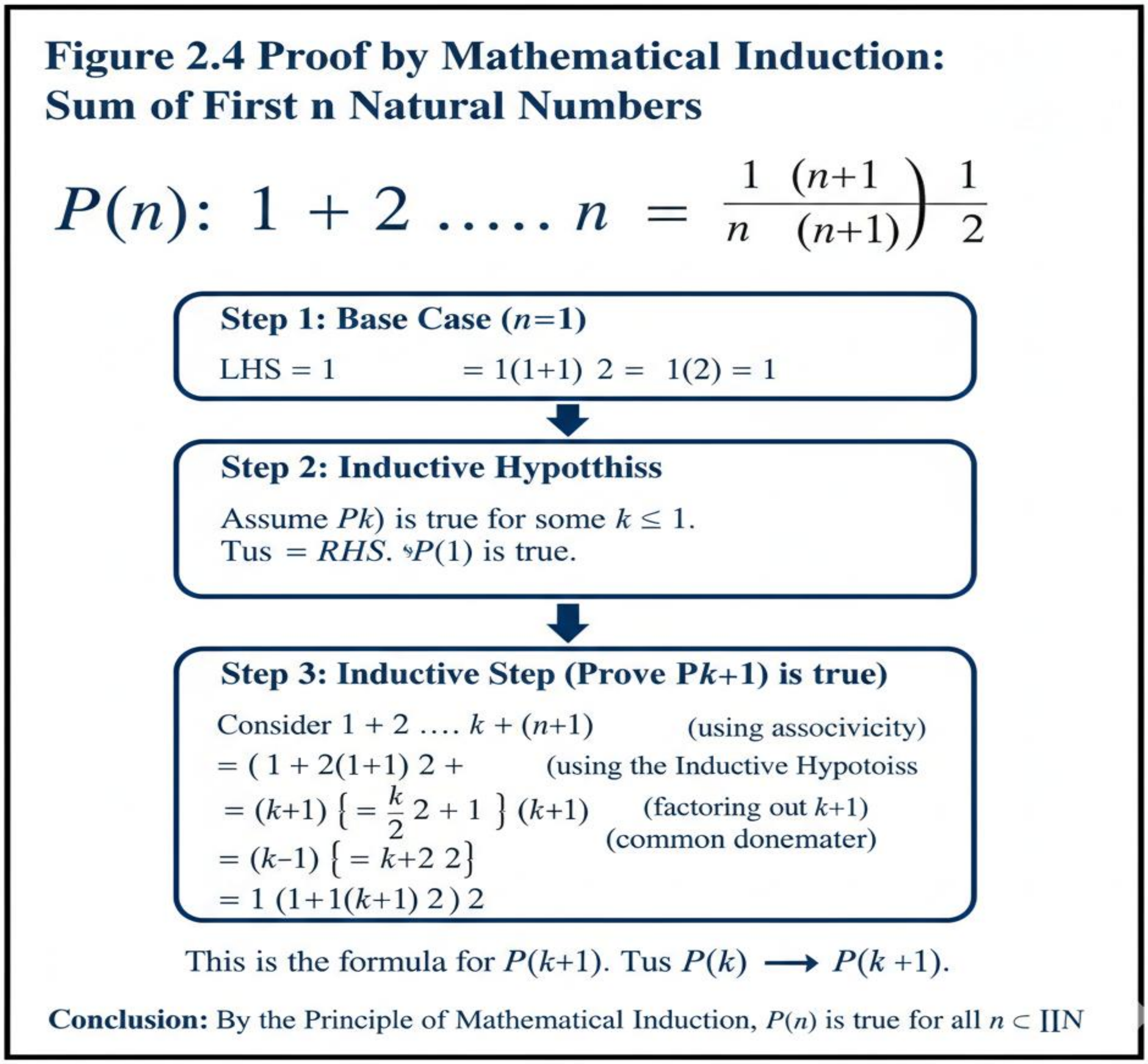
Step	Name	Action
1	**Base Case**	Show that the statement holds for the first value, usually n = 1.
2	**Inductive Hypothesis**	Assume the statement is true for an arbitrary integer k (i.e., assume P(k) is true).
3	**Inductive Step**	Prove that if the statement is true for n = k, then it must also be true for n = k + 1.

Text box summarising the three steps of mathematical induction

2.3.3 Applications of induction in number theory

Mathematical induction is widely used in number theory to prove formulas and properties. For example, it can be used to prove that the sum of the first n natural numbers is n(n + 1)/2. It can also prove divisibility properties, such as showing that $2^n - 1$ is divisible by 3 for certain values of n.

Fill in the blank: The formula for the sum of the first n natural numbers is ______.



A diagram showing theinduction proof sum of natural numbers"

Pilot questions 2.3

What is the first step in a proof by induction? a) Inductive hypothesis b) Inductive step c) Base case d) Conclusion

What does the inductive hypothesis assume? a) The statement is true for $n = 1$ b) The statement is true for $n = k$ c) The statement is true for $n = k + 1$ d) The statement is false for $n = k$

Which of the following formulas can be proved using induction? a) $n(n + 1)/2$ b) $a^2 + b^2 = c^2$ c) $E = mc^2$ d) $\sin^2\theta + \cos^2\theta = 1$

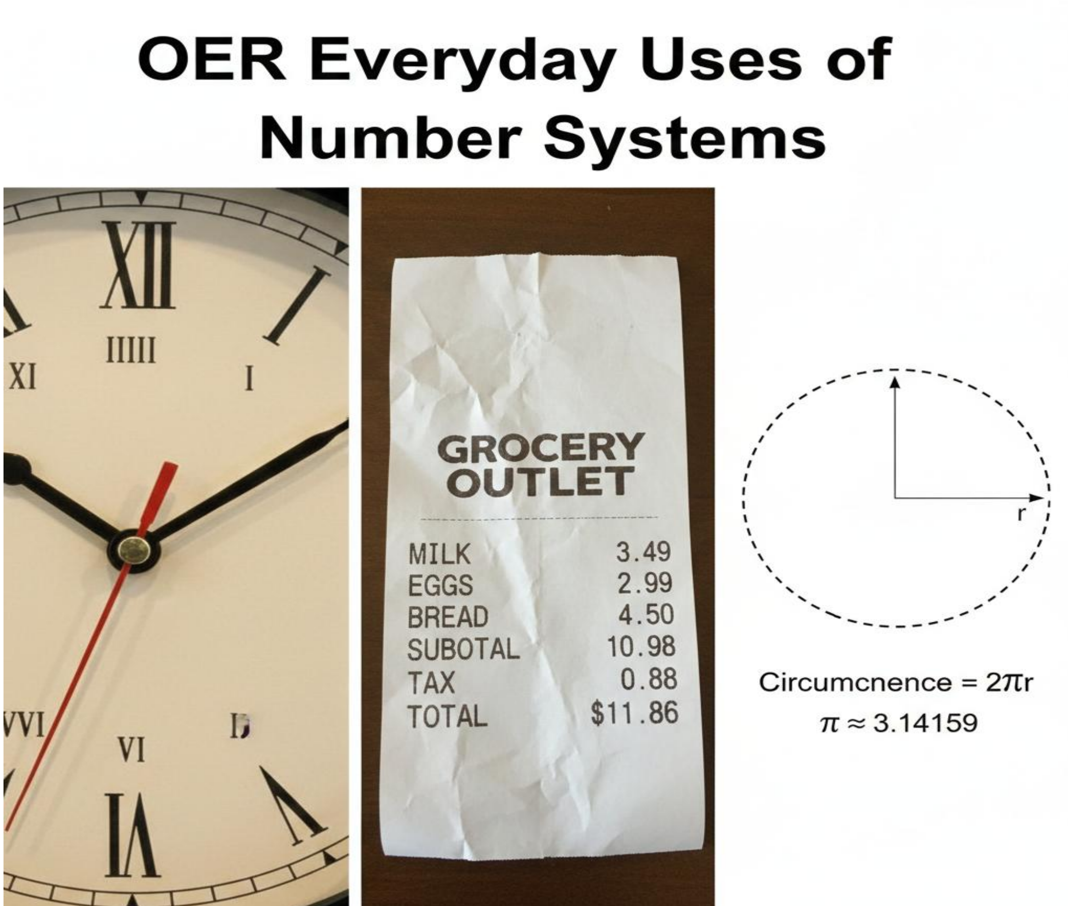
What is the purpose of the inductive step? a) To verify the base case b) To assume the statement is true for $n = k$ c) To show the statement holds for $n = k + 1$ d) To conclude the proof

Which branch of mathematics frequently uses induction to prove properties? a) Geometry b) Number theory c) Calculus d**2.4 Applications of Number Systems and Sequences**

2.4.1 Everyday uses of number systems

Number systems are used constantly in daily life. For example, when you check the time, measure distances, or calculate expenses, you are applying the properties of integers, rational numbers, and real numbers. Rational numbers are especially important in financial transactions, where fractions and decimals represent prices, interest rates, and discounts. Irrational numbers such as π are used in geometry and engineering to calculate areas and circumferences.

Fill in the blank: The number π is commonly used in calculating the ______ of circles.

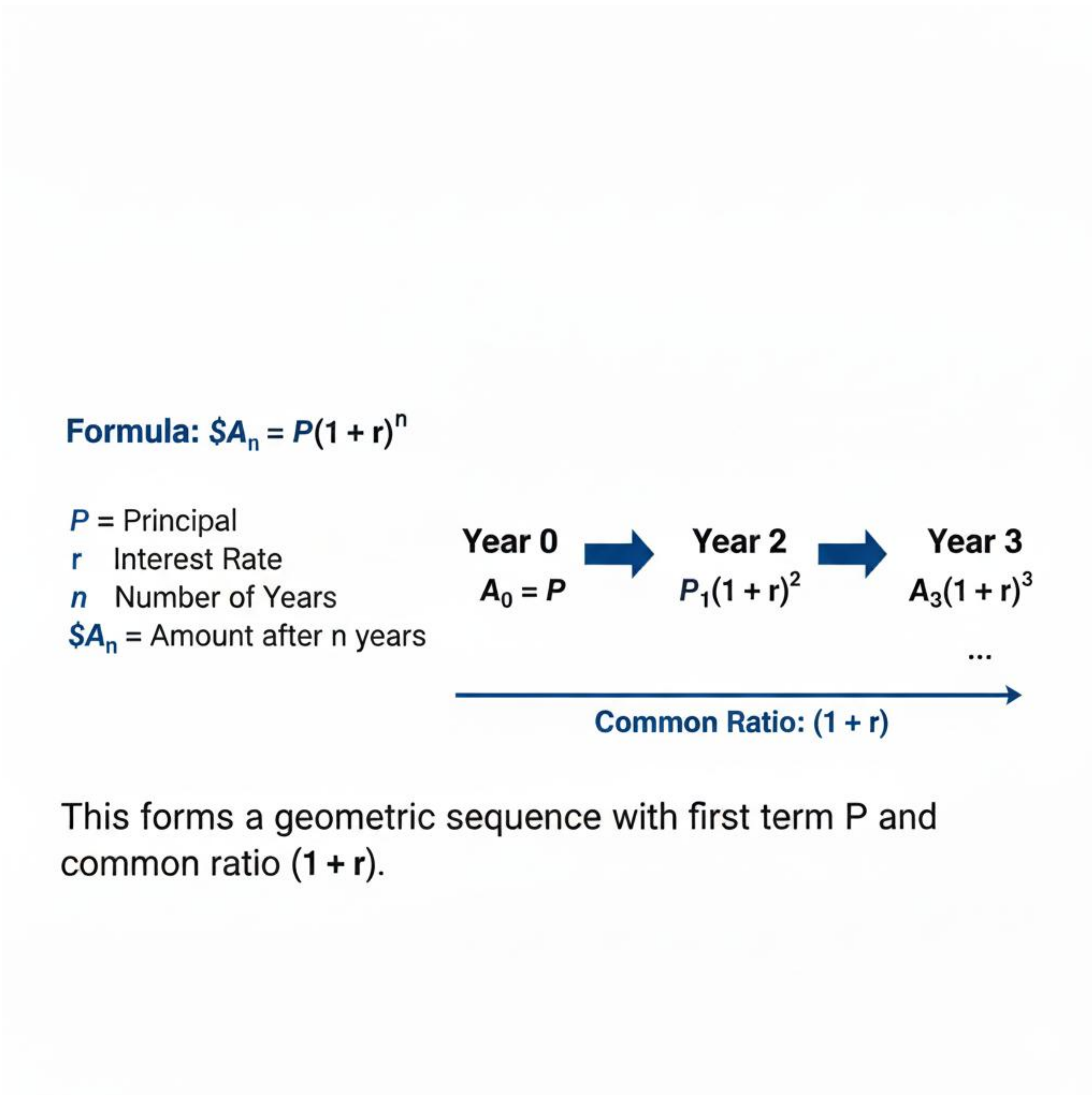


A diagram showing everyday uses of number systems

2.4.2 Problem-solving with sequences and series

Sequences and series are powerful tools for solving problems in mathematics and beyond. For example, arithmetic sequences can be used to calculate the total cost of items when prices increase by a fixed amount, while geometric sequences are applied in calculating compound interest. Series are also used in physics to approximate values and in computer science for algorithms.

Fill in the blank: Compound interest problems are solved using ______ sequences.



Application of geometric sequences in compound interest

Pilot questions 2.4

Which of the following is an everyday use of number systems? a) Calculating expenses b) Drawing graphs of functions c) Solving trigonometric identities d) Differentiating functions

Which number is commonly used in geometry to calculate areas and circumferences? a) √2 b) π c) 1/2 d) 0

Arithmetic sequences can be applied to: a) Compound interest b) Increasing costs by a fixed amount c) Calculating derivatives d) Solving trigonometric equations

Geometric sequences are useful in solving: a) Linear equations b) Compound interest problems c) Quadratic equations d) Probability distributions

Which field uses series to approximate values? a) Physics b) Literature c) History d) Geography

Series Summary

This Series has guided you through the essential concepts of number systems, sequences, and mathematical induction. Its purpose was to strengthen your ability to classify numbers, recognise patterns, and prove statements that apply to infinitely many cases. These skills are central to algebra and trigonometry, and they also provide a foundation for advanced mathematical reasoning.

We began by exploring the number systems, including real numbers, integers, rational and irrational numbers, and their properties. Then we moved on to sequences and series, where you learnt how to define, classify, and compute arithmetic and geometric progressions, as well as use summation notation. Following that, we examined the principle of mathematical induction, focusing on the base case, inductive hypothesis, and inductive step, and saw how it is applied in number theory. Finally, we considered practical applications of number systems and sequences in everyday life, finance, and science. By completing this Series, you should now be able to classify numbers (SLO 3.1), describe and compute terms of sequences (SLO 3.2), evaluate series (SLO 3.3), demonstrate proofs using induction (SLO 3.4), and apply these concepts to solve practical problems (SLO 3.5).

Glossary of Terms

Arithmetic sequence: A sequence in which each term is obtained by adding a fixed number, called the common difference.

Associativity: A property of numbers where grouping does not affect the result, e.g., $(a + b) + c = a + (b + c)$.

Closure: A property stating that operations on numbers within a set produce results that are also within the set.

Common difference: The fixed number added to each term in an arithmetic sequence.

Common ratio: The fixed number multiplied to each term in a geometric sequence.

Distributivity: A property where multiplication distributes over addition, e.g., $a(b + c) = ab + ac$.

Geometric sequence: A sequence in which each term is obtained by multiplying by a fixed number, called the common ratio.

Inductive hypothesis: The assumption that a statement holds true for $n = k$ in a proof by induction.

Inductive step: The process of showing that if a statement holds for $n = k$, then it also holds for $n = k + 1$.

Integers: Whole numbers including positive numbers, negative numbers, and zero.

Irrational numbers: Numbers that cannot be expressed as fractions, with non-terminating and non-repeating decimal expansions.

Mathematical induction: A method of proving statements for all natural numbers using a base case and inductive step.

Rational numbers: Numbers that can be expressed as fractions p/q, where p and q are integers and $q \neq 0$.

Real numbers: All numbers that can be represented on the number line, including rational and irrational numbers.

Sequence: An ordered list of numbers arranged according to a specific rule.

Series: The sum of the terms of a sequence.

Summation notation: A mathematical symbol (Σ) used to represent the sum of terms.

Concept Check Questions [Series 2]

Objective questions

Which of the following numbers is irrational? (Linked to SLO 3.1) a) 3/4 b) $\sqrt{2}$ c) -5 d) 0

What is the fixed number added to each term in an arithmetic sequence called? (Linked to SLO 3.2) a) Common ratio b) Common difference c) Term d) Series

Which symbol is used in summation notation? (Linked to SLO 3.3) a) ∩ b) ∪ c) Σ d) π

Short-answer questions

Define the inductive step in mathematical induction. (Linked to SLO 3.4)

Give one everyday example of how number systems are applied. (Linked to SLO 3.5)

Long-answer questions

Demonstrate how to prove by induction that the sum of the first n natural numbers is n(n + 1)/2. (Linked to SLO 3.4)

Analyse how geometric sequences can be applied in calculating compound interest. (Linked to SLO 3.5)

Answers to Concept Check Questions [Series 2]

Objective questions

b) √2

b) Common difference

c) Σ

Short-answer questions

The inductive step shows that if a statement holds for n = k, then it also holds for n = k + 1.

Checking the time on a clock or calculating expenses in a shop are everyday applications of number systems.

Long-answer questions

Basic response: To prove the formula for the sum of the first n natural numbers, begin with the base case n = 1, where the sum is 1, which equals 1(1 + 1)/2. Assume the formula holds for n = k, then show it holds for n = k + 1 by adding (k + 1) to the assumed sum. *Value-added response*: Beyond the basic proof, this formula is widely used in computer science for analysing algorithms and in physics for calculating series of measurements.

Basic response: A geometric sequence models compound interest because each term is obtained by multiplying by (1 + r), where r is the interest rate. *Value-added response*: This principle is applied in finance to predict long-term growth of investments, and in economics to model population growth or resource consumption.

Answers to Pilot Questions [Series 2]

Part 2.1

b) √2

c) Associativity

c) 1/3

b) {..., -3, -2, -1, 0, 1, 2, 3, ...}

d) Distributivity

Part 2.2

b) Term

b) {3, 6, 9, 12}

b) Common ratio

c) Summation

b) 2 + 4 + 6

Part 2.3

c) Base case

b) The statement is true for n = k

a) n(n + 1)/2

c) To show the statement holds for n = k + 1

b) Number theory

Part 2.4

a) Calculating expenses

b) π

b) Increasing costs by a fixed amount

b) Compound interest problems

a) Physics

Series 3 Quadratic Equations and the Binomial Theorem

Series 3 Outline

3.1 Quadratic equations

3.1.1 Definition and standard form

3.1.2 Methods of solving quadratic equations (factorisation, completing the square, quadratic formula)

3.1.3 Nature of roots and discriminant

3.2 Applications of quadratic equations

3.2.1 Word problems involving quadratic equations

3.2.2 Graphical representation of quadratic functions

3.3 The binomial theorem

3.3.1 Statement of the binomial theorem

3.3.2 Expansion using the theorem

3.3.3 General term and middle term

3.4 Applications of the binomial theorem

3.4.1 Approximation using the binomial theorem

3.4.2 Problem-solving in algebra and probability

Introduction

Quadratic equations and the binomial theorem are two pillars of algebra that connect abstract reasoning with practical problem-solving. A quadratic equation is a polynomial equation of degree two, and it appears in diverse contexts such as physics, economics, and engineering. The binomial theorem, on the other hand, provides a systematic way of expanding powers of binomial expressions, which is essential in algebra, probability, and approximation.

The aim of this Series is to equip you with the skills to recognise, solve, and apply quadratic equations, and to expand and approximate expressions using the binomial theorem.

We begin by defining quadratic equations and exploring methods of solving them, including factorisation, completing the square, and the quadratic formula. We then examine the nature of roots and discriminants, followed by practical applications such as word problems and graphical representations. Next, we introduce the binomial theorem, focusing on its statement, expansion techniques, and identification of general and middle terms. Finally, we conclude with applications of the binomial theorem in approximation and probability, showing how these algebraic tools are used in real-world contexts.

Upon completing this Series, you should be able to:

- **SLO 4.1** define quadratic equations and express them in standard form,
- **SLO 4.2** demonstrate methods of solving quadratic equations, including factorisation, completing the square, and the quadratic formula,
- **SLO 4.3** analyse the nature of roots of quadratic equations using the discriminant,

SLO 4.4 apply quadratic equations to solve word problems and represent quadratic functions graphically,

SLO 4.5 state the binomial theorem and use it to expand binomial expressions,

SLO 4.6 identify and compute the general term and middle term in a binomial expansion, and

SLO 4.7 apply the binomial theorem to approximation and probability problems in practical contexts.

3.1 Quadratic Equations

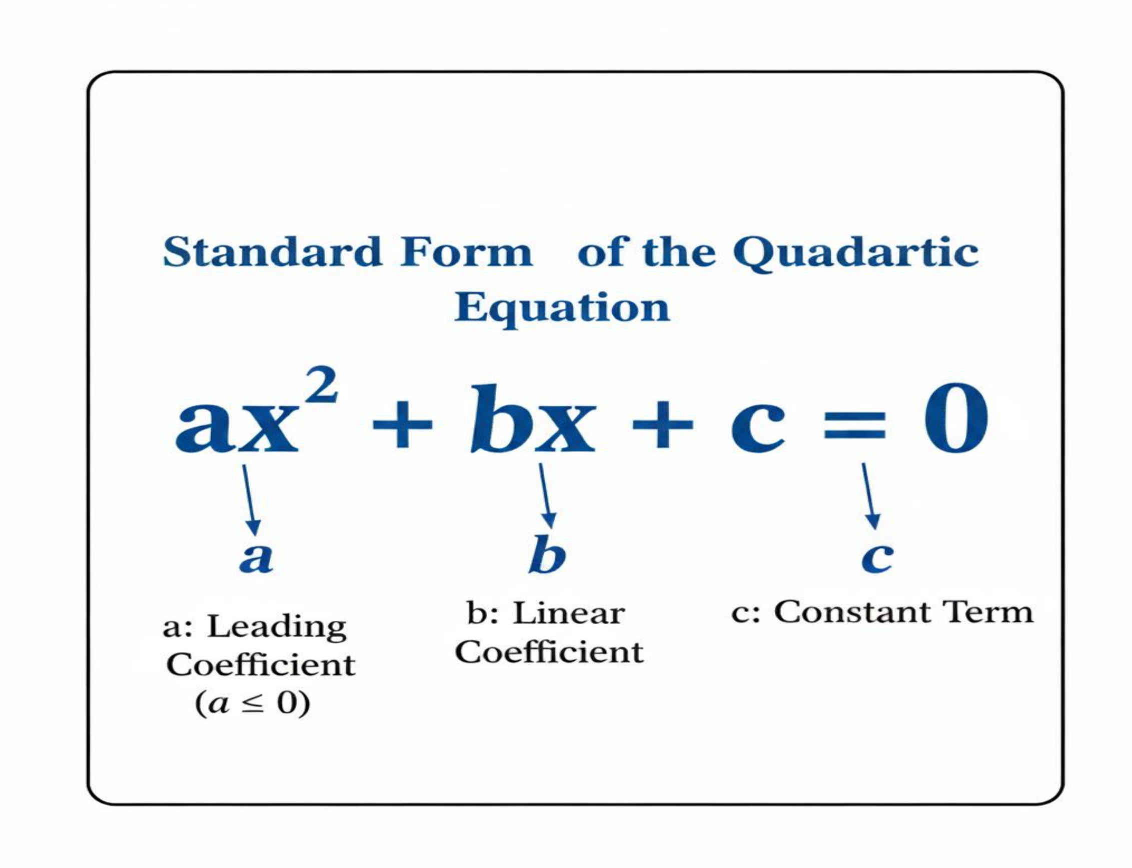
3.1.1 Definition and standard form

A quadratic equation is a polynomial equation of degree two, usually written in the standard form:

$$ax2+bx+c=0$$

where a,b,care real numbers and a0. The solutions to a quadratic equation are called **roots**.

Fill in the blank: A quadratic equation is a polynomial equation of degree ______



Standard form of a quadratic equation

3.1.2 Methods of solving quadratic equations

Quadratic equations can be solved using three main methods:

Factorisation: Expressing the quadratic as a product of two linear factors.

Completing the square: Rearranging the equation into a perfect square form.

Quadratic formula: Using the formula

$$x=-b\pm b2-4ac2a$$

Fill in the blank: The quadratic formula is used to find the ______ of a quadratic equation.

[Insert box here] Caption: Box 3.1 Methods of solving quadratic equations Content:

Factorisation

Completing the square

Quadratic formula

Method	***When to Use***	***Example***
Factoring	***When the quadratic expression can be easily broken into two binomials.***	$x^2 + 5x + 6 = 0 \rightarrow (x+2)(x+3) = 0.$ ***Roots:*** $x = -2, -3$
Square Root Property	***When the equation is in the form*** $x^2 = d$ ***or*** $(x-h)^2 = k$.	$x^2 = 25 \rightarrow x = \pm 5$
Completing the Square	***Useful for deriving the quadratic formula or converting to vertex form.***	$x^2 + 6x = 7 \rightarrow x^2 + 6x + 9 = 16 \rightarrow (x+3)^2 = 16$
Quadratic Formula	***A universal method that works for any quadratic equation, including those with complex roots.***	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Text box summarising methods of solving quadratic equations with examples." Search string

3.1.3 Nature of roots and discriminant

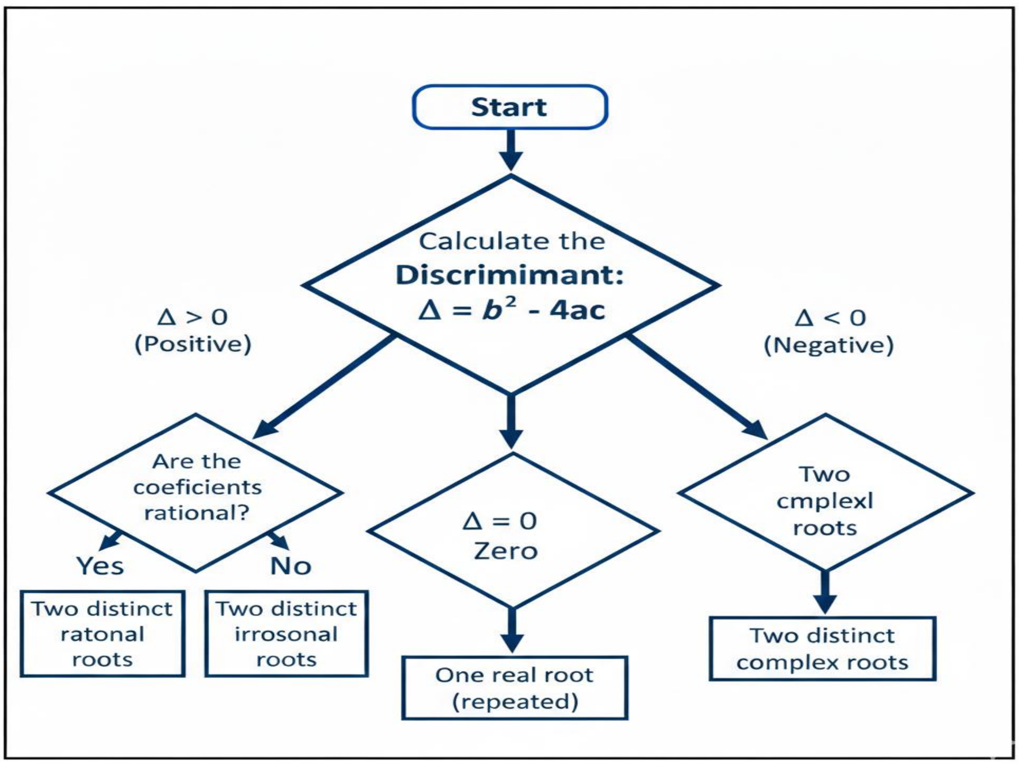
The discriminant of a quadratic equation is given by D=b2-4ac. It determines the nature of the roots:

If D>0, the equation has two distinct real roots.

If D=0, the equation has two equal real roots.

If D<0, the equation has two complex roots.

Fill in the blank: If the discriminant is less than zero, the quadratic equation has ______ roots.



A flowchart showing discriminant values and corresponding nature of roots.

Pilot questions 3.1

What is the standard form of a quadratic equation? a) $ax + b = 0$ b) $ax^2 + bx + c = 0$ c) $ax^3 + bx^2 + c = 0$ d) $ax^4 + bx^3 + c = 0$

Which of the following is NOT a method of solving quadratic equations? a) Factorisation b) Completing the square c) Quadratic formula d) Differentiation

What is the discriminant of a quadratic equation? a) $b^2 - 4ac$ b) $a^2 - 4bc$ c) $c^2 - 4ab$ d) $ab - 4c$

If the discriminant is zero, what is the nature of the roots? a) Two distinct real roots b) Two equal real roots c) Two complex roots d) No roots

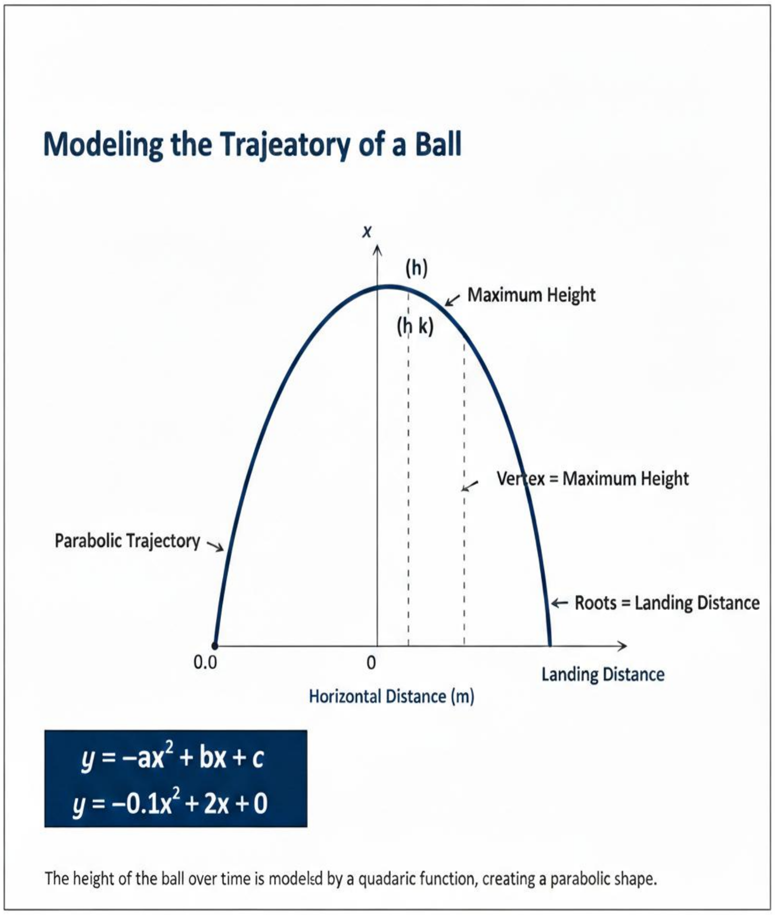
Which formula is used to solve quadratic equations directly? a) Binomial theorem b) Quadratic formula c) Pythagoras theorem d) Factor theorem

3.2 Applications of Quadratic Equations

3.2.1 Word problems involving quadratic equations

Quadratic equations often arise in real-life problem situations. For example, if you throw a ball upwards, its height at any time tcan be modelled by a quadratic equation. Similarly, profit and cost functions in business can lead to quadratic equations when maximising or minimising values.

Fill in the blank: The height of a ball thrown upwards can be modelled by a ______ equation.



A diagram showing the trajectory of a ball with its height modelled by a quadratic equation. Caption

3.2.2 Graphical representation of quadratic functions

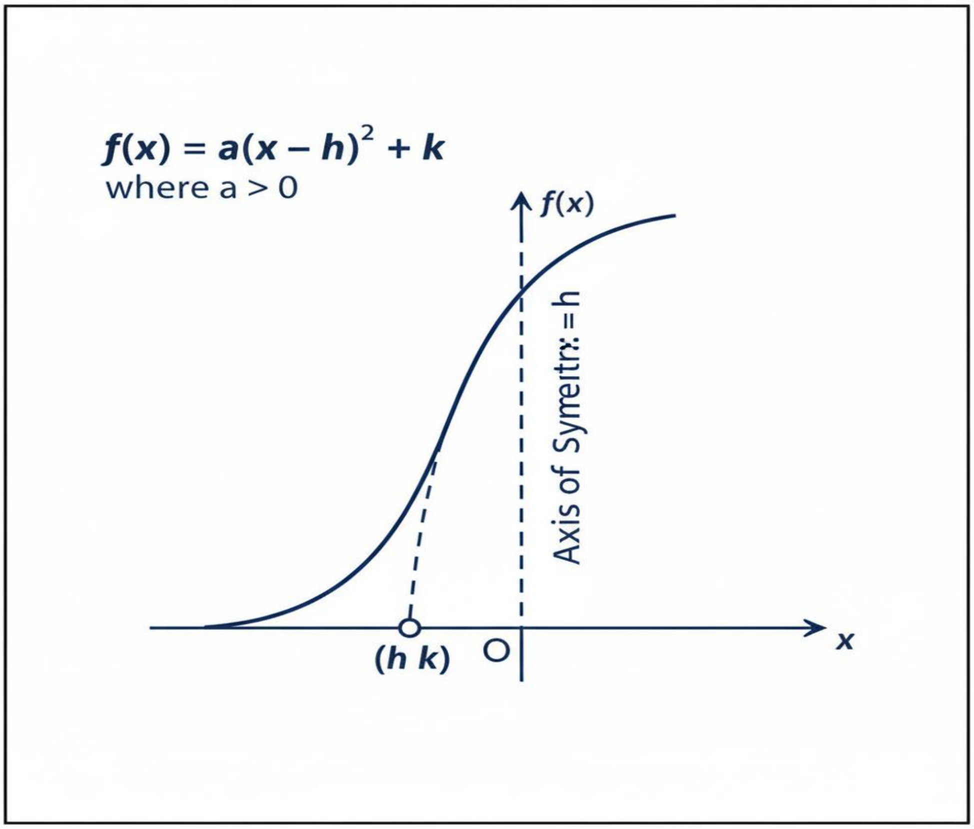
Quadratic functions are represented graphically as parabolas. The general shape of the parabola depends on the coefficient a:

If $a>0$, the parabola opens upwards.

If $a<0$, the parabola opens downwards.

The vertex of the parabola represents the maximum or minimum point, and the axis of symmetry passes through the vertex.

Fill in the blank: The graph of a quadratic function is called a ______.



A graph showing a parabola opening upwards with labelled vertex and axis of symmetry.

Pilot questions 3.2

Which of the following real-life situations can be modelled by a quadratic equation? a) The trajectory of a ball b) The growth of bacteria c) The calculation of compound interest d) The rotation of the earth

What is the shape of the graph of a quadratic function? a) Circle b) Line c) Parabola d) Ellipse

If the coefficient ain a quadratic function is positive, the parabola opens: a) Downwards b) Upwards c) Sideways d) Flat

The vertex of a parabola represents: a) The axis of symmetry b) The maximum or minimum point c) The discriminant d) The slope of the tangent

Which part of the parabola passes through the vertex? a) Axis of symmetry b) Discriminant c) Quadratic formula d) Factorisation line

3.3 The Binomial Theorem

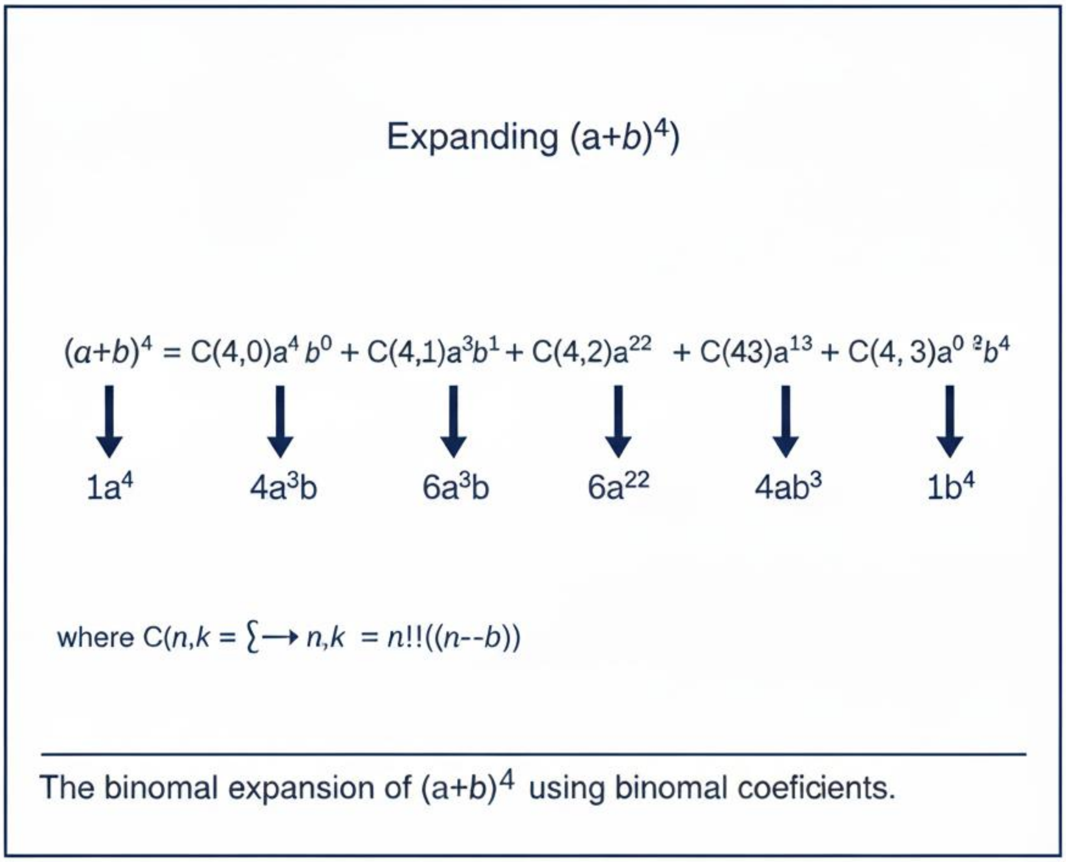
3.3.1 Statement of the binomial theorem

The **binomial theorem** provides a formula for expanding expressions of the form (a+b)n, where nis a positive integer. It states:

$$(a+b)n=k=0n(nk)an-kbk$$

Here, (nk)represents the **binomial coefficient**, calculated as n!k!(n-k)!.

Fill in the blank: The binomial theorem expands expressions of the form (a+b)



The binomal expansion of (a+b)4 using binomal coeficients.

A diagram showing the expansion of (a+b)4using binomial coefficients

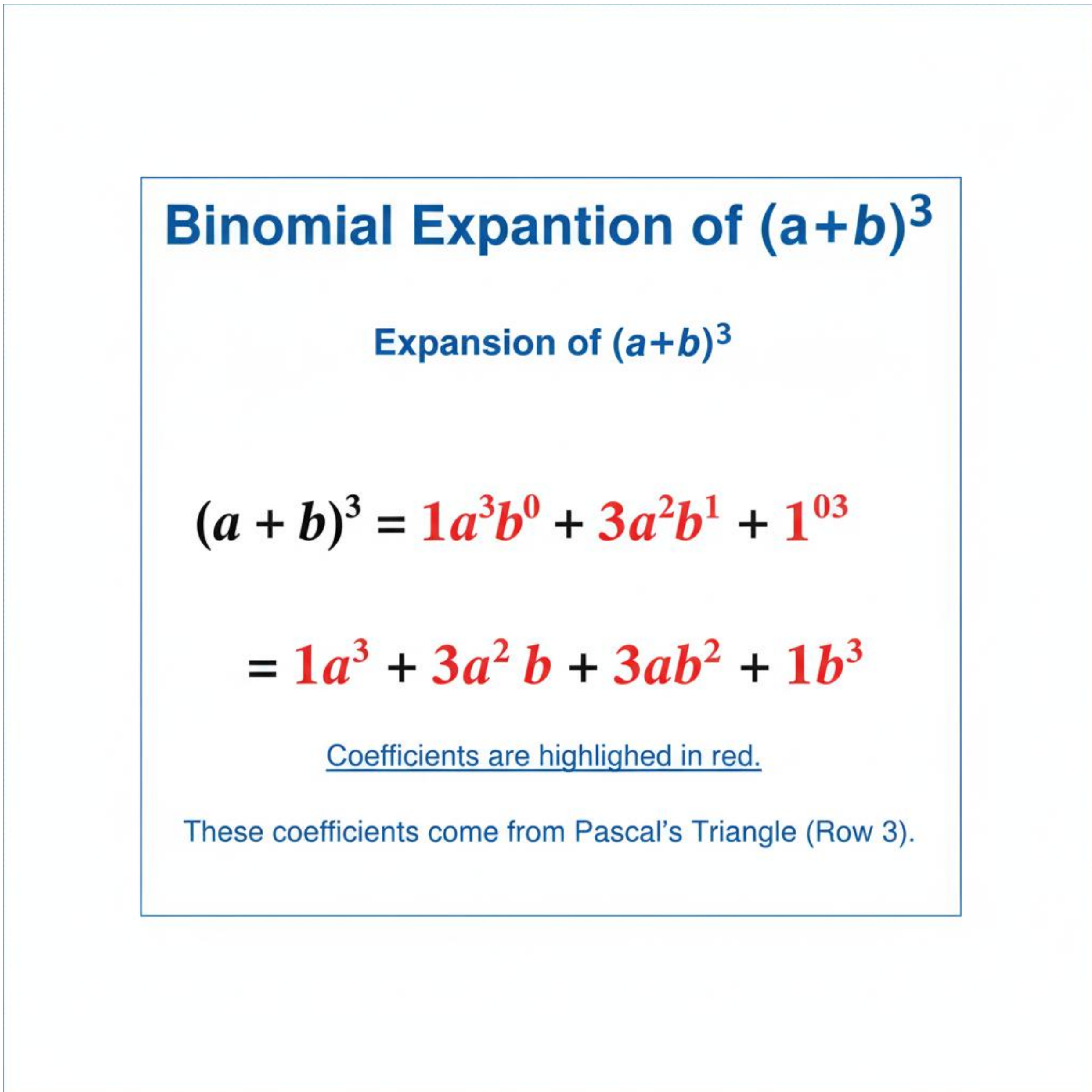
3.3.2 Expansion using the theorem

Using the binomial theorem, we can expand expressions systematically. For example:

$$(a+b)3=a3+3a2b+3ab2+b3$$

This avoids manual multiplication and provides a quick way to expand higher powers.

Fill in the blank: The coefficient of a^2b in $(a+b)^3$ is ______.



Example of binomial expansion Content:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

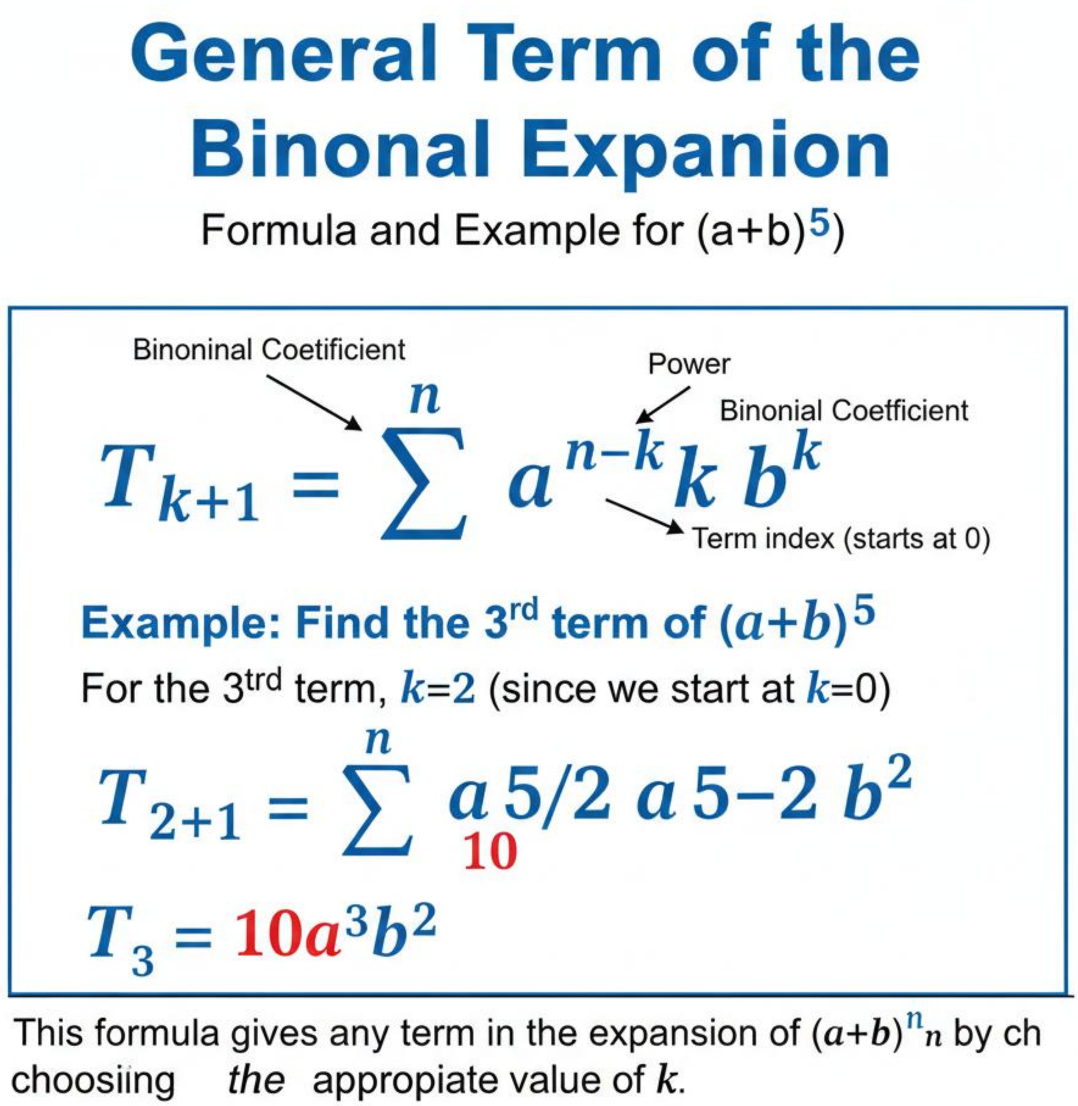
3.3.3 General term and middle term

The **general term** in the expansion of $(a+b)^n$ is given by:

$$T_{k+1} = \binom{n}{k} a^{n-k} b^k$$

The **middle term** occurs when k=n2if nis even, or when there are two middle terms if nis odd.

Fill in the blank: The general term in the expansion of (a+b)nis denoted by T_+1.



Pilot questions 3.3

What does the binomial theorem expand? a) (a − b)^n b) (a + b)^n c) (a × b)^n d) (a ÷ b)^n

What is the binomial coefficient (nk)equal to? a) n!k!(n-k)!b) k!n!(n-k)!c) n!(n+k)!d) n!k!

Expand (a+b)2. a) a2+b2b) a2+2ab+b2c) a2+ab+b2d) a2+3ab+b2

What is the coefficient of a2bin (a+b)3? a) 1 b) 2 c) 3 d) 4

If n=4, how many terms are in the expansion of (a+b)4? a) 3 b) 4 c) 5 d) 6

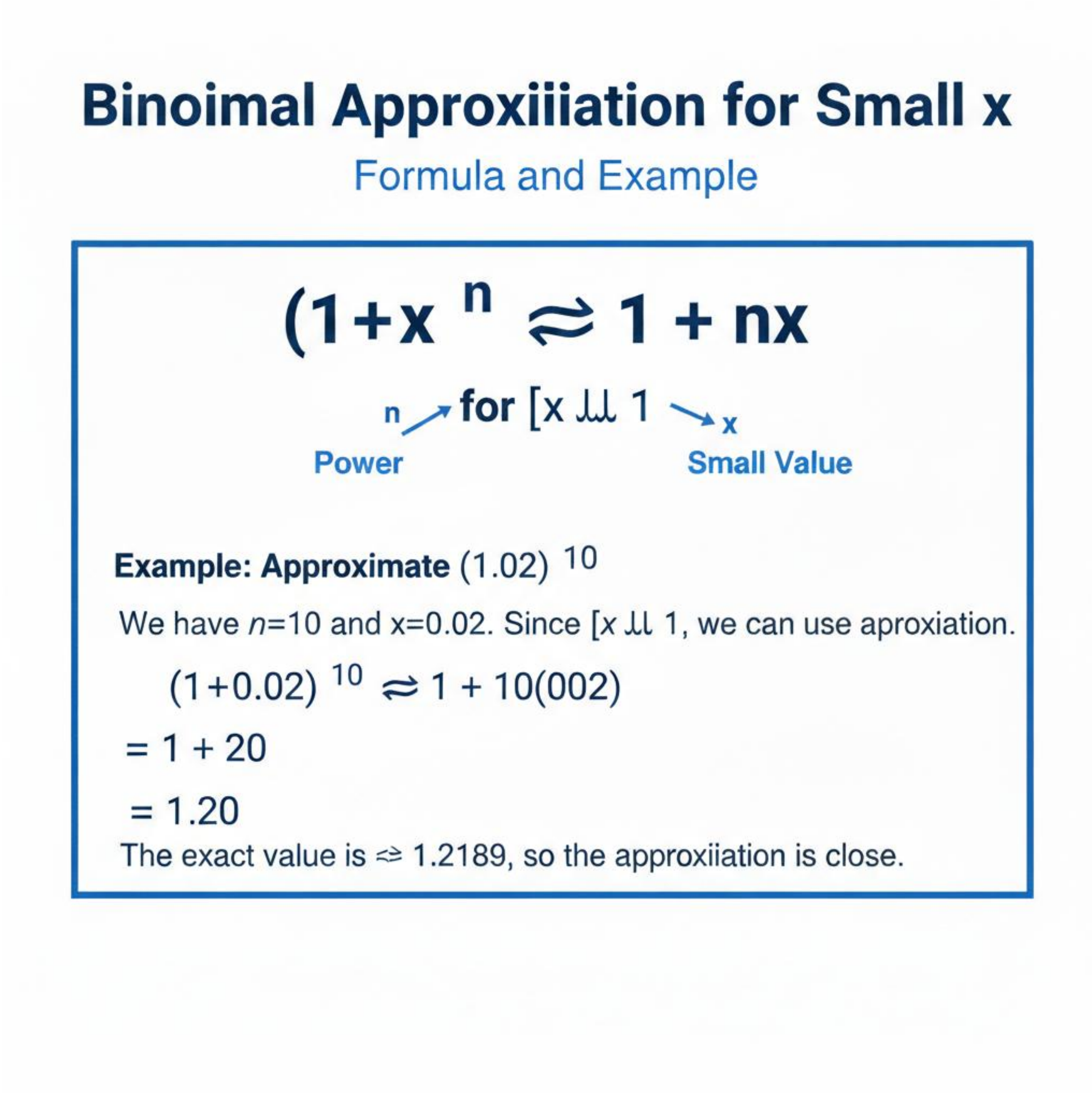
3.4 Applications of the Binomial Theorem

3.4.1 Approximation using the binomial theorem

The binomial theorem can be used to approximate values when nis large or when dealing with expressions like (1+x)nwhere xis small. For example, (1+x)n≈1+nxwhen xis close to zero. This is useful in physics, engineering, and economics for simplifying calculations.

Fill in the blank: When xis small, (1+x)ncan be approximated by

______



A diagram showing approximation of (1+x)nfor small values of x.
Caption:

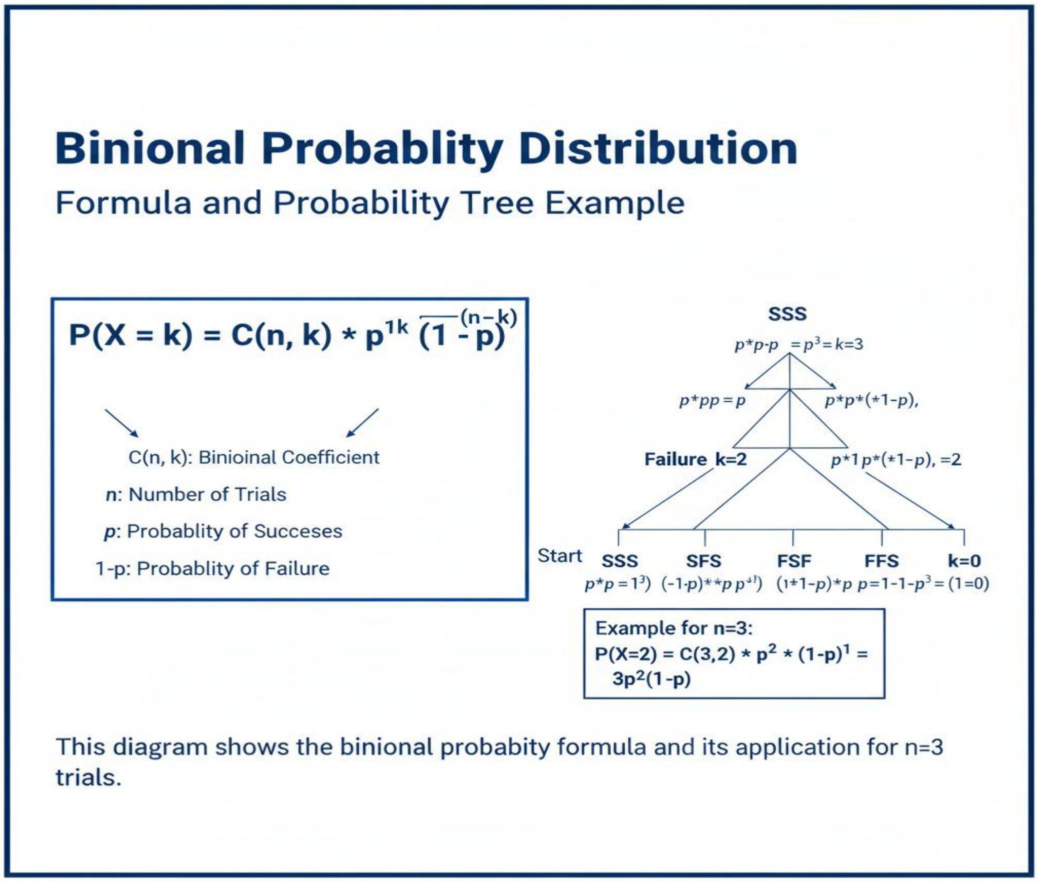
3.4.2 Problem-solving in algebra and probability

The binomial theorem is also applied in probability, especially in calculating probabilities in binomial distributions. For example, the probability of obtaining exactly ksuccesses in nindependent trials is given by:

$$P(X=k)=(nk)pk(1-p)n-k$$

This formula is directly connected to the binomial expansion.

Fill in the blank: The probability of obtaining exactly ksuccesses in ntrials is given by the ______ distribution.



A diagram showing a probability tree and binomial distribution formula. Caption

Pilot questions 3.4

Which of the following is an application of the binomial theorem? a) Calculating derivatives b) Approximating values of expressions c) Solving trigonometric identities d) Drawing graphs of functions

When xis small, (1+x)ncan be approximated by: a) 1+nxb) 1+xnc) n+xd) nx2

The binomial theorem is applied in probability to calculate: a) Compound interest b) Binomial distribution c) Linear regression d) Circle area

What does (nk)represent in probability? a) Number of trials b) Number of successes c) Binomial coefficient d) Probability of failure

Which field frequently uses binomial approximations for simplifying calculations? a) Physics b) Literature c) History d) Geography

Series Summary

This Series has introduced you to quadratic equations and the binomial theorem, two central topics in algebra. The purpose was to strengthen your ability to solve equations of degree two and to expand binomial expressions systematically, both of which are widely applied in mathematics and beyond.

We began by defining quadratic equations and exploring methods of solving them, including factorisation, completing the square, and the quadratic formula. We then examined the discriminant to analyse the nature of roots, followed by applications such as word problems and graphical representations of parabolas. Next, we studied the binomial theorem, learning how to expand expressions, identify the general term and middle term, and apply the theorem to approximation and probability. By completing this Series, you should now be able to define and solve quadratic equations (SLO 4.1, SLO 4.2), analyse roots (SLO 4.3), apply quadratic equations in practical contexts (SLO 4.4), state and use the binomial theorem (SLO 4.5), compute general and middle terms (SLO 4.6), and apply the theorem in approximation and probability problems (SLO 4.7).

Glossary of Terms

Axis of symmetry: A vertical line that divides a parabola into two mirror-image halves.

- **Binomial coefficient**: A number (nk)representing the coefficients in binomial expansions, calculated as n!k!(n-k)!.
- **Binomial theorem**: A formula for expanding expressions of the form (a+b)n.
- **Completing the square**: A method of solving quadratic equations by rewriting them as perfect squares.
- **Discriminant**: The expression b2-4acused to determine the nature of roots of a quadratic equation.
- **Factorisation**: A method of solving quadratic equations by expressing them as a product of two linear factors.
- **General term**: The formula for the kth term in a binomial expansion.
- **Middle term**: The central term(s) in a binomial expansion, depending on whether nis even or odd.
- **Parabola**: The U-shaped graph of a quadratic function.
- **Quadratic equation**: A polynomial equation of degree two, expressed in the form ax2+bx+c=0.
- **Quadratic formula**: The formula used to solve quadratic equations: -b±b2-4ac2a.
- **Roots**: The solutions of a quadratic equation.

Concept Check Questions [Series 3]

Objective questions

1. What is the standard form of a quadratic equation? (Linked to SLO 4.1) a) ax+b=0b) ax2+bx+c=0c) ax3+bx2+c=0d) ax4+bx3+c=0
2. Which of the following is NOT a method of solving quadratic equations? (Linked to SLO 4.2) a) Factorisation b) Completing the square c) Quadratic formula d) Differentiation

What does the discriminant determine? (Linked to SLO 4.3) a) The axis of symmetry b) The nature of roots c) The vertex of the parabola d) The slope of the tangent

Short-answer questions

Define a parabola and explain its orientation based on the coefficient a. (Linked to SLO 4.4)

State the binomial theorem. (Linked to SLO 4.5)

Long-answer questions

Derive the quadratic formula by completing the square. (Linked to SLO 4.2)

Analyse how the binomial theorem can be applied in probability distributions. (Linked to SLO 4.7)

Answers to Concept Check Questions [Series 3]

Objective questions

b) $ax2+bx+c=0$

d) Differentiation

b) The nature of roots

Short-answer questions

A parabola is the graph of a quadratic function. If $a>0$, it opens upwards; if $a<0$, it opens downwards.

The binomial theorem states that $(a+b)n=k=0n(nk)an-kbk$.

Long-answer questions

Basic response: Starting with $ax2+bx+c=0$, divide through by a, rearrange terms, and complete the square to derive the quadratic formula. *Value-added response*: Beyond the derivation, the quadratic formula is widely used in engineering, physics, and economics to solve optimisation and modelling problems.

Basic response: The binomial theorem underpins the binomial distribution formula, which calculates probabilities of successes in trials. *Value-added response*: In statistics, this connection is used to model real-world phenomena such as genetics, quality control, and risk analysis.

Answers to Pilot Questions [Series 3]

Part 3.1

b) ax2+bx+c=0

d) Differentiation

a) b2-4ac

b) Two equal real roots

b) Quadratic formula

Part 3.2

a) The trajectory of a ball

c) Parabola

b) Upwards

b) The maximum or minimum point

a) Axis of symmetry

Part 3.3

b) (a + b)^n

a) n!k!(n-k)!

b) a2+2ab+b2

c) 3

c) 5

Part 3.4

b) Approximating values of expressions

a) 1+nx

b) Binomial distribution

c) Binomial coefficient

a) Physics

Series 4 Trigonometry and its Applications

Series Outline

4.1 Fundamentals of trigonometry

4.1.1 Definition of trigonometric ratios

4.1.2 Values of trigonometric ratios for standard angles

4.1.3 Trigonometric identities

4.2 Trigonometric equations

4.2.1 Solving basic trigonometric equations

4.2.2 Applications in problem-solving

4.3 Applications of trigonometry

4.3.1 Heights and distances

4.3.2 Navigation and physics problems

4.4 Graphs of trigonometric functions

4.4.1 Graphs of sine, cosine, and tangent functions

4.4.2 Properties and transformations of trigonometric graphs

This Series will connect algebraic reasoning with geometry, showing how trigonometry is used to model periodic phenomena, solve real-world problems, and understand mathematical identities.

Introduction

Trigonometry is the branch of mathematics that studies the relationships between the angles and sides of triangles. It extends far beyond geometry, providing tools to model periodic phenomena such as waves, oscillations, and circular motion. From navigation and astronomy to engineering and physics, trigonometry is one of the most practical and widely applied areas of mathematics.

The aim of this Series is to equip you with the ability to define and use trigonometric ratios, solve trigonometric equations, apply trigonometry to real-world problems, and interpret graphs of trigonometric functions.

We begin with the fundamentals of trigonometry, including the definition of trigonometric ratios, values for standard angles, and key identities. Next, we explore trigonometric equations and their applications in problem-solving. We then move into practical applications such as calculating heights and distances, navigation, and physics problems. Finally, we examine the graphs of sine, cosine, and tangent functions, along with their properties and transformations.

Series Learning Outcomes (SLOs) for Series 4: Trigonometry and its Applications

By the end of this Series, you should be able to:

- **SLO 5.1** define trigonometric ratios and compute their values for standard angles.
- **SLO 5.2** apply trigonometric identities to simplify expressions and solve problems.
- **SLO 5.3** solve basic trigonometric equations and apply them in mathematical problem-solving.
- **SLO 5.4** use trigonometry to calculate heights, distances, and angles in real-world contexts.
- **SLO 5.5** apply trigonometry in navigation, physics, and other applied sciences.
- **SLO 5.6** sketch and interpret graphs of sine, cosine, and tangent functions.
- **SLO 5.7** analyse transformations of trigonometric graphs, including shifts, stretches, and reflections.

4.1 Fundamentals of Trigonometry

4.1.1 Definition of trigonometric ratios

Trigonometric ratios relate the angles of a right-angled triangle to the lengths of its sides. For an angle θ in a right triangle:

Sine (sin θ) = Opposite / Hypotenuse

Cosine (cos θ) = Adjacent / Hypotenuse

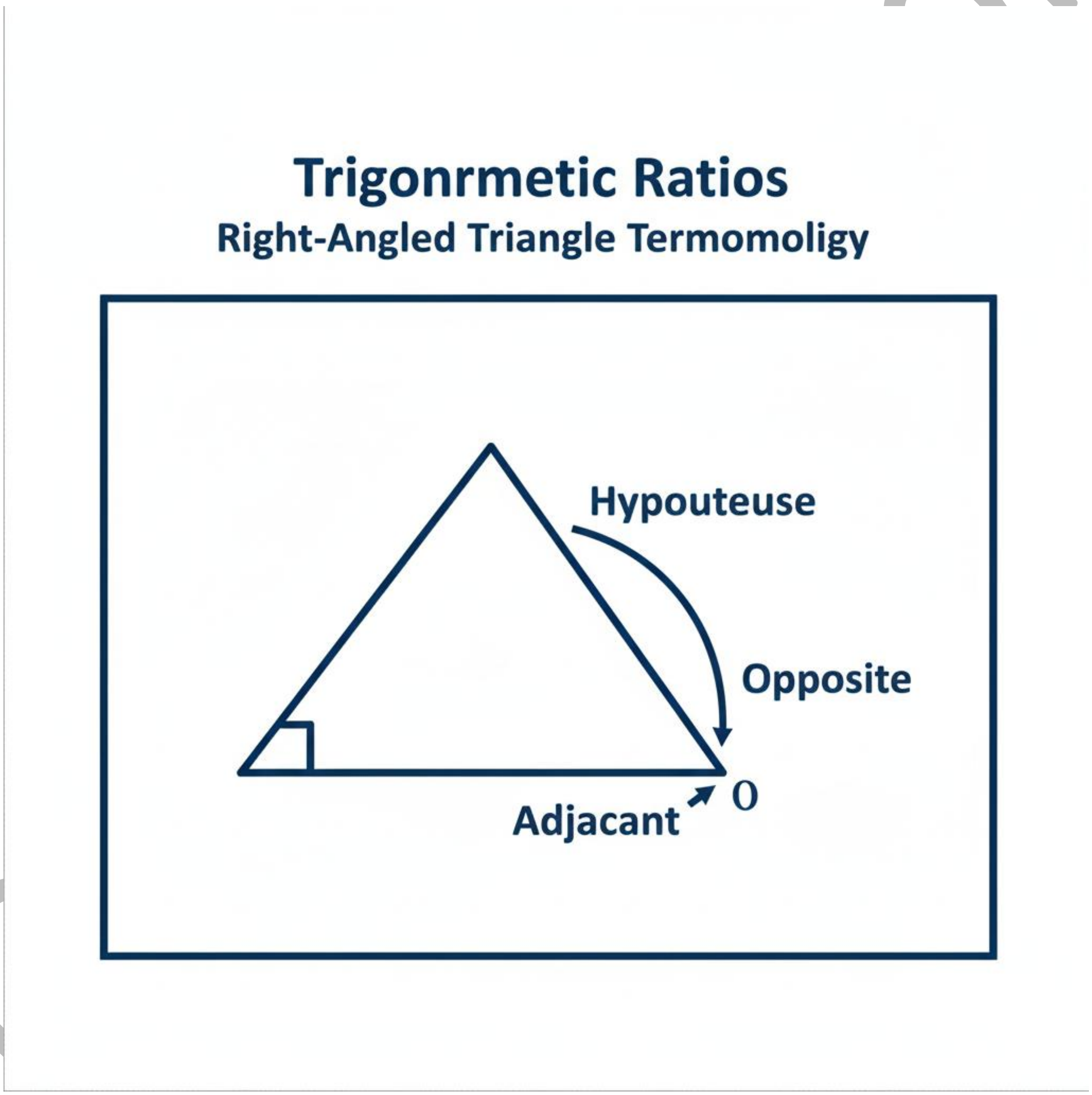
Tangent (tan θ) = Opposite / Adjacent

Cosecant (csc θ) = Hypotenuse / Opposite

Secant (sec θ) = Hypotenuse / Adjacent

Cotangent (cot θ) = Adjacent / Opposite

Fill in the blank: The ratio of opposite side to hypotenuse is called ______



A right-angled triangle labelled with sides (opposite, adjacent, hypotenuse) and angle θ.

4.1.2 Values of trigonometric ratios for standard angles

The values of trigonometric ratios for standard angles (0°, 30°, 45°, 60°, 90°) are widely used in problem-solving. For example:

$\sin 30° = 1/2$

$\cos 60° = 1/2$

$\tan 45° = 1$

Fill in the blank: The value of sin 30° is ______.

Caption: Table 4.1 Values of trigonometric ratios for standard angles

Angle	sin θ	cos θ	tan θ
0°	0	1	0
30°	1/2	√3/2	1/√3
45°	√2/2	√2/2	1
60°	√3/2	1/2	√3
90°	1	0	Undefined

4.1.3 Trigonometric identities

Trigonometric identities are equations involving trigonometric ratios that hold true for all values of θ. The most important ones are:

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Fill in the blank: The identity $\sin^2\theta + \cos^2\theta =$ ______ is called the Pythagorean identity.

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Pilot questions 4.1

Which ratio is defined as opposite/hypotenuse? a) Cos θ b) Sin θ c) Tan θ d) Sec θ

What is the value of cos 60°? a) 0 b) 1/2 c) √3/2 d) 1

Which trigonometric ratio is undefined at 90°? a) Sin θ b) Cos θ c) Tan θ d) Sec θ

What does the identity $\sin^2\theta + \cos^2\theta = 1$ represent? a) Law of sines b) Pythagorean identity c) Law of cosines d) Tangent rule

What is the value of tan 45°? a) 0 b) 1 c) √2 d) Undefined

4.2 Trigonometric Equations

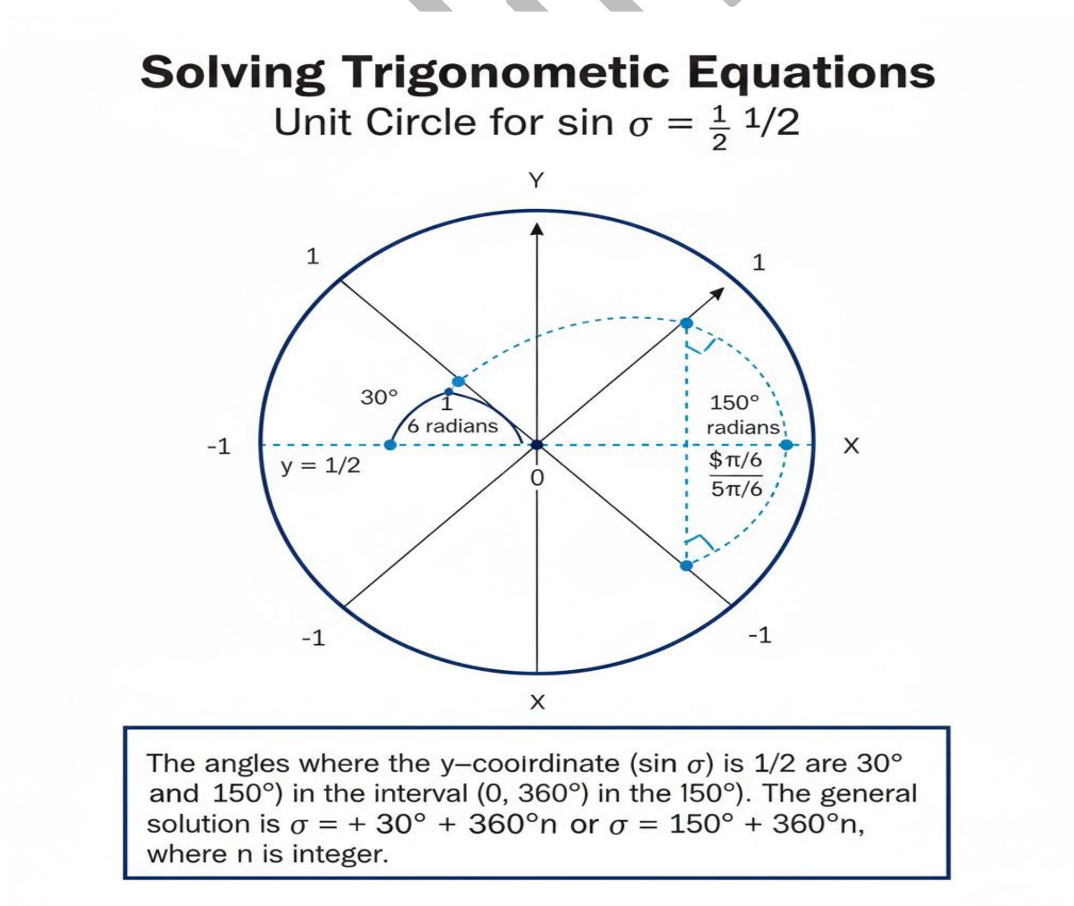
4.2.1 Solving basic trigonometric equations

A **trigonometric equation** is an equation involving trigonometric functions of an angle. To solve such equations, we often use identities and the known values of trigonometric ratios.

Example: Solve sin θ = 1/2.

sin θ = 1/2 when θ = 30° or θ = 150° (within 0° ≤ θ ≤ 180°).

Fill in the blank: The equation sin θ = 1/2 has solutions θ = ______ and θ = ______ in the interval 0° ≤ θ ≤ 180°.

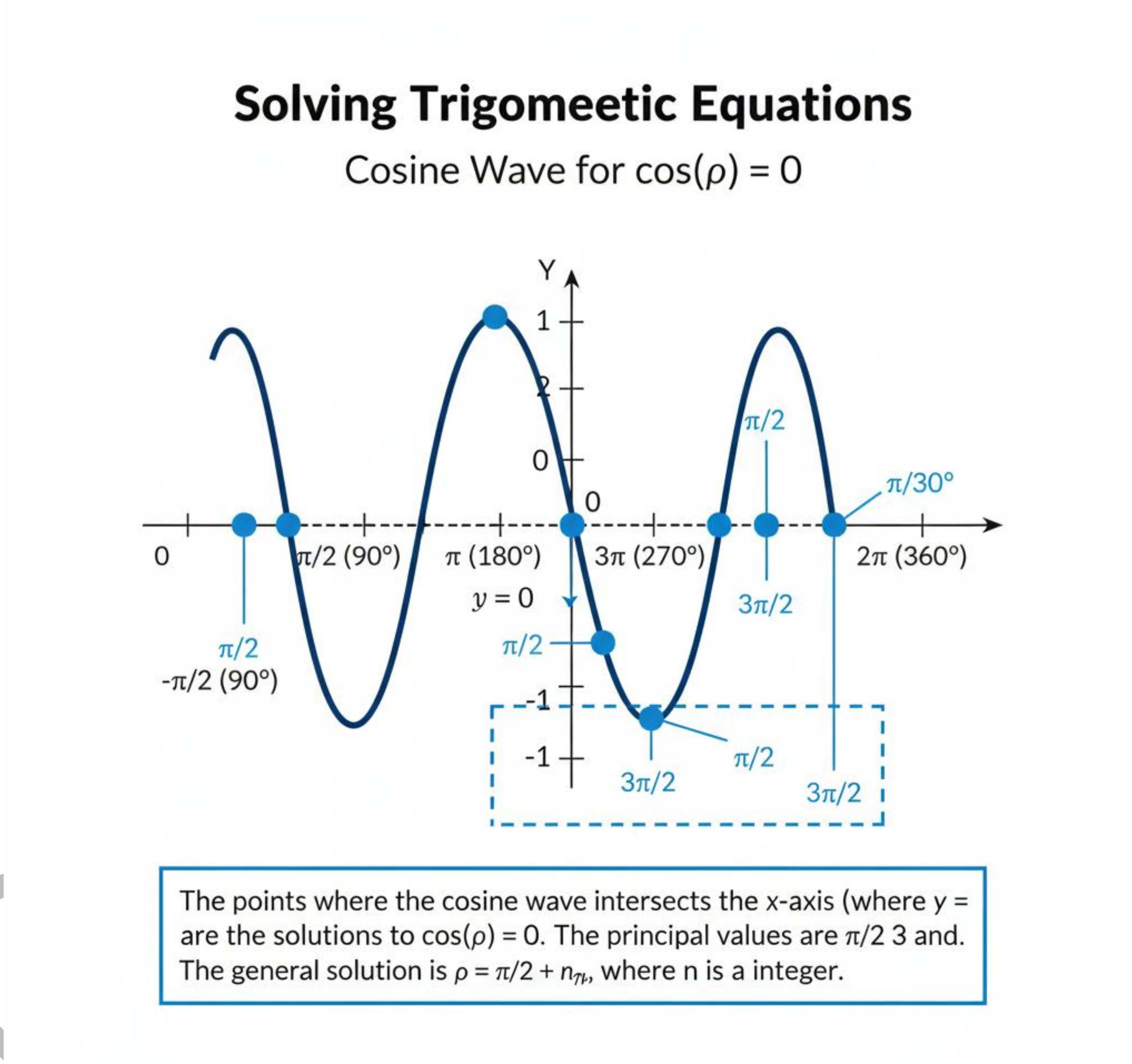


Solutions of sin θ = 1/2 on the unit

4.2.2 Applications in problem-solving

Trigonometric equations are applied in physics, engineering, and navigation. For example, solving equations like $\cos\theta = 0$ helps determine angles of oscillations, while $\tan\theta = \sqrt{3}$ can be used in surveying problems.

Fill in the blank: The equation $\cos\theta = 0$ has solutions θ = ______ and θ = ______ within $0° \leq \theta \leq 360°$.



Application of trigonometric equations in oscillations

Pilot questions 4.2

Which of the following is a trigonometric equation? a) $x^2 + 2x + 1 = 0$ b) $\sin\theta = 1/2$ c) $y = mx + c$ d) $a^2 + b^2 = c^2$

Solve $\sin\theta = 0$ in the interval $0° \le \theta \le 360°$. a) $\theta = 0°, 180°, 360°$ b) $\theta = 90°, 270°$ c) $\theta = 45°, 135°$ d) $\theta = 30°, 150°$

Solve $\cos\theta = 0$ in the interval $0° \le \theta \le 360°$. a) $\theta = 0°, 180°$ b) $\theta = 90°, 270°$ c) $\theta = 45°, 135°$ d) $\theta = 30°, 150°$

Which field uses trigonometric equations to study oscillations? a) Literature b) Physics c) History d) Geography

Solve $\tan\theta = \sqrt{3}$ in the interval $0° \le \theta \le 180°$. a) $\theta = 30°$ b) $\theta = 45°$ c) $\theta = 60°$ d) $\theta = 90°$

4.3 Applications of Trigonometry

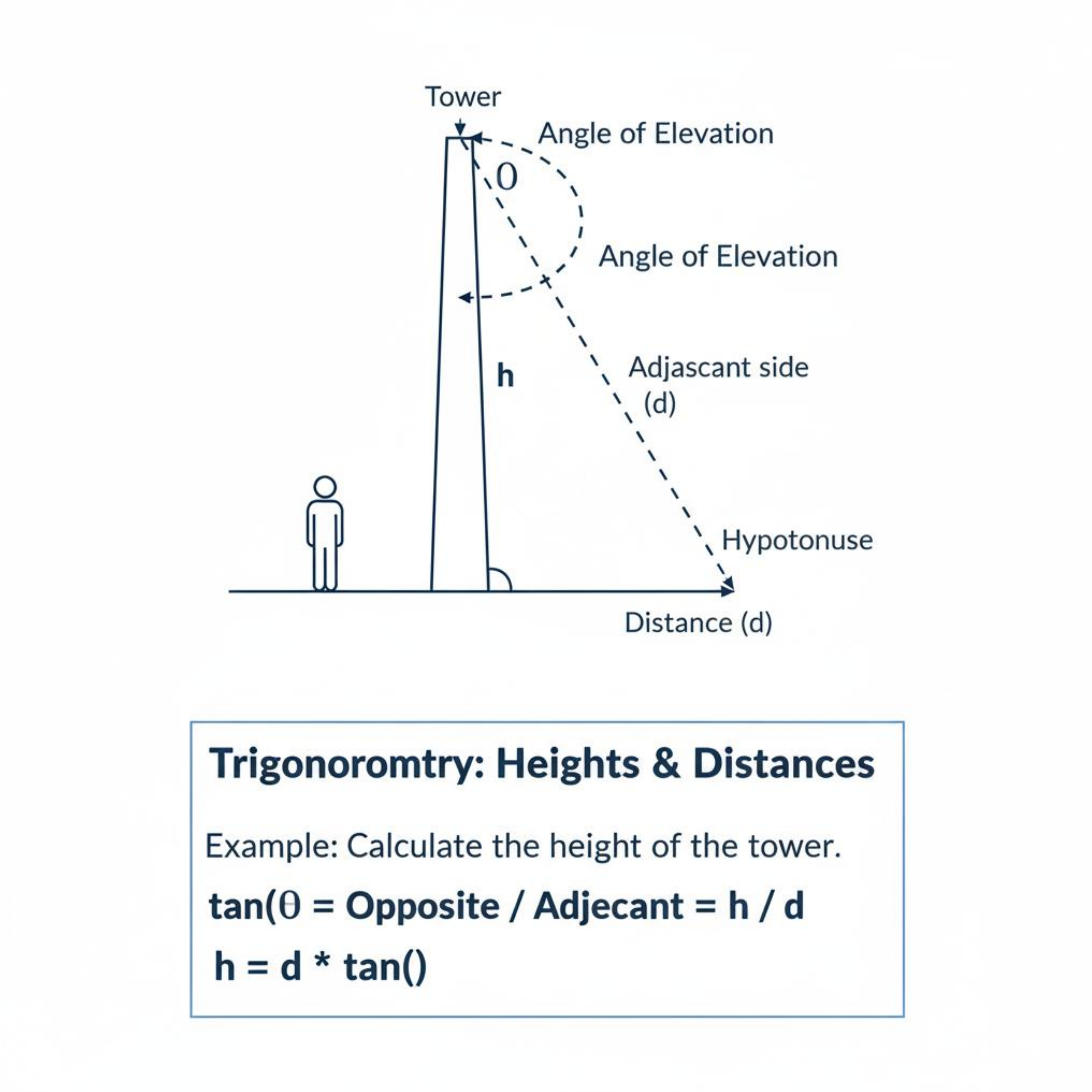
4.3.1 Heights and distances

Trigonometry is widely used to calculate heights and distances that cannot be measured directly. By applying trigonometric ratios to right-angled triangles formed in real-life situations, we can determine unknown lengths.

Example: If the angle of elevation to the top of a tower is 30° and the distance from the observer to the base is 50 m, then the height of the tower can be calculated using:

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} \quad \tan 30° = \frac{h}{50}$$

Fill in the blank: The ratio used to calculate height from angle of elevation and base distance is ______



Application of trigonometry in calculating heights

4.3.2 Navigation and physics problems

Trigonometry is essential in navigation, physics, and engineering. For example:

In navigation, bearings and directions are calculated using trigonometric ratios.

In physics, trigonometry is used to resolve forces into components, analyse wave motion, and study oscillations.

Fill in the blank: In physics, trigonometry is used to resolve ______ into horizontal and vertical components.

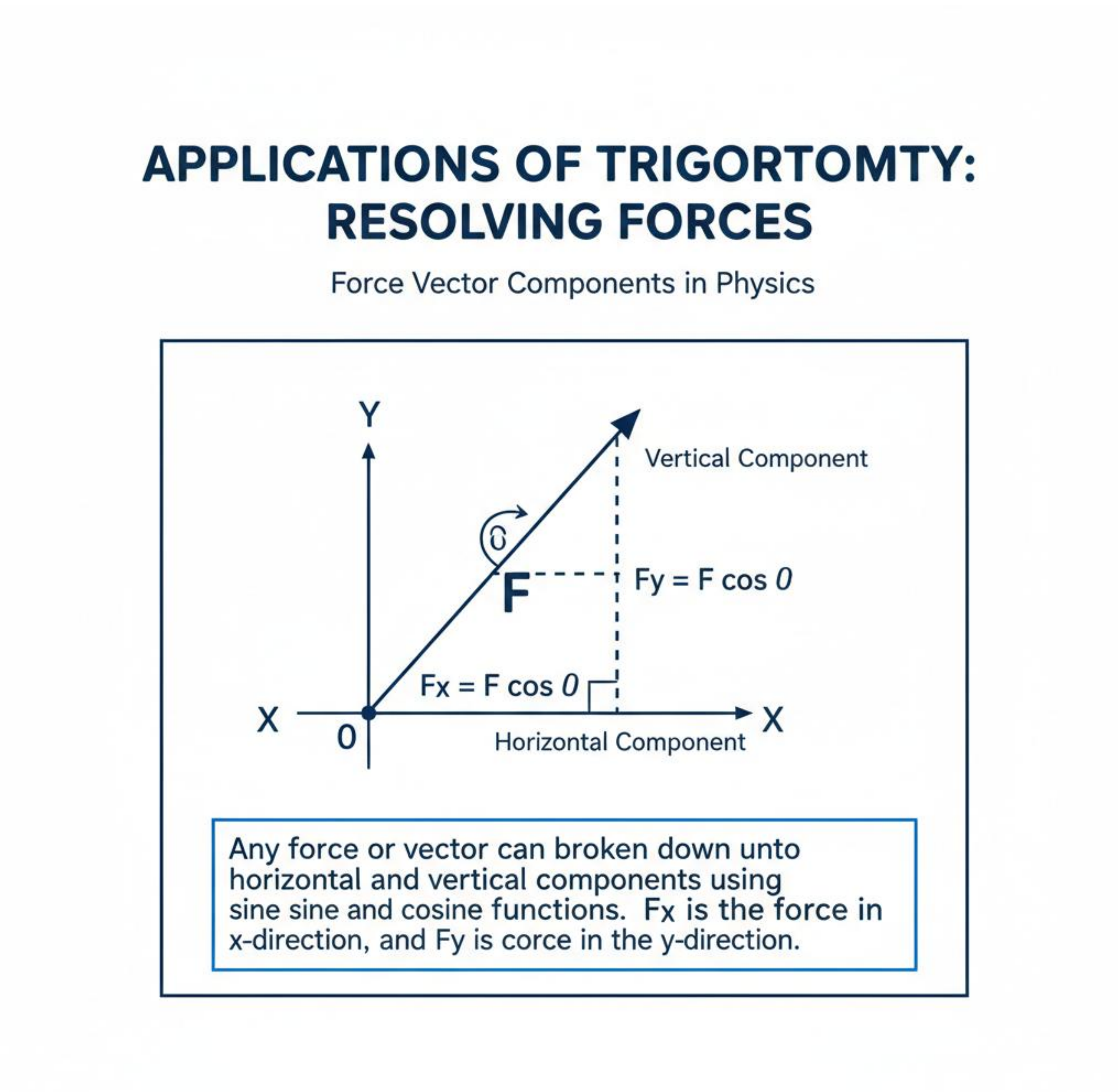


Figure 4.5 Application of trigonometry in resolving forces

Pilot questions 4.3

Which of the following real-life problems can be solved using trigonometry? a) Calculating heights of towers b) Measuring distances across rivers c) Navigation bearings d) All of the above

If the angle of elevation to the top of a building is 45° and the distance to the base is 20 m, what is the height of the building? a) 10 m b) 20 m c) 30 m d) 40 m

In navigation, trigonometry is used to calculate: a) Bearings and directions b) Compound interest c) Population growth d) Literary analysis

In physics, trigonometry is used to resolve: a) Forces b) Speeds c) Masses d) Temperatures

Which trigonometric ratio is used when calculating height from angle of elevation and base distance? a) Sine b) Cosine c) Tangent d) Secant

4.4 Graphs of Trigonometric Functions

4.4.1 Graphs of sine, cosine, and tangent functions

The graphs of trigonometric functions show their periodic nature:

Sine function (y = sin θ):

Starts at 0, oscillates between −1 and 1.

Period = 360° (or 2πradians).

Cosine function (y = cos θ):

Starts at 1, oscillates between −1 and 1.

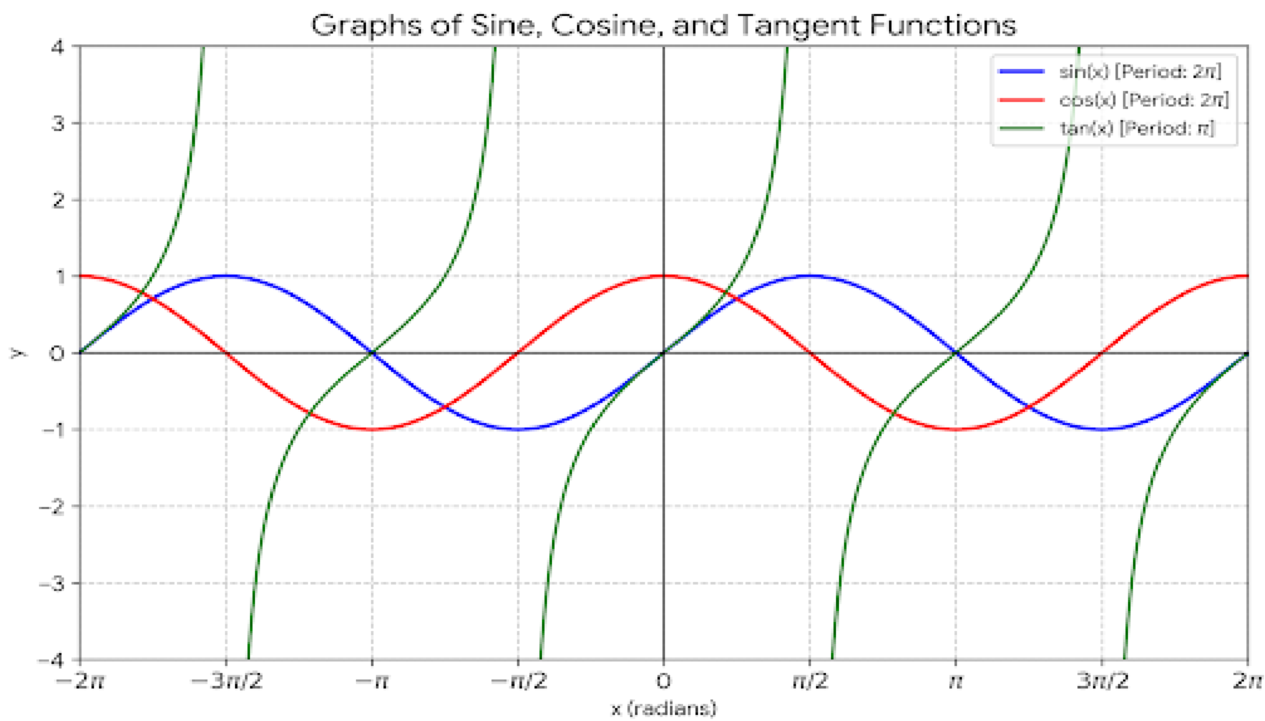
Period = 360° (or 2πradians).

Tangent function (y = tan θ):

Starts at 0, increases to infinity, then repeats.

Period = 180° (or radians).

Fill in the blank: The period of the sine and cosine functions is ______ degrees.



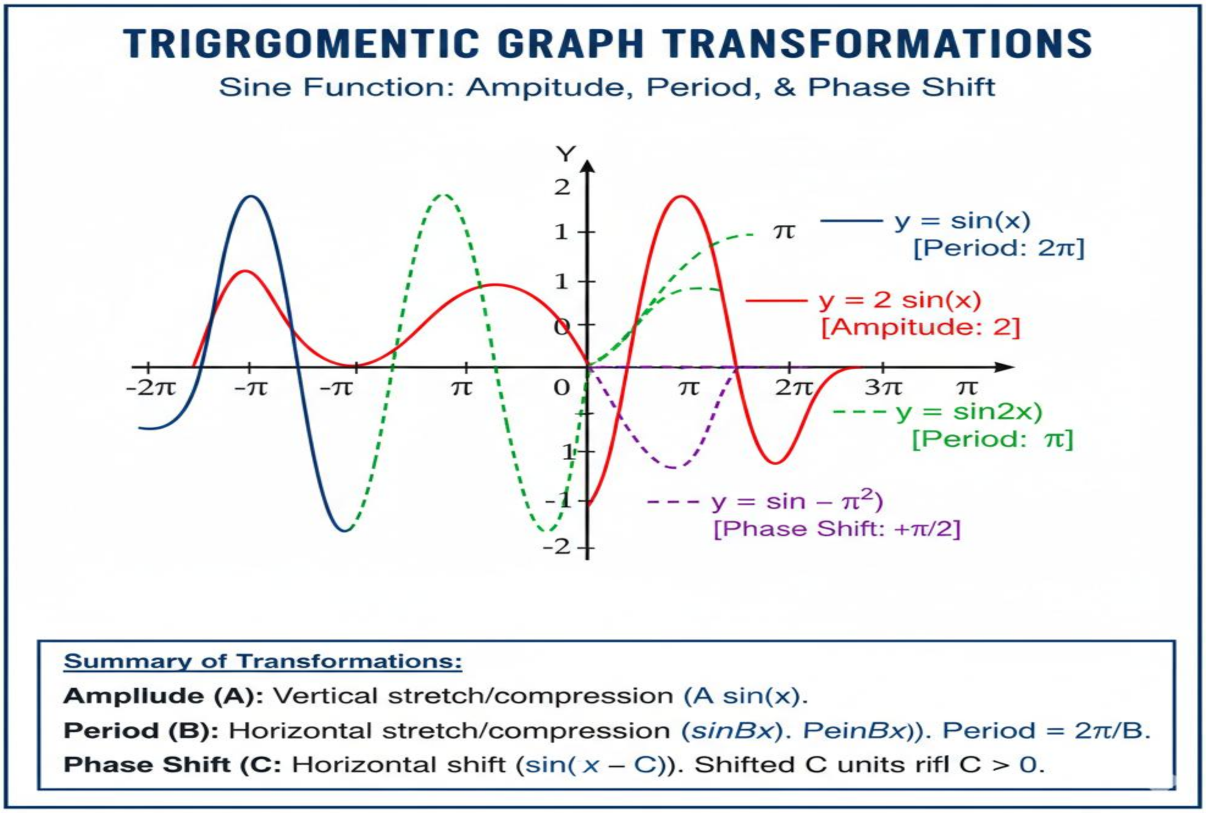
Graphs of sine, cosine, and tangent functions

4.4.2 Properties and transformations of trigonometric graphs

Trigonometric graphs can be transformed by changing amplitude, period, or phase shift:

- **Amplitude**: The maximum displacement from the axis (e.g., $y = 2\sin\theta$ has amplitude 2).
- **Period change**: $y = \sin(k\theta)$ changes the period to $360°/k$.
- **Phase shift**: $y = \sin(\theta + \varphi)$ shifts the graph horizontally.

Fill in the blank: The maximum displacement of a trigonometric graph from its axis is called ______



Transformations of trigonometric graphs

Pilot questions 4.4

What is the period of the sine function? a) 90° b) 180° c) 360° d) 720°

What is the amplitude of $y = 3\sin\theta$? a) 1 b) 2 c) 3 d) 4

The period of $y = \cos(2\theta)$ is: a) 360° b) 180° c) 90° d) 720°

A phase shift in a trigonometric graph represents: a) Vertical stretch b) Horizontal shift c) Change in amplitude d) Change in slope

Which trigonometric function has a period of 180°? a) Sine b) Cosine c) Tangent d) Secant

Series 4 Summary

This Series explored the fundamentals and applications of trigonometry. We began with the definition of trigonometric ratios, their values for standard angles, and key identities. We then moved into solving trigonometric equations and applying them in problem-solving contexts. Next, we examined practical applications such as calculating heights and distances, navigation, and resolving forces in physics. Finally, we studied the graphs of sine, cosine, and tangent functions, along with their properties and transformations.

By completing this Series, you should now be able to define and compute trigonometric ratios (SLO 5.1), apply identities (SLO 5.2), solve trigonometric equations (SLO 5.3), use trigonometry in real-world contexts (SLO 5.4, SLO 5.5), and sketch and interpret trigonometric graphs with transformations (SLO 5.6, SLO 5.7).

Glossary of Terms

Angle of elevation: The angle formed when looking upward from the horizontal to an object.

Angle of depression: The angle formed when looking downward from the horizontal to an object.

Amplitude: The maximum displacement of a trigonometric graph from its axis.

Bearing: A direction measured clockwise from the north line.

Cosecant (csc θ): Reciprocal of sine, defined as hypotenuse/opposite.

Cosine (cos θ): Ratio of adjacent side to hypotenuse in a right triangle.

Cotangent (cot θ): Reciprocal of tangent, defined as adjacent/opposite.

Period: The interval after which a trigonometric function repeats its values.

Phase shift: Horizontal displacement of a trigonometric graph.

Sine (sin θ): Ratio of opposite side to hypotenuse in a right triangle.

Tangent (tan θ): Ratio of opposite side to adjacent side in a right triangle.

Trigonometric identity: An equation involving trigonometric ratios that holds true for all values of θ.

Concept Check Questions [Series 4]

Objective questions

What is the value of $\sin 30°$? (Linked to SLO 5.1) a) 0 b) 1/2 c) $\sqrt{3}/2$ d) 1

Which identity is known as the Pythagorean identity? (Linked to SLO 5.2) a) $\sin^2\theta + \cos^2\theta = 1$ b) $\tan^2\theta + \sec^2\theta = 1$ c) $\cot^2\theta + \csc^2\theta = 1$ d) $\sin^2\theta - \cos^2\theta = 1$

Solve $\cos\theta = 0$ in the interval $0° \leq \theta \leq 360°$. (Linked to SLO 5.3) a) $\theta = 0°, 180°$ b) $\theta = 90°, 270°$ c) $\theta = 45°, 135°$ d) $\theta = 30°, 150°$

Short-answer questions

Define angle of elevation and give one example of its application. (Linked to SLO 5.4)

Explain how trigonometry is used to resolve forces in physics. (Linked to SLO 5.5)

Long-answer questions

Sketch the graph of y = sin θ for 0° ≤ θ ≤ 360° and describe its properties. (Linked to SLO 5.6)

Analyse how amplitude and period changes affect the graph of y = cos θ. (Linked to SLO 5.7)

Answers to Concept Check Questions [Series 4]

Objective questions

b) 1/2

a) $\sin^2\theta + \cos^2\theta = 1$

b) θ = 90°, 270°

Short-answer questions

Angle of elevation is the angle formed when looking upward from the horizontal to an object. Example: measuring the height of a tower.

In physics, trigonometry is used to resolve a force vector into horizontal and vertical components, simplifying analysis of motion.

Long-answer questions

Basic response: The graph of y = sin θ oscillates between −1 and 1, with period 360°. *Value-added response*: This property is used in modelling sound waves, alternating current, and other periodic phenomena.

Basic response: Increasing amplitude stretches the graph vertically, while changing the coefficient of θ alters the period. *Value-added response*: These transformations are applied in engineering to model oscillations with varying intensity and frequency.

Answers to Pilot Questions [Series 4]

Part 4.1

b) Sin θ

b) 1/2

c) Tan θ

b) Pythagorean identity

b) 1

Part 4.2

b) sin θ = 1/2

a) θ = 0°, 180°, 360°

b) θ = 90°, 270°

b) Physics

c) 60°

Part 4.3

d) All of the above

b) 20 m

a) Bearings and directions

a) Forces

c) Tangent

Part 4.4

c) 360°

c) 3

b) 180°

b) Horizontal shift

c) Tangent

Series 5 Coordinate Geometry

Series Outline

5.1 Fundamentals of coordinate geometry

5.1.1 Cartesian plane and coordinates

5.1.2 Distance between two points

5.1.3 Midpoint of a line segment

5.2 Straight lines

5.2.1 Equation of a straight line

5.2.2 Slope of a line

5.2.3 Parallel and perpendicular lines

5.3 Circles

5.3.1 Equation of a circle

5.3.2 Properties of circles in coordinate geometry

5.4 Applications of coordinate geometry

5.4.1 Solving geometric problems using coordinates

5.4.2 Real-life applications in navigation, mapping, and design

This Series will connect algebra with geometry, showing how coordinates are used to represent geometric figures, solve problems, and apply mathematics in real-world contexts like navigation and design.

Introduction

Coordinate geometry, also known as analytic geometry, is the study of geometry using a coordinate system. It bridges algebra and geometry by representing geometric figures with equations and coordinates. This allows us to solve geometric problems using algebraic methods and to visualise algebraic relationships geometrically.

The aim of this Series is to equip you with the ability to use coordinates to represent points, lines, and circles, and to apply these concepts in solving both theoretical and practical problems.

We begin with the fundamentals of coordinate geometry, including the Cartesian plane, distance between two points, and midpoint of a line segment. Next, we study straight lines, focusing on their equations, slopes, and conditions for parallelism and perpendicularity. We then move into circles, deriving their equations and exploring their properties. Finally, we examine applications of coordinate geometry in solving geometric problems and in real-life contexts such as navigation, mapping, and design.

Series Learning Outcomes (SLOs)

By the end of this Series, you should be able to:

SLO 6.1 describe the Cartesian plane and represent points using coordinates.

SLO 6.2 calculate the distance between two points using the distance formula.

SLO 6.3 determine the midpoint of a line segment using coordinates.

SLO 6.4 derive and apply the equation of a straight line.

SLO 6.5 compute the slope of a line and use it to analyse parallel and perpendicular lines.

SLO 6.6 state and apply the equation of a circle in coordinate geometry.

SLO 6.7 solve geometric problems using coordinates.

SLO 6.8 apply coordinate geometry in real-life contexts such as navigation, mapping, and design.

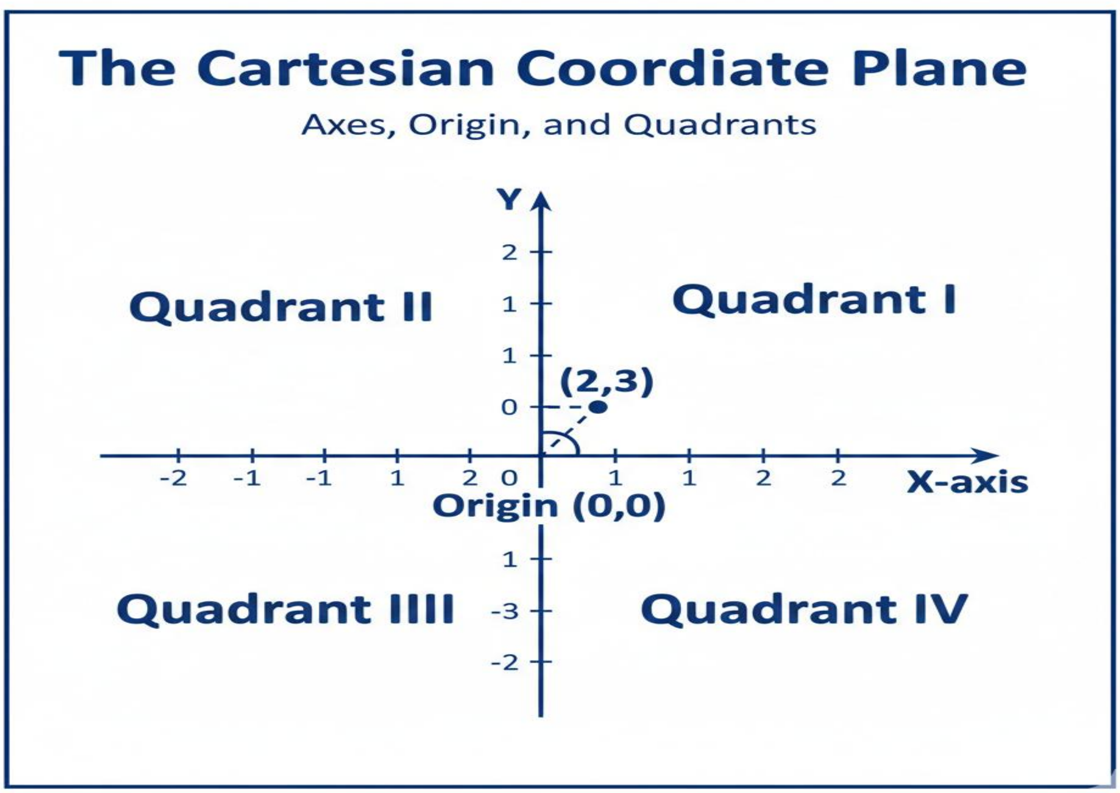
This gives us a clear set of skills to master in Series 5, aligned with each Part of the outline.

5.1 Fundamentals of Coordinate Geometry

5.1.1 Cartesian plane and coordinates

The **Cartesian plane** is formed by two perpendicular axes: the horizontal axis (x-axis) and the vertical axis (y-axis). Any point in the plane can be represented by an ordered pair (x,y), where xis the horizontal distance and yis the vertical distance from the origin (0,0).

Fill in the blank: The point where the x-axis and y-axis intersect is called the ______.



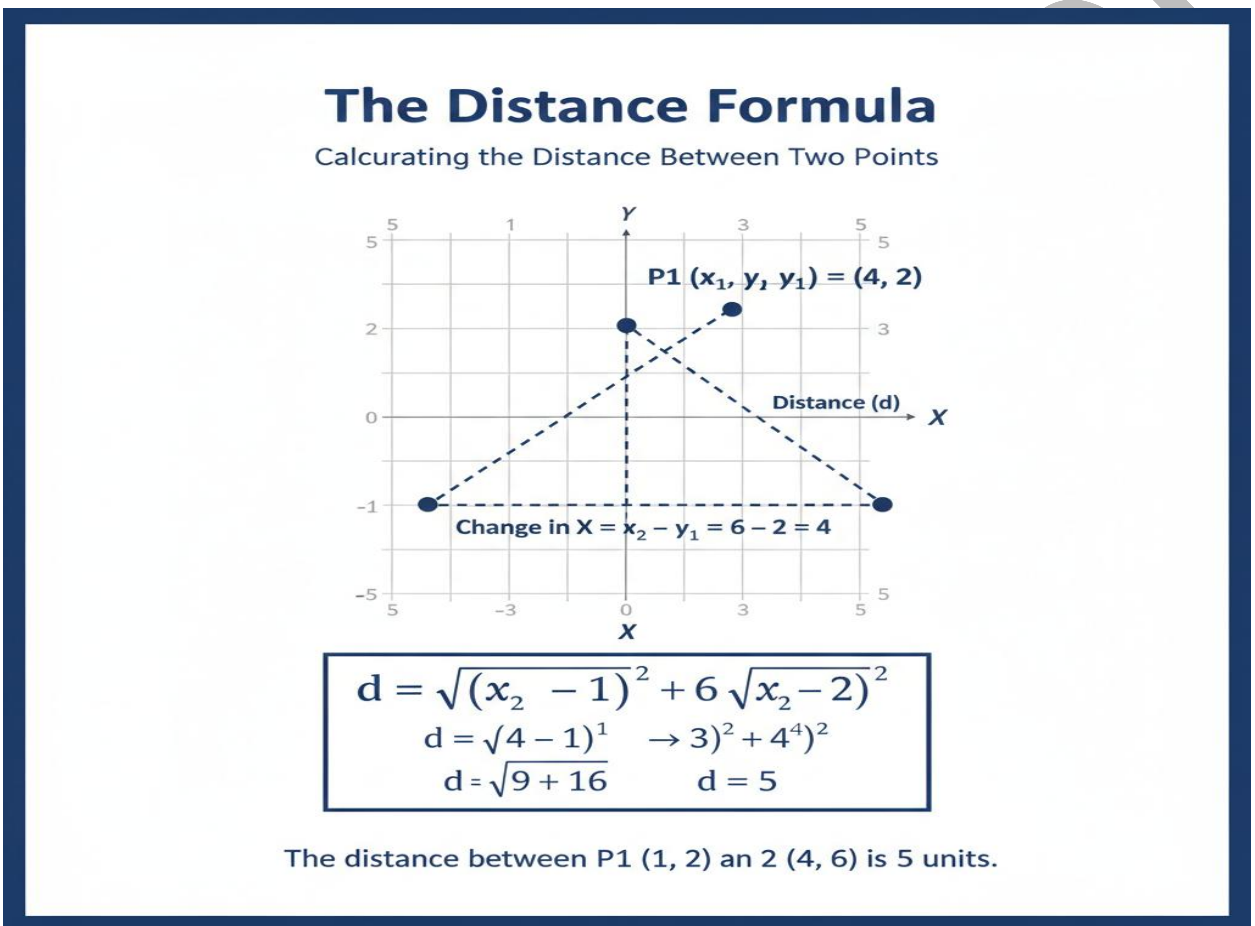
Cartesian plane and coordinates

5.1.2 Distance between two points

The distance between two points (x1,y1)and (x2,y2)is given by the formula:

$$d=(x2-x1)2+(y2-y1)2$$

Fill in the blank: The distance formula is derived from the ______ theorem.



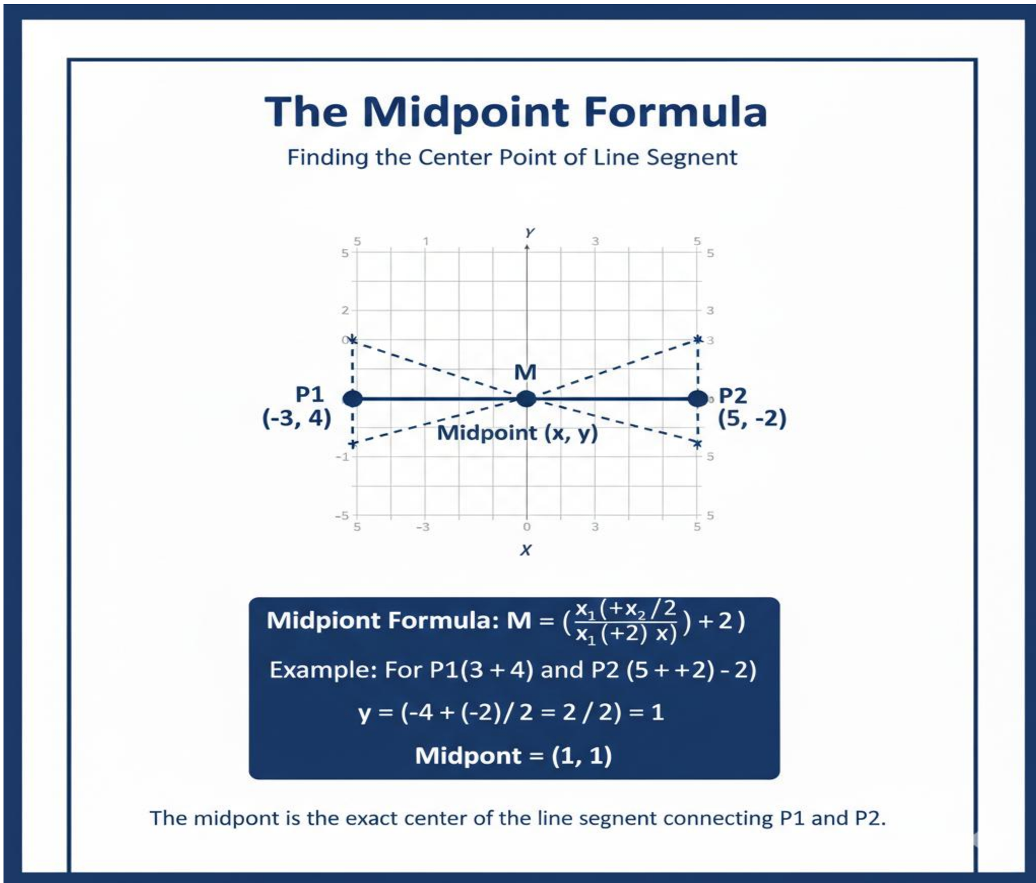
Distance between two points using coordinates

5.1.3 Midpoint of a line segment

The midpoint of a line segment joining two points (x1,y1)and (x2,y2)is given by:

$$M=(x1+x22,y1+y22)$$

Fill in the blank: The midpoint formula averages the ______ of the endpoints.



Midpoint of a line segment

Pilot questions 5.1

What is the point where the x-axis and y-axis intersect? a) Vertex b) Origin c) Midpoint d) Centre

The distance between points (2,3) and (5,7) is: a) 5 b) $\sqrt{25}$ c) $\sqrt{(3^2 + 4^2)}$ d) $\sqrt{(9 + 16)}$

The distance formula is derived from which theorem? a) Binomial theorem b) Pythagoras theorem c) Trigonometric theorem d) Midpoint theorem

What is the midpoint of (2,4) and (6,8)? a) (4,6) b) (3,5) c) (5,7) d) (2,8)

The coordinates of a point are represented as: a) (x,y) b) (y,x) c) (x+y) d) (x:y)

5.2 Straight Lines

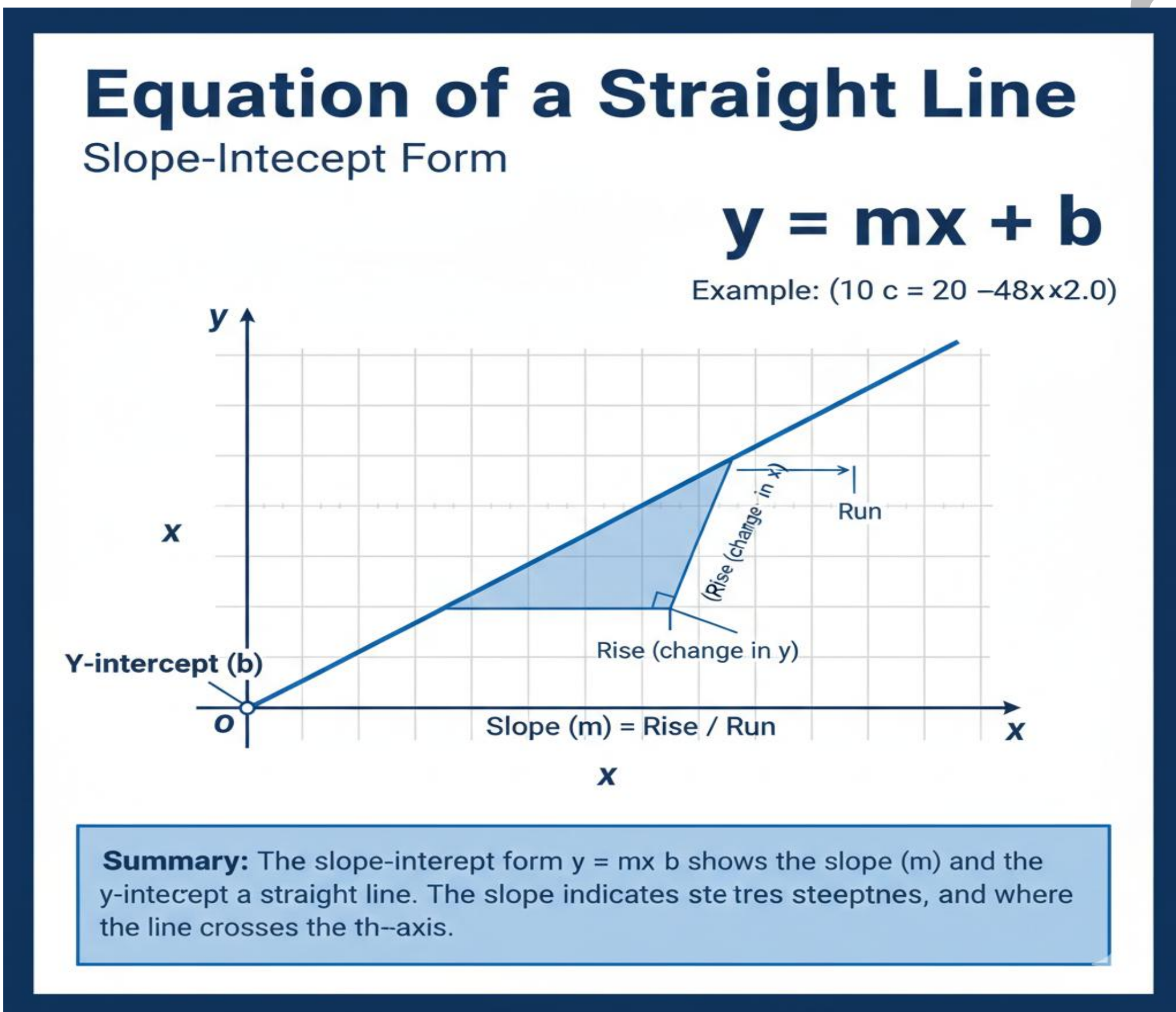
5.2.1 Equation of a straight line

The general equation of a straight line in the Cartesian plane is:

$$y=mx+c$$

where mis the slope (gradient) of the line and cis the y-intercept.

Fill in the blank: In the equation y=mx+c, mrepresents the ______ of the line.



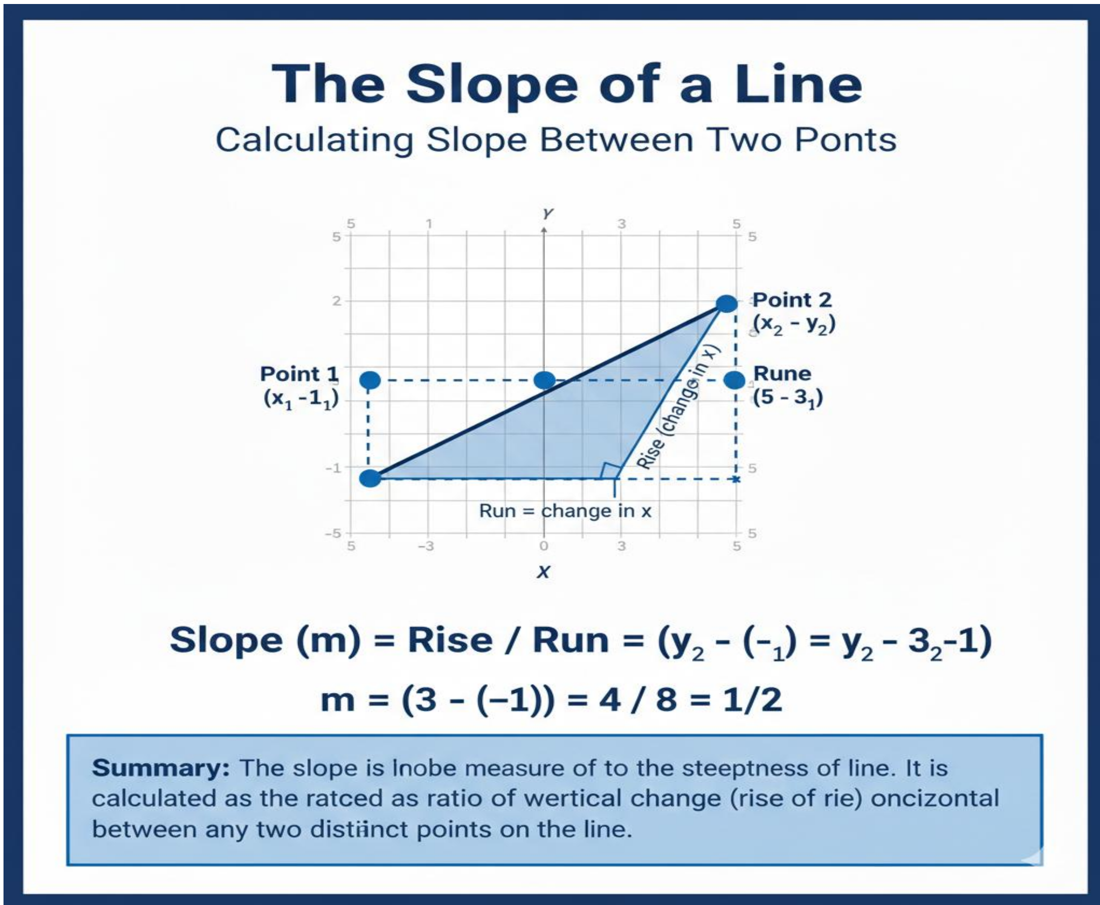
Equation of a straight line

5.2.2 Slope of a line

The slope of a line passing through two points (x1,y1)and (x2,y2)is given by:

$$m=y2-y1x2-x1$$

Fill in the blank: The slope of a line is the ratio of the change in ______ to the change in ______



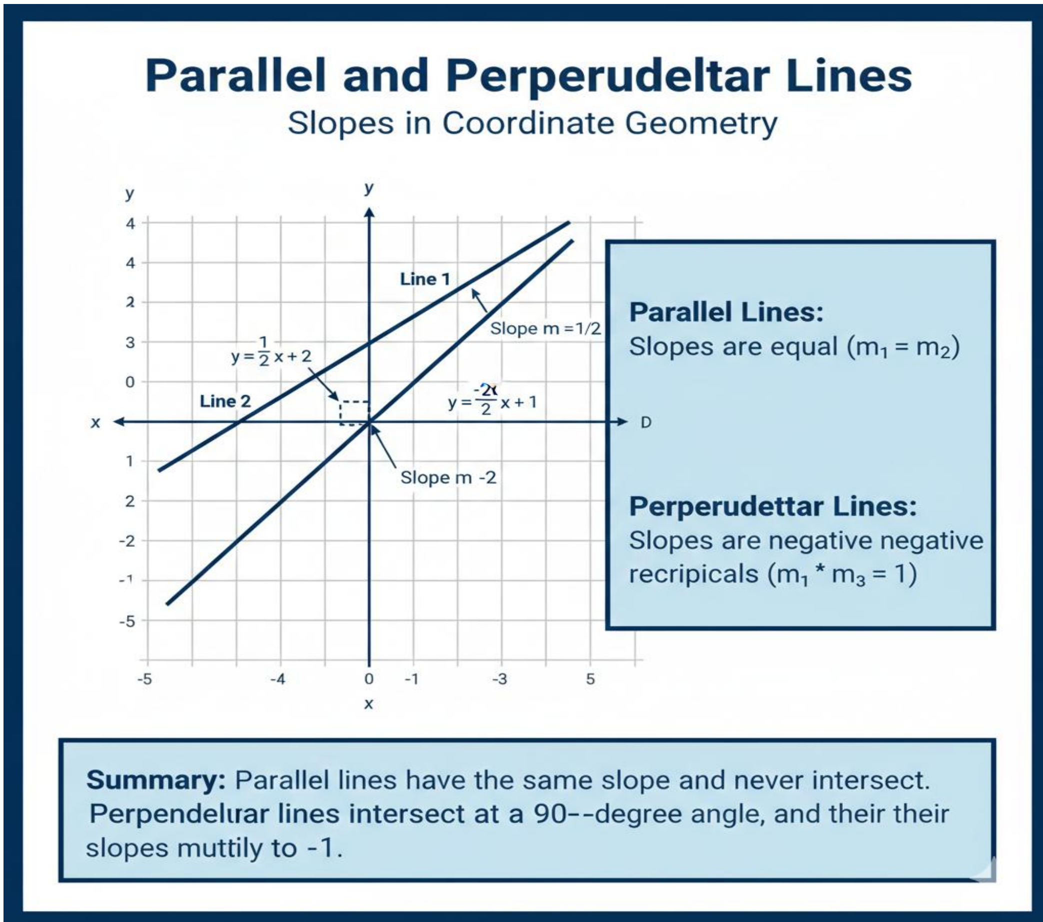
Slope of a line between two points

5.2.3 Parallel and perpendicular lines

Parallel lines: Two lines are parallel if they have the same slope.

Perpendicular lines: Two lines are perpendicular if the product of their slopes is -1.

Fill in the blank: Two lines are perpendicular if the product of their slopes is ______



Parallel and perpendicular lines AI prompt suggestion

Pilot questions 5.2

What is the general equation of a straight line? a) y = mx + c b) y = ax² + bx + c c) y² = x² + c d) y = m/x + c

In the equation y = mx + c, what does c represent? a) Slope b) Origin c) y-intercept d) Midpoint

The slope of a line through (2,3) and (5,9) is: a) 2 b) 3/2 c) 6/3 d) 9/5

Two lines are parallel if: a) Their slopes are equal b) Their slopes multiply to −1 c) Their intercepts are equal d) They intersect at the origin

Two lines are perpendicular if: a) Their slopes are equal b) Their slopes multiply to −1 c) Their intercepts are equal d) They never meet

5.3 Circles

5.3.1 Equation of a circle

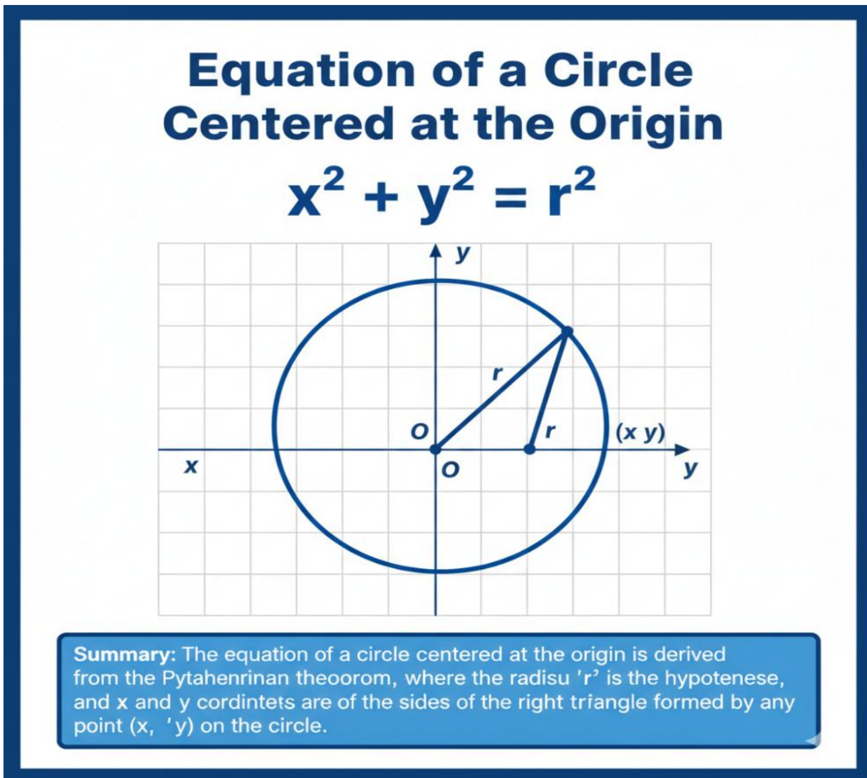
The general equation of a circle with centre (h,k)and radius ris:

$$(x-h)2+(y-k)2=r2$$

If the circle is centred at the origin (0,0), the equation becomes:

$$x2+y2=r2$$

Fill in the blank: The equation of a circle with centre (0,0)and radius ris ______



Equation of a circle centred at the origin

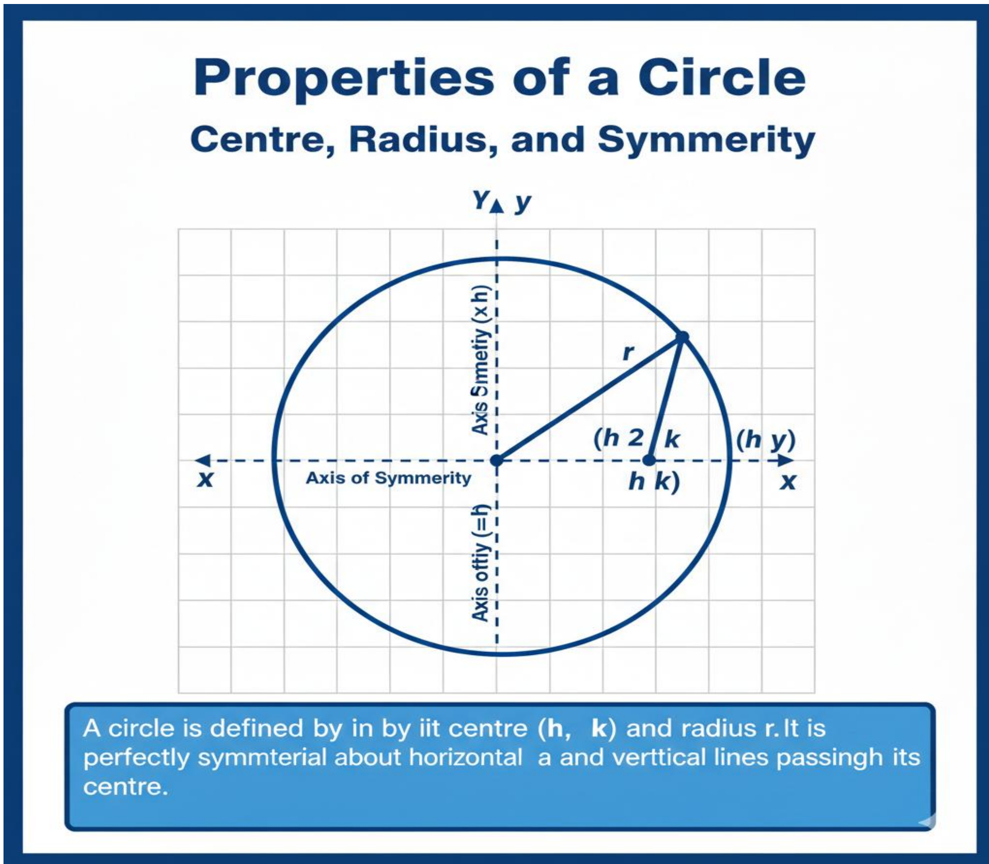
Key properties of circles in coordinate geometry include:

The centre is represented by coordinates (h,k).

The radius is the distance from the centre to any point on the circle.

The circle is symmetric about both axes if centred at the origin.

Fill in the blank: The radius of a circle is the distance from the ______ to any point on the circle.



] A diagram showing a circle with labelled centre, radius, and symmetry about axes. Caption:

Pilot questions 5.3

What is the general equation of a circle with centre (h,k)and radius r? a) (x+h)2+(y+k)2=r2b) (x-h)2+(y-k)2=r2c) (x-h)2+(y+k)2=r2d) (x+h)2+(y-k)2=r2

What is the equation of a circle centred at the origin with radius 5? a) x2+y2=10b) x2+y2=25c) x2+y2=5d) x2+y2=50

The centre of a circle is represented by: a) (x, y) b) (h, k) c) (r, k) d) (h, r)

The radius of a circle is the distance from: a) Origin to x-axis b) Centre to any point on the circle c) x-axis to y-axis d) Midpoint to intercept

A circle centred at the origin is symmetric about: a) x-axis only b) y-axis only c) both axes d) neither axis

5.4 Applications of Coordinate Geometry

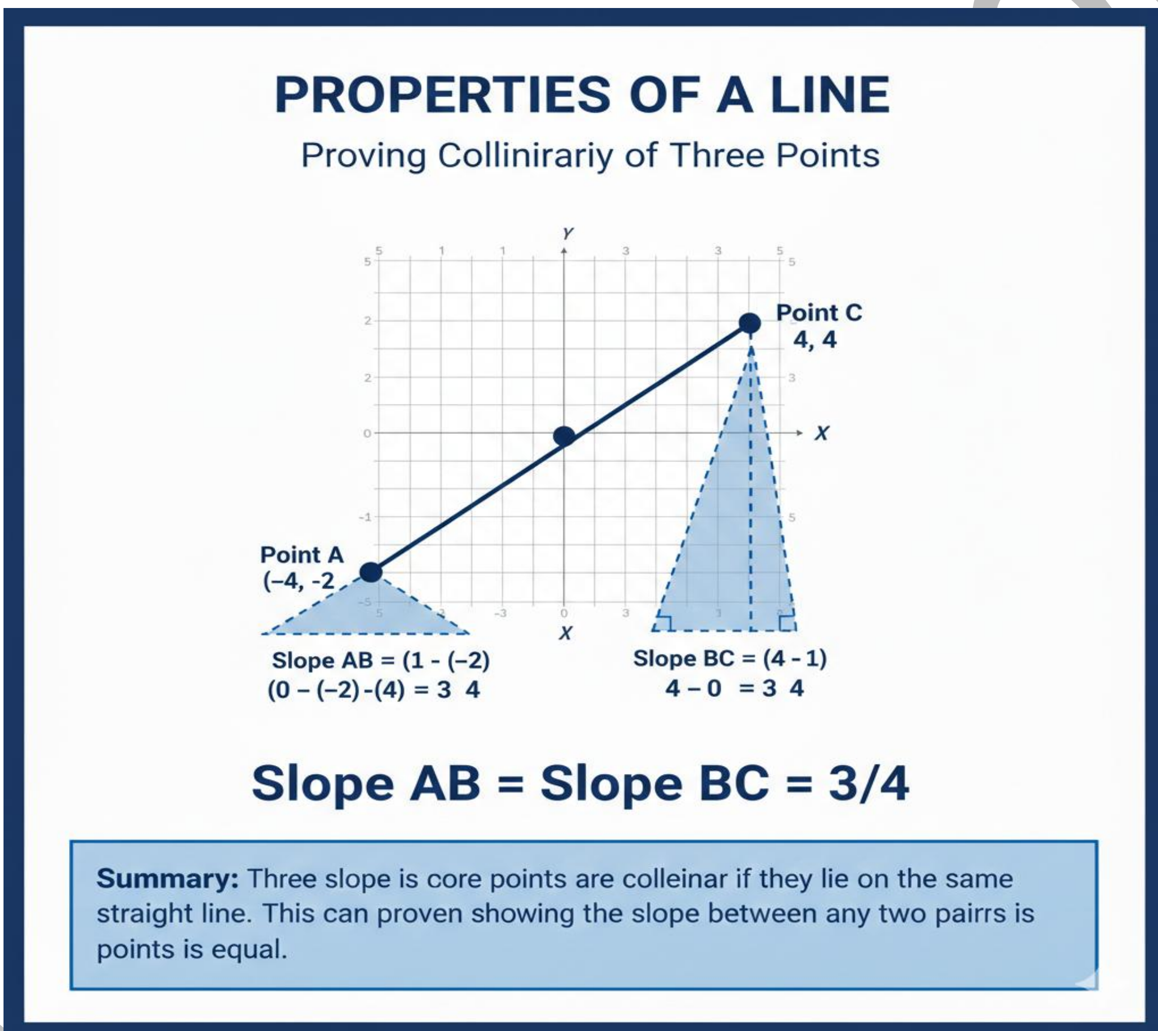
5.4.1 Solving geometric problems using coordinates

Coordinate geometry allows us to solve geometric problems by representing shapes with equations.

Example: To prove that a triangle is right-angled, we can calculate slopes of its sides. If the product of two slopes is -1, the triangle has a right angle.

Example: To check if points are collinear, we can verify if the slopes between them are equal.

Fill in the blank: Points are collinear if the slopes between them are ______



Using coordinates to check collinearity

5.4.2 Real-life applications in navigation, mapping, and design

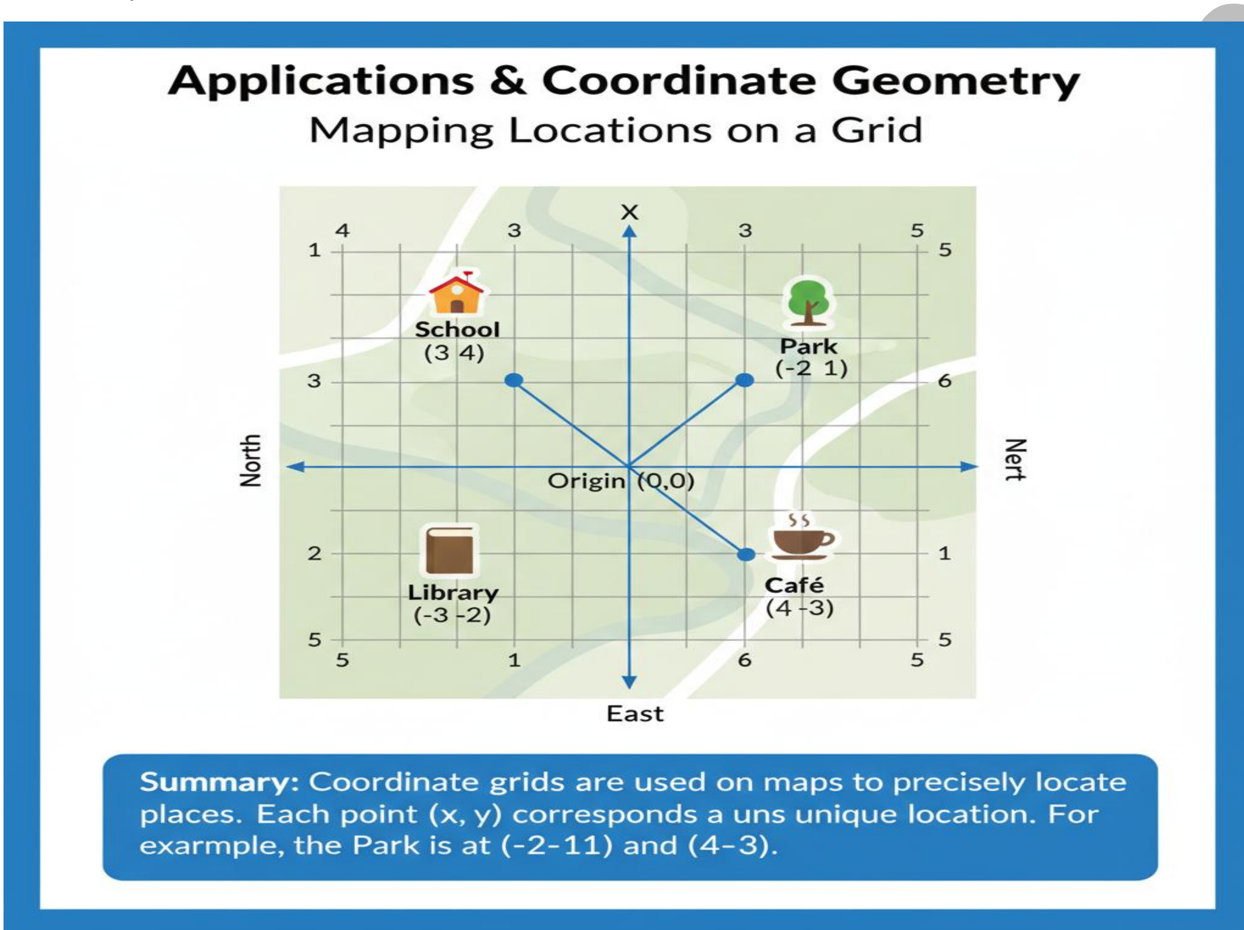
Coordinate geometry is applied in many real-life contexts:

Navigation: Coordinates are used to determine positions and directions.

Mapping: Maps use coordinate systems to represent locations accurately.

Design: Architects and engineers use coordinates to plan structures and layouts.

Fill in the blank: In mapping, coordinate geometry is used to represent ______ accurately.



Application of coordinate geometry in mapping AI prompt suggestion: "

Pilot questions 5.4

Which of the following problems can be solved using coordinate geometry? a) Checking collinearity of points b) Proving a triangle is right-angled c) Finding the midpoint of a line segment d) All of the above

Points are collinear if: a) Their slopes are equal b) Their slopes multiply to −1 c) Their intercepts are equal d) They form a circle

In navigation, coordinate geometry is used to determine: a) Speeds b) Positions and directions c) Masses d) Temperatures

In mapping, coordinate geometry is used to represent: a) Locations b) Colours c) Languages d) Populations

Architects and engineers use coordinate geometry to: a) Plan structures and layouts b) Write equations c) Solve quadratic equations d) Analyse poetry

Series 5 Summary

This Series explored the fundamentals and applications of coordinate geometry. We began with the Cartesian plane, distance formula, and midpoint formula. We then studied straight lines, focusing on their equations, slopes, and conditions for parallelism and perpendicularity. Next, we examined circles, deriving their equations and exploring their properties. Finally, we looked at applications of coordinate geometry in solving geometric problems and in real-life contexts such as navigation, mapping, and design.

By completing this Series, you should now be able to represent points on the Cartesian plane (SLO 6.1), calculate distances and midpoints (SLO 6.2, SLO 6.3), derive and apply equations of straight lines (SLO 6.4), compute slopes and analyse line relationships (SLO 6.5), state and apply the equation of a circle (SLO 6.6), solve geometric problems using coordinates (SLO 6.7), and apply coordinate geometry in real-life contexts (SLO 6.8).

Glossary of Terms

Cartesian plane: A plane defined by two perpendicular axes (x-axis and y-axis).

Coordinates: Ordered pairs (x,y)representing points on the Cartesian plane.

Distance formula: Formula used to calculate the distance between two points.

Midpoint formula: Formula used to find the midpoint of a line segment.

Slope (gradient): Measure of steepness of a line, calculated as rise/run.

Equation of a line: Algebraic representation of a straight line, typically y=mx+c.

Parallel lines: Lines with equal slopes that never meet.

Perpendicular lines: Lines whose slopes multiply to −1.

Equation of a circle: (x-h)2+(y-k)2=r2, where (h,k)is the centre and ris the radius.

Collinearity: Condition where points lie on the same straight line.

Concept Check Questions [Series 5]

Objective questions

What is the point where the x-axis and y-axis intersect? (Linked to SLO 6.1) a) Vertex b) Origin c) Midpoint d) Centre

The distance between points (1,2) and (4,6) is: (Linked to SLO 6.2) a) 5 b) $\sqrt{25}$ c) $\sqrt{(3^2 + 4^2)}$ d) $\sqrt{(9 + 16)}$

What is the midpoint of (2,4) and (6,8)? (Linked to SLO 6.3) a) (4,6) b) (3,5) c) (5,7) d) (2,8)

In the equation $y = mx + c$, what does c represent? (Linked to SLO 6.4) a) Slope b) Origin c) y-intercept d) Midpoint

Two lines are perpendicular if: (Linked to SLO 6.5) a) Their slopes are equal b) Their slopes multiply to −1 c) Their intercepts are equal d) They never meet

Short-answer questions

Write the equation of a circle centred at (0,0) with radius 7. (Linked to SLO 6.6)

Explain how to check if three points are collinear using coordinates. (Linked to SLO 6.7)

Long-answer questions

Discuss how coordinate geometry is applied in navigation and mapping. (Linked to SLO 6.8)

Answers to Concept Check Questions [Series 5]

Objective questions

b) Origin

c) $\sqrt{(3^2 + 4^2)} = 5$

a) (4,6)

c) y-intercept

b) Their slopes multiply to −1

Short-answer questions

Equation: x2+y2=49.

To check collinearity, calculate slopes between pairs of points. If slopes are equal, the points are collinear.

Long-answer questions

Basic response: Coordinate geometry is used in navigation to determine positions and directions, and in mapping to represent locations accurately. *Value-added response*: In modern GPS systems, coordinate geometry underpins algorithms for route planning, distance measurement, and spatial analysis, making it essential in transportation, urban planning, and design.

Answers to Pilot Questions [Series 5]

Part 5.1

b) Origin

d) $\sqrt{(9 + 16)} = 5$

b) Pythagoras theorem

a) (4,6)

a) (x,y)

Part 5.2

a) $y = mx + c$

c) y-intercept

a) 2

a) Their slopes are equal

b) Their slopes multiply to -1

Part 5.3

b) $(x - h)^2 + (y - k)^2 = r^2$

b) $x^2 + y^2 = 25$

b) (h, k)

b) Centre to any point on the circle

c) Both axes

Part 5.4

d) All of the above

a) Their slopes are equal

b) Positions and directions

a) Locations

a) Plan structures and layouts