La-Net

MTH407

MATHEMATICAL METHODS

MTH 407: Mathematical Methods

Series 1: Calculus of Variation and Lagrange's Functional

Introduction

In mathematics and physics, many important problems involve finding a function rather than a single number that optimises a certain quantity. This is the essence of the calculus of variation, a powerful tool that extends the ideas of differential calculus to functionals. From determining the shortest path between two points to deriving the equations of motion in mechanics, variational methods provide a unifying framework that connects analysis, geometry, and physical principles. This Series introduces you to these foundational ideas, setting the stage for deeper exploration in later parts of the course.

The aim of this Series is to equip you with the ability to define and explain the calculus of variation, to describe the structure of Lagrange's functional and its associated density, and to apply necessary conditions for extremum to solve basic variational problems.

We shall commence this Series by introducing the calculus of variation, its scope, and some motivating examples. Next, we will examine Lagrange's functional and the concept of associated density, which form the building blocks of variational problems. Following that, we will study the necessary condition for a weak relative extremum, leading to the derivation and application of the Euler–Lagrange equation. Finally, we will wrap up by exploring Hamilton's principle and Lagrange's equations, with applications to mechanics and geodesic problems. This sequential approach will provide you with a clear pathway from fundamental definitions to practical applications.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 1.1 define the calculus of variation and illustrate its scope with basic examples,

SLO 1.2 explain the structure of Lagrange's functional and describe the role of its associated density,

SLO 1.3 derive the Euler–Lagrange equation as the necessary condition for a weak relative extremum,

SLO 1.4 apply the Euler–Lagrange equation to solve simple variational problems, and

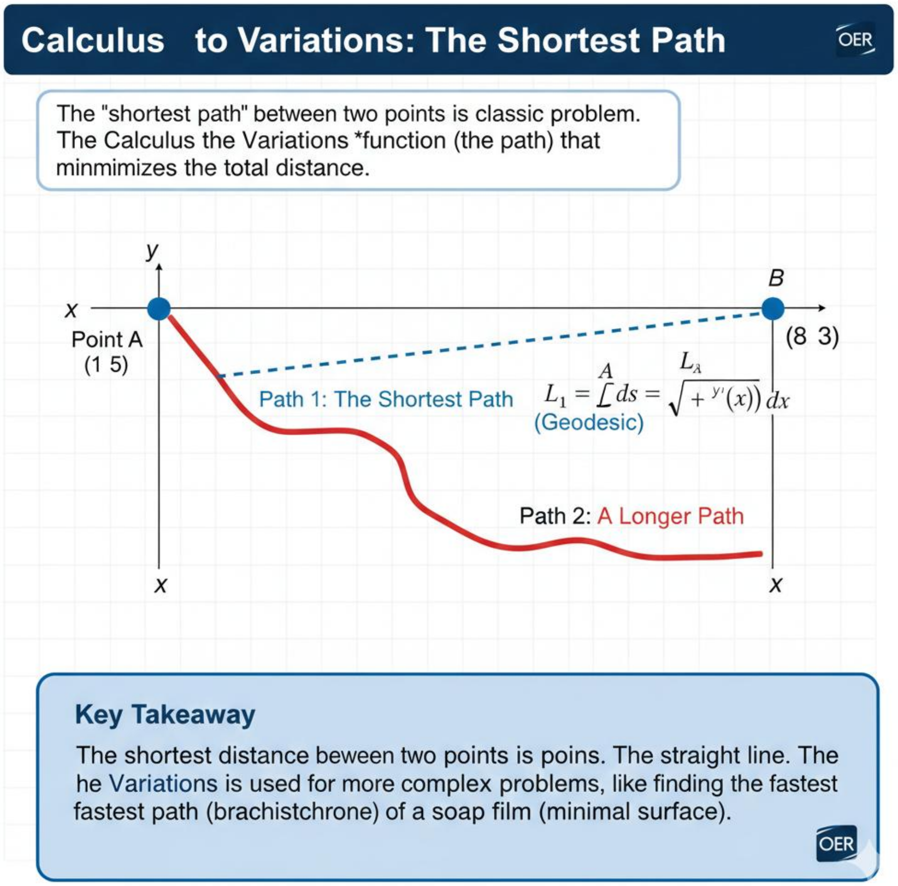
SLO 1.5 analyse Hamilton's principle and demonstrate how it leads to Lagrange's equations with applications to mechanics and geodesic problems.

1.1 Introduction To Calculus Of Variation

1.1.1 Definition and scope of calculus of variation

The calculus of variation is a branch of mathematics concerned with finding functions that optimise certain quantities, usually expressed as integrals. Unlike ordinary calculus, which deals with maxima and minima of functions, calculus of variation focuses on functionals, which are mappings from a set of functions to real numbers. For example, determining the shortest path between two points on a plane is a variational problem, since the path itself is a function and the length is a functional.

Fill in the blank: Calculus of variation deals with optimising __________ rather than ordinary functions.

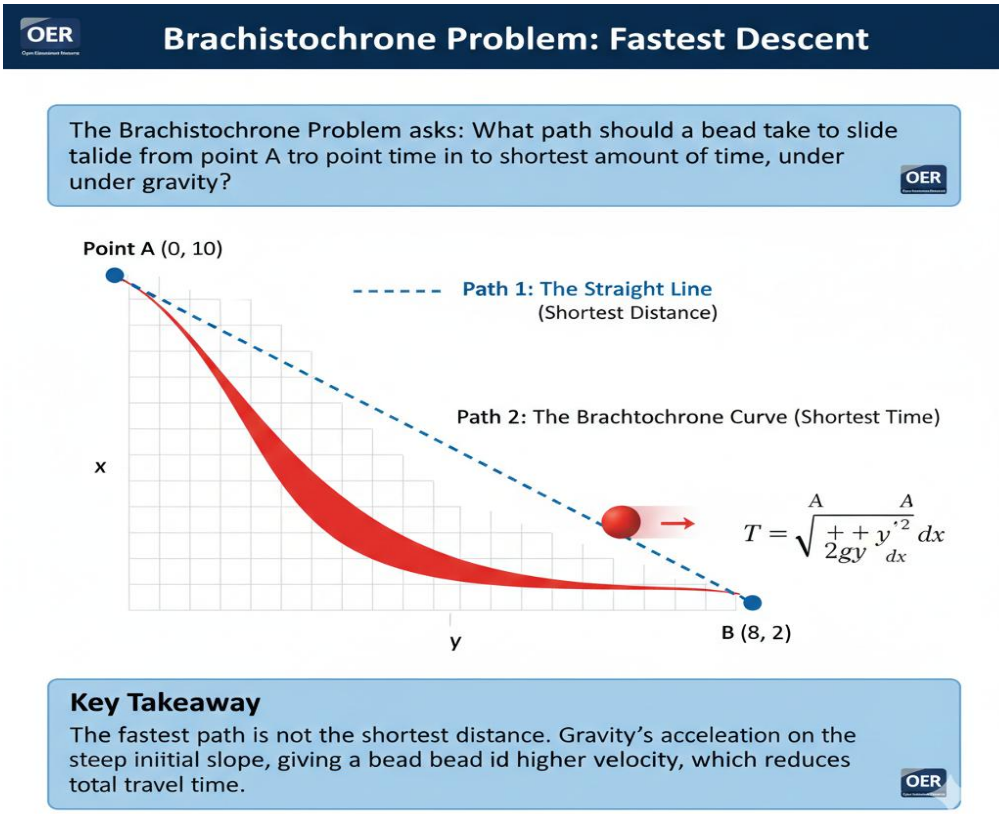


Example of a variational problem involving shortest paths.

1.1.2 Historical background and motivation

The origins of calculus of variation can be traced to problems posed by mathematicians such as Euler and Lagrange in the eighteenth century. One famous example is the brachistochrone problem, which asks for the curve along which a particle will descend under gravity in the shortest time. This problem motivated the development of systematic methods for dealing with functionals.

Fill in the blank: The brachistochrone problem seeks the curve of __________ descent under gravity.



The brachistochrone curve as a classical variational problem.

1.1.3 Basic examples of variational problems

Several classical problems illustrate the scope of calculus of variation:

Finding the shortest path between two points (geodesics).

Determining the shape of a hanging chain (catenary).

Minimising the action in mechanics (Hamilton's principle).

Each of these problems involves optimising a functional defined by an integral.

Fill in the blank: The shape of a hanging chain is known as a __________.

Examples of variational problems

Problem	Functional	Outcome
Shortest path	Length integral	Geodesic
Hanging chain	Potential energy integral	Catenary
Particle motion	Action integral	Hamilton's principle

Pilot questions 1.1

Calculus of variation deals with optimising: A. Functionals B. Ordinary functions C. Random numbers D. Undefined sets

The brachistochrone problem asks for: A. The curve of fastest descent under gravity B. The longest path between two points C. A random curve D. Undefinedness

The shape of a hanging chain is called: A. Catenary B. Geodesic C. Random curve D. Undefined

Functionals are mappings from: A. Functions to real numbers B. Real numbers to functions C. Random sets to random sets D. Undefinedness

A geodesic represents: A. The shortest path between two points B. The longest path between two points C. A random path D. Undefinedness

1.2 Lagrange's Functional And Associated Density

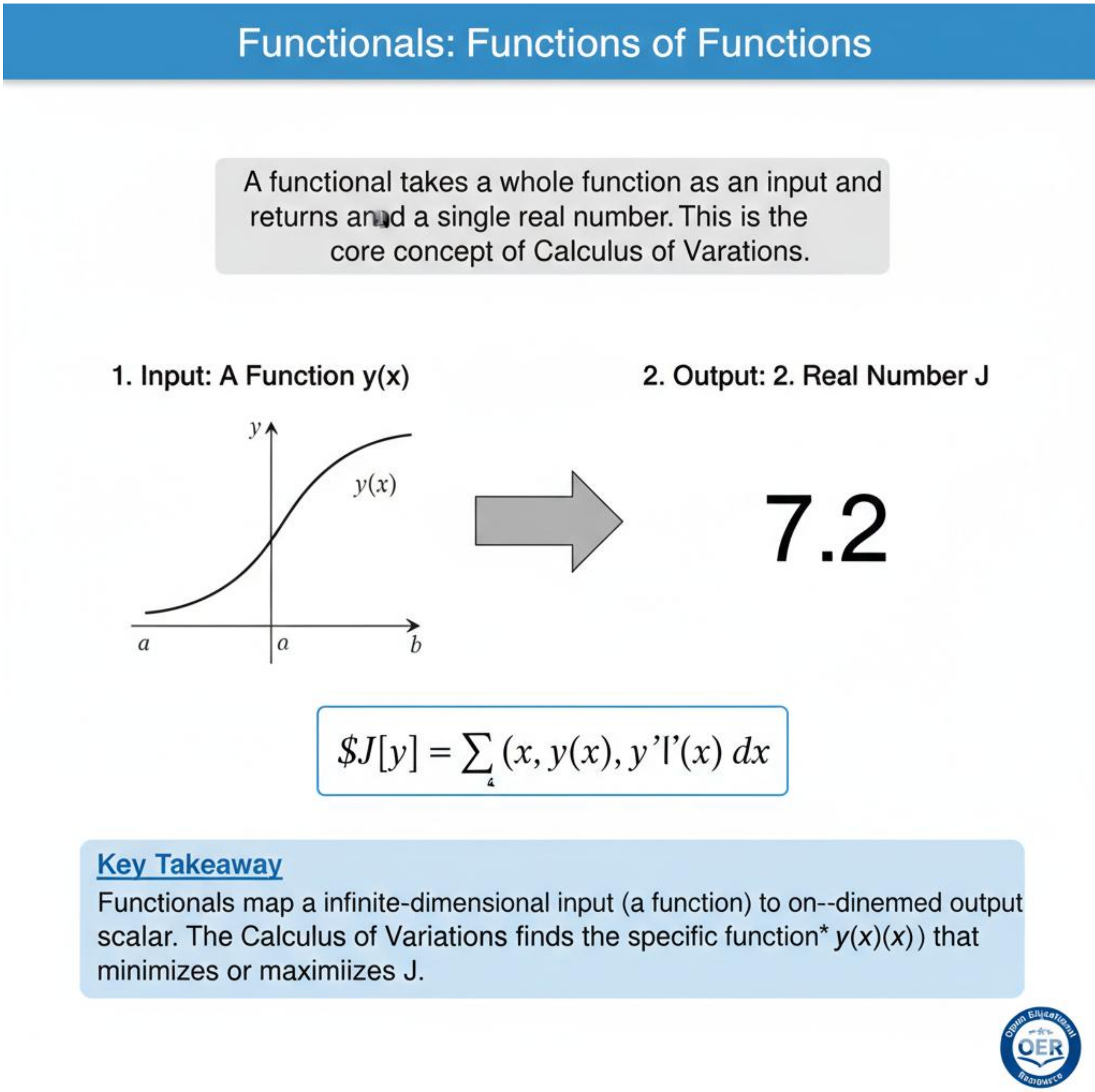
1.2.1 Definition of functional

A functional is a mapping from a set of functions to real numbers. In calculus of variation, functionals are usually expressed as integrals involving a function and its derivatives. For example,

$$J[y]=abF(x,y(x),y'(x))\ dx$$

is a functional, where $y(x)$is the function being varied, and Fis called the integrand or density.

Fill in the blank: A functional maps a set of __________ to real numbers.



: A functional mapping functions to real numbers. AI prompt

1.2.2 Lagrange's functional form

The Lagrange's functional is a specific type of functional expressed as:

$$J[y]=abL(x,y(x),y'(x))\ dx$$

Here, Lis called the Lagrangian, and it depends on the independent variable x, the function y(x), and its derivative y'(x). This form is central to variational problems, as it provides the framework for deriving necessary conditions for extrema.

Fill in the blank: In Lagrange's functional, the integrand is called the __________.

Components of Lagrange's functional

Symbol	Meaning
J[y]	Functional
L(x,y,y')	Lagrangian
y(x)	Function being varied
y'(x)	Derivative of the function

1.2.3 Associated density and its role in variational problems

The associated density refers to the integrand L(x,y,y')in the functional. It determines how the function and its derivative contribute to the value of the functional. By varying the function y(x), we seek to minimise or maximise the integral defined by this density. The associated density is therefore the key element that links the functional to physical or geometric problems.

Fill in the blank: The associated density in Lagrange's functional is represented by the __________.

Role of associated density

Term	**Symbol**	**Role & Description**
Functional	J[y]	The "total quantity" we want to minimize (e.g., total time, total energy, or total length).
Associated Density	L(x, y, y')	The **Lagrangian**. It describes the local "cost" or "density" at every point along the path.
The Action Integral	\int_a^b L dx	The mathematical process of summing

Term	Symbol	Role & Description
		up the density L from start to finish.
Euler-Lagrange Eq.	\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0	The fundamental "test" that the density must pass to ensure the path is an extremum.

Determines contribution of function and derivative

Defines the structure of the functional

Provides basis for deriving Euler–Lagrange equation

Pilot questions 1.2

A functional is: A. A mapping from functions to real numbers B. A mapping from numbers to functions C. A random set D. Undefined

Lagrange's functional is expressed as: A. J[y]=abL(x,y,y')dxB. J[y]=aby(x)dxC. J[y]=abxdxD. Undefined

In Lagrange's functional, the integrand is called: A. Lagrangian B. Functional C. Derivative D. Random term

The associated density refers to: A. The integrand L(x,y,y')B. The function y(x)C. The derivative y'(x)D. Undefined

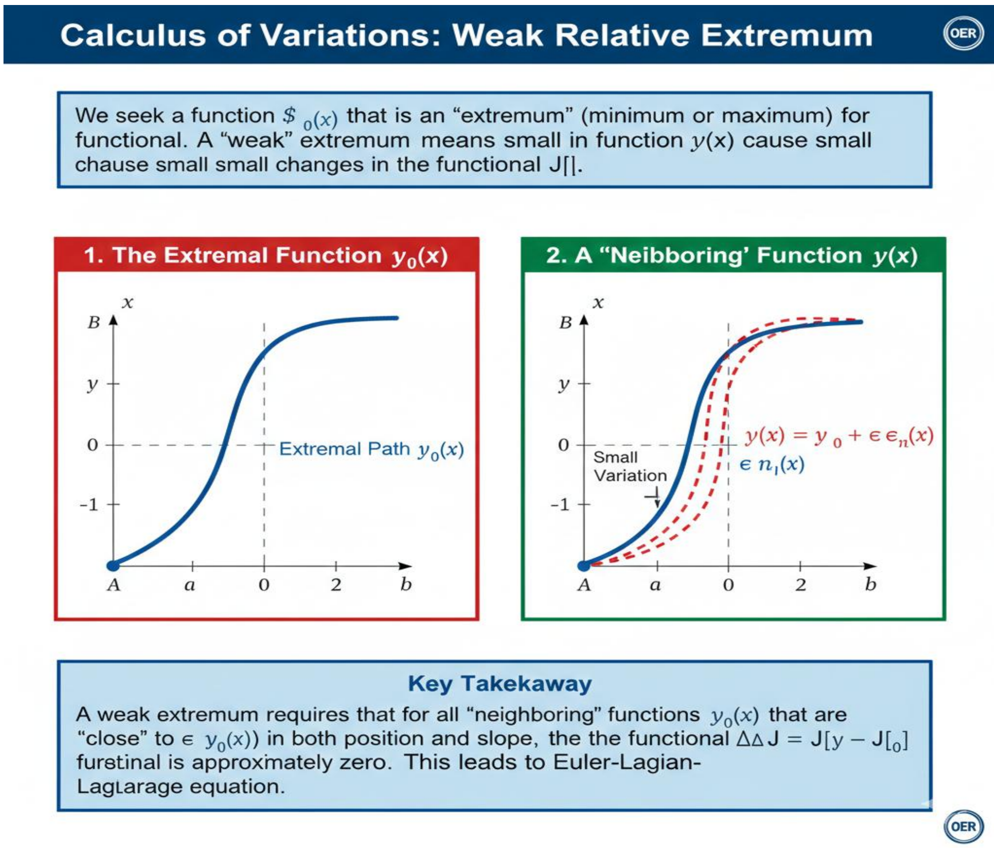
The associated density is important because it: A. Determines contributions of function and derivative B. Randomly assigns values C. Is irrelevant D. Undefined

1.3 Necessary Condition For A Weak Relative Extremum

1.3.1 Concept of extremum in variational calculus

In calculus of variation, an extremum refers to a function that minimises or maximises a given functional. A weak relative extremum means that the extremum holds for small variations of the function within a restricted neighbourhood. This concept is crucial because it allows us to determine whether a candidate function is optimal under slight perturbations.

Fill in the blank: A weak relative extremum holds for small __________ of the function.



Weak relative extremum illustrated with small variations around a function.

1.3.2 Derivation of Euler–Lagrange equation

The Euler–Lagrange equation provides the necessary condition for a weak relative extremum. Starting with the functional

$$J[y]=abL(x,y(x),y'(x))\ dx,$$

we consider a variation $y(x)+\varepsilon\eta(x)$, where $\eta(x)$is a small perturbation vanishing at the endpoints. By differentiating with respect to and setting the derivative to zero, we obtain:

Ly-ddx(Ly')=0.

This is the Euler–Lagrange equation, which must be satisfied by any extremal function.

Fill in the blank: The Euler–Lagrange equation is derived by considering small __________ of the function.

[Insert box here] Caption: Box 1.2: Euler–Lagrange equation

Ly-ddx(Ly')=0

1.3.3 Applications of Euler–Lagrange equation to simple problems

The Euler–Lagrange equation can be applied to solve classical variational problems:

Shortest path (geodesic): Minimising the length functional leads to straight lines in Euclidean space.

Catenary problem: The shape of a hanging chain is derived by minimising potential energy.

Optics (Fermat's principle): Light follows the path that minimises travel time, which can be derived using the Euler–Lagrange equation.

Fill in the blank: The Euler–Lagrange equation applied to optics leads to __________ principle.

Applications of Euler–Lagrange equation

Problem	Functional	Result
Shortest path	Length integral	Straight line (geodesic)
Hanging chain	Potential energy	Catenary curve
Optics	Travel time	Fermat's principle

Pilot questions 1.3

A weak relative extremum holds for: A. Small variations of the function B. Large variations of the function C. Random changes D. Undefinedness

The Euler–Lagrange equation is derived by: A. Considering small variations of the function B. Differentiating random functions C. Ignoring variations D. Undefinedness

The Euler–Lagrange equation is expressed as: A. Ly-ddx(Ly')=0B. y'=0C. L=yD. Undefined

Applying the Euler–Lagrange equation to the shortest path problem gives: A. Straight line (geodesic) B. Random curve C. Undefined path D. Circle

In optics, the Euler–Lagrange equation leads to: A. Fermat's principle B. Random principle C. Undefined principle D. Newton's principle

1.4 Hamilton's Principle And Lagrange's Equations

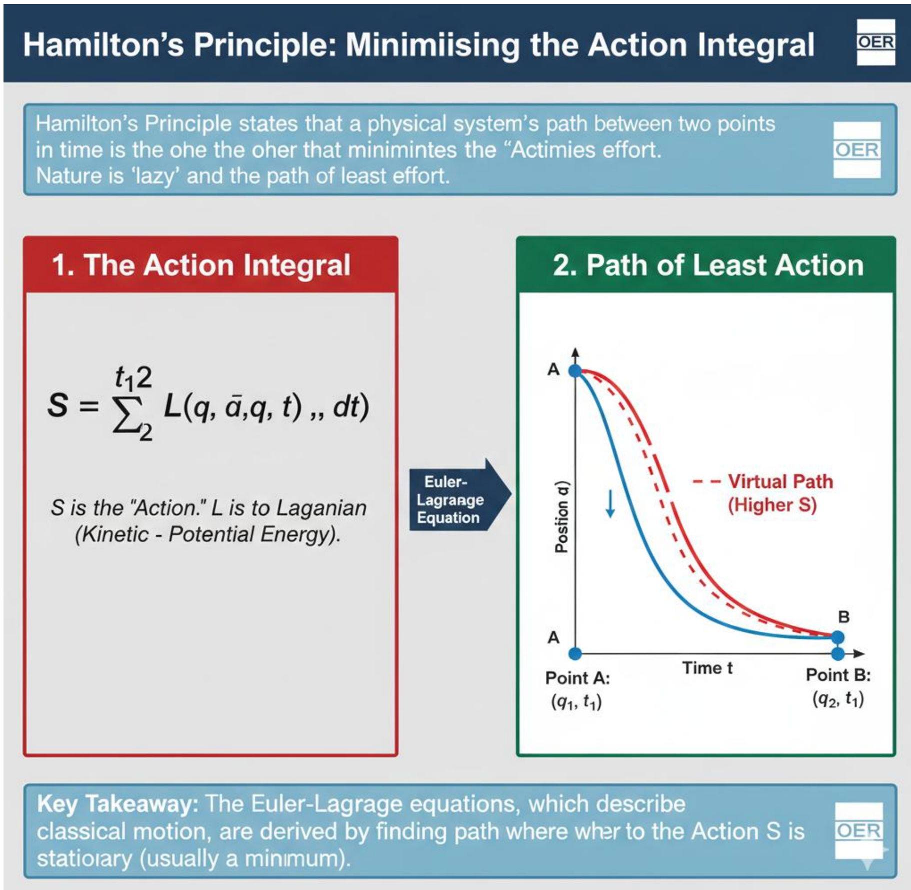
1.4.1 Statement of Hamilton's principle

Hamilton's principle states that the actual path taken by a physical system between two configurations is the one that makes the action integral stationary. The action is defined as the integral of the Lagrangian over time:

$$S=t1t2L(q,q,t)\ dt$$

where qrepresents the generalised coordinates and qtheir time derivatives. This principle provides a variational foundation for mechanics.

Fill in the blank: Hamilton's principle states that the actual path makes the __________ integral stationary.



Hamilton's principle illustrated with a particle path. AI prompt:

1.4.2 Derivation of Lagrange's equations from Hamilton's principle

By applying Hamilton's principle to the action functional, we obtain the Lagrange's equations of motion:

ddt(Lqi)-Lqi=0

for each generalised coordinate qi. These equations describe the dynamics of mechanical systems and are equivalent to Newton's laws, but expressed in terms of energy functions rather than forces.

Fill in the blank: Lagrange's equations are derived from Hamilton's principle by varying the __________ functional.

Lagrange's equations of motion

ddt(Lqi)-Lqi=0

1.4.3 Applications to mechanics and geodesic problems

Lagrange's equations have wide applications:

- Classical mechanics: They provide a systematic way to derive equations of motion for systems with constraints.
- Geodesic problems: In geometry, minimising length functionals leads to geodesics, which are the shortest paths on surfaces.
- Field theory: In physics, Hamilton's principle extends to fields, leading to equations such as Maxwell's equations in electromagnetism.

Fill in the blank: In geometry, minimising length functionals leads to __________ on surfaces.

Applications of Lagrange's equations

Area	Application	Outcome
Mechanics	Equations of motion	System dynamics
Geometry	Length functional	Geodesics
Field theory	Action principle	Field equations

Pilot questions 1.4

- Hamilton's principle states that the actual path makes the: A. Action integral stationary B. Energy integral maximum C. Random integral stationary D. Undefined integral
- The action is defined as: A. Integral of the Lagrangian over time B. Integral of force over distance C. Random integral D. Undefined
- Lagrange's equations are derived from: A. Hamilton's principle B. Newton's second law directly C. Random principle D. Undefined
- In geometry, minimising length functionals leads to: A. Geodesics B. Random curves C. Undefined paths D. Circles only
- Lagrange's equations can be applied to: A. Mechanics, geometry, and field theory B. Random sets only C. Undefined problems D. Arithmetic only

Series summary

In this Series, we introduced the calculus of variation, a branch of mathematics concerned with finding functions that optimise functionals. We began with the definition, scope, and historical motivation, highlighting classical problems such as the brachistochrone and catenary. We then examined Lagrange's functional and its associated density, which form the foundation of variational problems. Next, we derived the Euler–Lagrange equation as the necessary condition for a weak relative extremum and applied it to problems in geometry and optics. Finally, we studied Hamilton's principle and showed how it leads to Lagrange's equations of motion, with applications in mechanics, geodesics, and field theory. This Series has provided you with the essential tools to analyse variational problems and prepared you for more advanced topics in subsequent Series.

Glossary of terms

Functional: A mapping from a set of functions to real numbers, often expressed as an integral.

Calculus of variation: A branch of mathematics concerned with optimising functionals.

Weak relative extremum: A function that minimises or maximises a functional under small variations.

Euler–Lagrange equation: The necessary condition for a weak relative extremum, expressed as Ly-ddx(Ly')=0.

Lagrangian: The integrand in Lagrange's functional, depending on x, y(x), and y'(x).

Associated density: The integrand of the functional, determining contributions of the function and its derivative.

Hamilton's principle: States that the actual path of a system makes the action integral stationary.

Action: The integral of the Lagrangian over time.

Lagrange's equations: Equations of motion derived from Hamilton's principle, expressed as ddt(Lqi)-Lqi=0.

Geodesic: The shortest path between two points on a surface.

Catenary: The curve formed by a hanging chain under gravity.

Concept check questions 1

Upon completing this Series, you should be able to:

(SLO 1.1) Define the calculus of variation and give one classical example of a variational problem.

(SLO 1.2) Explain the structure of Lagrange's functional and identify its associated density.

(SLO 1.3) Derive the Euler–Lagrange equation from a given functional.

(SLO 1.4) Apply the Euler–Lagrange equation to solve the catenary problem.

(SLO 1.5) Analyse Hamilton's principle and demonstrate how it leads to Lagrange's equations of motion.

Answers to pilot questions 1

Pilot questions 1.1: 1A, 2A, 3A, 4A, 5A Pilot questions 1.2: 1A, 2A, 3A, 4A, 5A
Pilot questions 1.3: 1A, 2A, 3A, 4A, 5A Pilot questions 1.4: 1A, 2A, 3A, 4A, 5A

Series 2: Extremum Conditions, Hamilton's Principle And Geodesic Problems

2.1 Extremum conditions in variational problems

2.1.1 Weak and strong extrema
2.1.2 First variation and necessary condition
2.1.3 Second variation and sufficient condition

2.2 Hamilton's principle revisited

2.2.1 Restating Hamilton's principle

2.2.2 Connection to Euler–Lagrange equation
2.2.3 Physical interpretation and applications

2.3 Lagrange's equations and applications

2.3.1 Derivation of Lagrange's equations
2.3.2 Applications in mechanics
2.3.3 Applications in geometry and field theory

2.4 Geodesic problems in variational calculus

2.4.1 Definition of geodesics
2.4.2 Variational formulation of geodesics
2.4.3 Applications of geodesics

Introduction

In the previous Series, we explored the foundations of the calculus of variation, including Lagrange's functional, the Euler–Lagrange equation, and Hamilton's principle. Building on that foundation, this Series focuses on the conditions under which functionals attain extrema, and how these conditions connect to physical and geometric problems.

The aim of this Series is to enable you to analyse extremum conditions in variational problems, to apply Hamilton's principle in mechanics, to derive and use Lagrange's equations, and to solve geodesic problems using variational methods.

We shall commence this Series by examining extremum conditions in variational problems, clarifying the difference between weak and strong extrema. Next, we will revisit Hamilton's principle in greater detail, showing its central role in mechanics. Following that, we will study Lagrange's equations and their applications to constrained systems. Finally, we will conclude with geodesic problems, where variational methods are used to determine shortest paths on surfaces. This progression will provide you with both theoretical insight and practical problem-solving skills.

Series learning outcomes

Upon completing this Series, you should be able to:

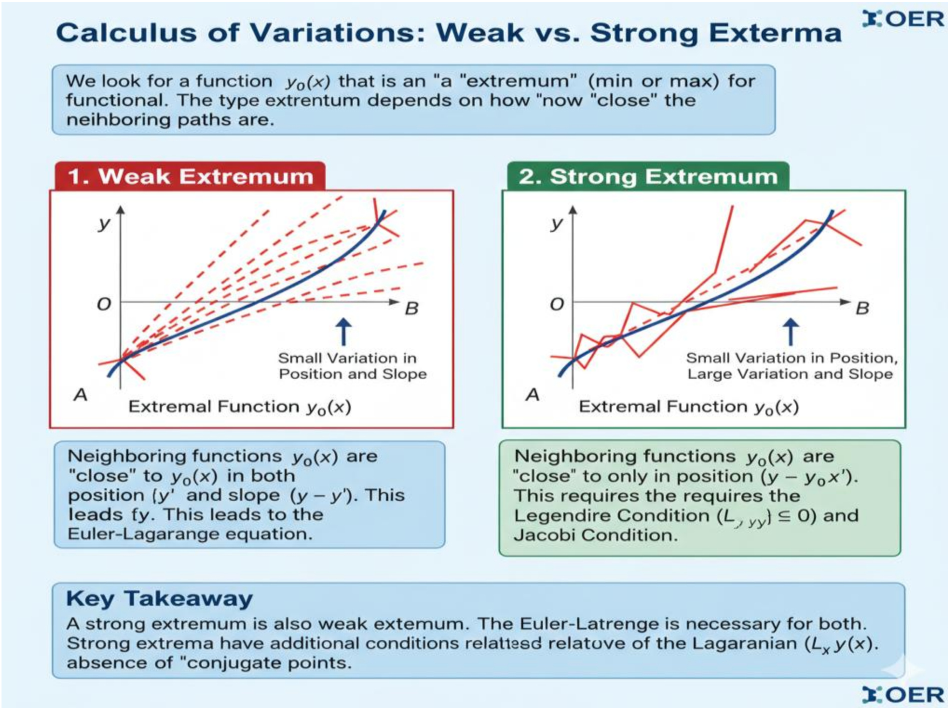
SLO 2.1 analyse extremum conditions in variational problems, distinguishing between weak and strong extrema, **SLO 2.2** apply Hamilton's principle to derive equations of motion in mechanics, **SLO 2.3** demonstrate the use of Lagrange's equations in solving constrained systems, and **SLO 2.4** solve geodesic problems using variational methods.

2.1 Extremum Conditions In Variational Problems

2.1.1 Weak and strong extrema

An **extremum** in variational calculus refers to a function that minimises or maximises a functional. A **weak extremum** occurs when the extremal property holds for small variations of the function in a restricted neighbourhood. In contrast, a **strong extremum** requires the extremal property to hold for all admissible variations, not just small ones. This distinction is important because many physical problems only require weak extrema, while certain geometric problems demand strong extrema.

Fill in the blank: A strong extremum requires the extremal property to hold for all __________ variations.



Comparison of weak and strong extrema in variational problems."

2.1.2 First variation and necessary condition

The **first variation** of a functional measures how the functional changes under small perturbations of the function. If the first variation vanishes, the function is a candidate for an extremum. Mathematically, for a functional

$$J[y]=abL(x,y,y')\ dx,$$

the first variation is given by

$$\delta J=ab(Ly\eta+Ly'')dx,$$

where $\eta(x)$is the perturbation. Setting $\delta J=0$leads to the Euler–Lagrange equation.

Fill in the blank: The first variation of a functional must __________ for a function to be extremal.

First variation condition

$$\delta J=0\Rightarrow Ly\text{-}ddx(Ly')=0$$

2.1.3 Second variation and sufficient condition

The **second variation** provides information about whether the extremum is a minimum or maximum. If the second variation is positive for all admissible perturbations, the extremum is a minimum. If it is negative, the extremum is a maximum. This parallels the role of the second derivative in ordinary calculus.

Fill in the blank: A positive second variation indicates that the extremum is a __________.

Role of first and second variations

Variation	**Condition**	**Outcome**
First variation	Must vanish	Candidate extremum
Second variation positive	Sufficient condition	Minimum
Second variation negative	Sufficient condition	Maximum

Pilot questions 2.1

A weak extremum holds for: A. Small variations B. Large variations C. Random changes D. Undefinedness

A strong extremum requires: A. All admissible variations B. Only small variations C. Random variations D. Undefinedness

The first variation of a functional must: A. Vanish for extremum B. Be positive C. Be negative D. Undefined

A positive second variation indicates: A. Minimum B. Maximum C. Random outcome D. Undefined

The Euler–Lagrange equation arises from: A. Setting the first variation to zero B. Random substitution C. Ignoring variations D. Undefinedness

2.2 Hamilton's Principle Revisited

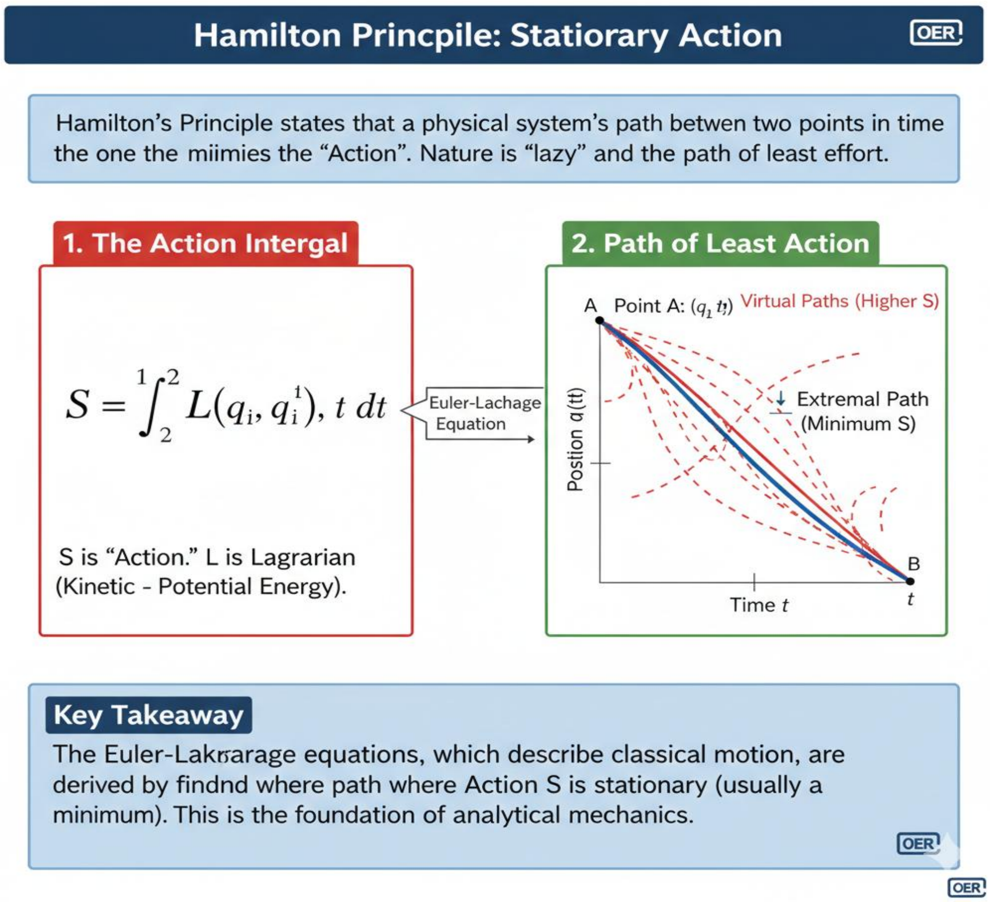
2.2.1 Restating Hamilton's principle

Hamilton's principle asserts that the actual motion of a system between two fixed times is such that the **action integral** is stationary. The action is defined as

$$S=t1t2L(q,q,t)\ dt,$$

where Lis the **Lagrangian**, qrepresents the generalised coordinates, and qtheir time derivatives. This principle provides a variational foundation for mechanics, unifying diverse physical laws under a single framework.

Fill in the blank: Hamilton's principle states that the actual motion makes the __________ integral stationary.



Hamilton's principle illustrated with a particle path. AI prompt:

2.2.2 Connection to Euler–Lagrange equation

By applying Hamilton's principle to the action functional, we obtain the **Euler–Lagrange equation** in mechanics:

ddt(Lqi)-Lqi=0.

This equation governs the dynamics of systems and generalises Newton's laws. It is particularly powerful because it accommodates constraints naturally through the choice of generalised coordinates.

Fill in the blank: The Euler–Lagrange equation in mechanics is derived from the __________ functional.

[Insert box here] Caption: Box 2.2: Euler–Lagrange equation in mechanics

$$ddt(Lqi)-Lqi=0$$

2.2.3 Physical interpretation and applications

Hamilton's principle has wide applications:

In **classical mechanics**, it provides a systematic way to derive equations of motion for systems with constraints.

In **optics**, Fermat's principle of least time is a special case of Hamilton's principle.

In **field theory**, Hamilton's principle extends to continuous systems, leading to equations such as Maxwell's equations in electromagnetism.

Fill in the blank: Fermat's principle in optics is a special case of __________ principle.

Applications of Hamilton's principle

Area	Application	Outcome
Mechanics	Equations of motion	System dynamics
Optics	Fermat's principle	Path of light
Field theory	Action principle	Field equations

Pilot questions 2.2

Hamilton's principle states that the actual motion makes the: A. Action integral stationary B. Energy integral maximum C. Random integral stationary D. Undefined integral

The action is defined as: A. Integral of the Lagrangian over time B. Integral of force over distance C. Random integral D. Undefined

The Euler–Lagrange equation in mechanics is derived from: A. The action functional B. Newton's second law directly C. Random principle D. Undefined

Fermat's principle in optics is a special case of: A. Hamilton's principle B. Newton's principle C. Random principle D. Undefined

Hamilton's principle can be applied to: A. Mechanics, optics, and field theory B. Random sets only C. Undefined problems D. Arithmetic onl

2.3 Lagrange’s Equations And Applications

2.3.1 Derivation of Lagrange’s equations

From Hamilton’s principle, we recall that the **action functional** is given by

$$S=t1t2L(q,q,t)\ dt,$$

where Lis the **Lagrangian**, qare the generalised coordinates, and qtheir time derivatives. By requiring the action to be stationary under small variations, we obtain the **Lagrange’s equations of motion**:

$$ddt(Lqi)-Lqi=0.$$

These equations generalise Newton’s laws and are particularly useful for systems with constraints.

Fill in the blank: Lagrange’s equations are derived by requiring the __________ to be stationary.

[Insert box here] Caption: Box 2.3: Lagrange’s equations of motion

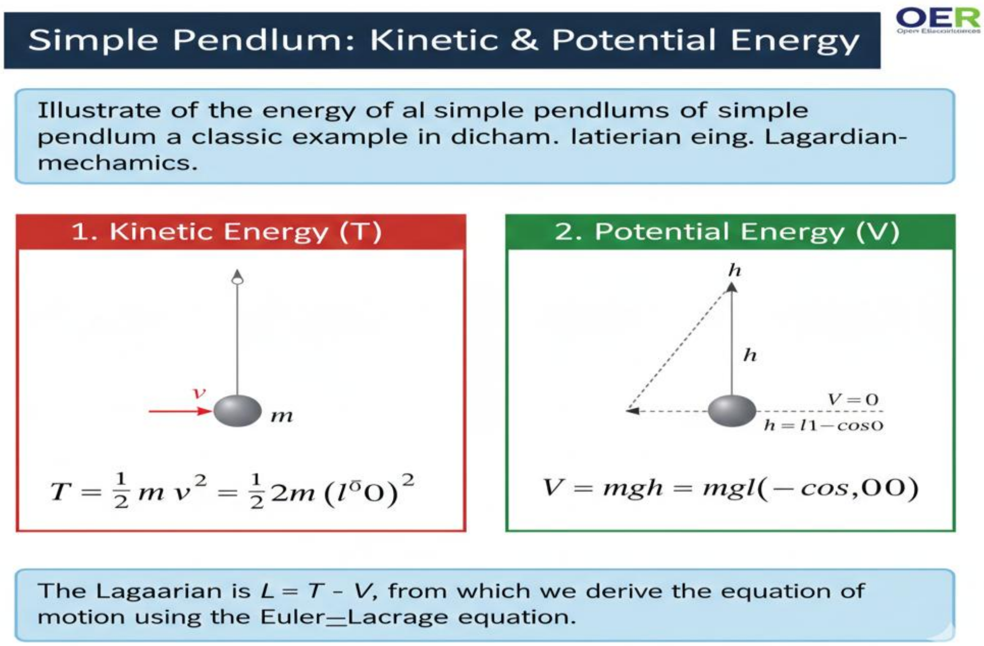
$$ddt(Lqi)-Lqi=0$$

2.3.2 Applications in mechanics

Lagrange’s equations are widely applied in mechanics:

- **Simple pendulum:** The motion of a pendulum can be derived using the Lagrangian L=T-V, where Tis kinetic energy and Vis potential energy.
- **Double pendulum:** Complex systems with multiple degrees of freedom can be analysed systematically.
- **Constrained systems:** Lagrange’s equations naturally incorporate constraints without requiring direct force analysis.

Fill in the blank: The Lagrangian is defined as the difference between __________ energy and potential energy.



Application of Lagrange's equations to a simple pendulum. AI prompt:

2.3.3 Applications in geometry and field theory

Beyond mechanics, Lagrange's equations extend to geometry and field theory:

Geodesics: Minimising length functionals on surfaces leads to geodesic equations.

Electromagnetism: Maxwell's equations can be derived from a Lagrangian density using Hamilton's principle.

General relativity: Einstein's field equations arise from a variational principle applied to the spacetime metric.

Fill in the blank: In general relativity, Einstein's field equations are derived from a __________ principle.

Applications of Lagrange's equations

Area	Application	Outcome
Mechanics	Pendulum motion	Equations of motion
Geometry	Length functional	Geodesics
Field theory	Lagrangian density	Maxwell's equations
Relativity	Variational principle	Einstein's field equations

Pilot questions 2.3

Lagrange's equations are derived by requiring the: A. Action to be stationary B. Energy to be maximum C. Random integral to vanish D. Undefined principle

The Lagrangian is defined as: A. Kinetic energy minus potential energy B. Potential energy minus kinetic energy C. Random energy D. Undefined

Lagrange's equations are particularly useful for: A. Constrained systems B. Random sets C. Undefined problems D. Arithmetic only

In geometry, minimising length functionals leads to: A. Geodesics B. Random curves C. Undefined paths D. Circles only

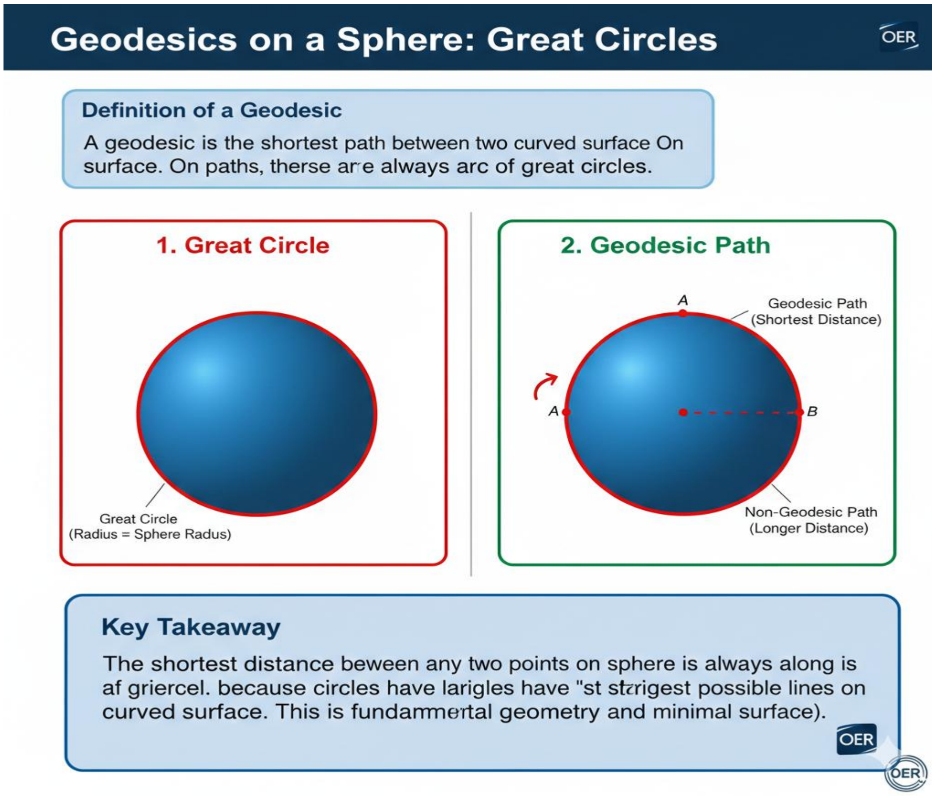
Einstein's field equations are derived from: A. A variational principle B. Newton's second law C. Random principle D. Undefined

2.4 Geodesic Problems In Variational Calculus

2.4.1 Definition of geodesics

A **geodesic** is the shortest path between two points on a surface. In Euclidean space, geodesics are straight lines, but on curved surfaces such as spheres, geodesics are arcs of great circles. Geodesics are central in geometry and physics because they generalise the concept of straight lines to curved spaces.

Fill in the blank: On a sphere, geodesics are arcs of __________ circles.



Geodesics on a sphere illustrated as arcs of great circles.

2.4.2 Variational formulation of geodesics

The geodesic problem can be formulated using the calculus of variation. The length of a curve on a surface is given by the functional

$$J[y]=ab1+(dydx)2\ dx.$$

Minimising this functional leads to the Euler–Lagrange equation, which provides the condition for geodesics. On curved surfaces, the functional incorporates the metric of the surface, making the problem more general.

Fill in the blank: The geodesic problem is solved by minimising the __________ functional.

[Insert box here] Caption: Box 2.4: Variational formulation of geodesics

$$J[y]=ab1+(dydx)2\ dx$$

2.4.3 Applications of geodesics

Geodesics have important applications:

Geometry: They define shortest paths on surfaces and are used in differential geometry.

Physics: In general relativity, geodesics describe the motion of particles under gravity in curved spacetime.

Navigation: Geodesics on Earth's surface correspond to great circle routes, which are used in air and sea travel.

Fill in the blank: In general relativity, geodesics describe the motion of particles in __________ spacetime.

[Insert table here] Caption: Table 2.4: Applications of geodesics

Area	**Application**	**Outcome**
Geometry	Shortest paths	Differential geometry
Physics	Motion of particles	Curved spacetime
Navigation	Great circle routes	Air and sea travel

Pilot questions 2.4

On a sphere, geodesics are arcs of: A. Great circles B. Random curves C. Undefined paths D. Small circles

The geodesic problem is solved by minimising the: A. Length functional B. Energy functional only C. Random integral D. Undefined functional

Geodesics in Euclidean space are: A. Straight lines B. Random curves C. Undefined paths D. Circles

In general relativity, geodesics describe: A. Motion of particles in curved spacetime B. Random motion C. Undefined motion D. Motion in flat space only

Geodesics on Earth's surface correspond to: A. Great circle routes B. Random paths C. Undefined routes D. Straight lines only

Series summary

In this Series, we examined the conditions under which functionals attain extrema in variational calculus. We began by distinguishing between **weak** and **strong extrema**, and introduced the concepts of first and second variations as necessary and sufficient conditions. We then revisited **Hamilton's principle**, restating its role in mechanics and showing how it leads to the Euler–Lagrange equation. Next, we studied **Lagrange's equations**, deriving them from Hamilton's principle and applying them to mechanics, geometry, and field theory. Finally, we explored **geodesic problems**, where variational methods are used to determine shortest paths on surfaces, with applications in geometry, physics, and navigation. This Series has strengthened your ability to analyse extremum conditions and apply variational principles to both physical and geometric problems.

Glossary of terms

Weak extremum: An extremum that holds for small variations of the function.

Strong extremum: An extremum that holds for all admissible variations.

First variation: The change in a functional under small perturbations, used to identify candidate extrema.

Second variation: Determines whether an extremum is a minimum or maximum.

Hamilton's principle: States that the actual motion of a system makes the action integral stationary.

Action: The integral of the Lagrangian over time.

Euler–Lagrange equation (mechanics): Derived from Hamilton's principle, governs system dynamics.

Lagrange's equations: Generalised equations of motion for systems with constraints.

Geodesic: The shortest path between two points on a surface, generalising straight lines to curved spaces.

Great circle: A geodesic on a sphere.

Concept check questions 2

Upon completing this Series, you should be able to:

(SLO 2.1) Analyse the difference between weak and strong extrema in variational problems.

(SLO 2.1) Explain the role of first and second variations in determining extrema.

(SLO 2.2) Apply Hamilton's principle to derive the Euler–Lagrange equation in mechanics.

(SLO 2.3) Demonstrate the use of Lagrange's equations in solving the motion of a simple pendulum.

(SLO 2.4) Solve a geodesic problem by minimising the length functional on a sphere.

Answers to pilot questions 2

Pilot questions 2.1: 1A, 2A, 3A, 4A, 5A **Pilot questions 2.2:** 1A, 2A, 3A, 4A, 5A
Pilot questions 2.3: 1A, 2A, 3A, 4A, 5A **Pilot questions 2.4:** 1A, 2A, 3A, 4A, 5A

Series 3: Du Bois-Raymond Equation, Corner Conditions And Variable End-Points

3.1 Du Bois-Raymond equation and its derivation

3.1.1 Background to the Du Bois-Raymond condition

3.1.2 Derivation of the Du Bois-Raymond equation
3.1.3 Significance of the Du Bois-Raymond equation

3.2 Corner conditions in variational problems

3.2.1 Introduction to corner conditions
3.2.2 Derivation of corner conditions
3.2.3 Applications of corner conditions

3.3 Variable end-points and transversality conditions

3.3.1 Introduction to variable end-points

3.3.2 Transversality conditions

3.3.3 Applications of transversality conditions

3.4 Applications of Du Bois-Raymond equation and corner conditions

Introduction

In the previous Series, we studied extremum conditions, Hamilton's principle, Lagrange's equations, and geodesic problems. These provided a strong foundation for understanding how variational methods apply to mechanics and geometry. In this Series, we move further into specialised conditions that arise in variational calculus, particularly when dealing with discontinuities and variable boundaries.

The aim of this Series is to equip you with the ability to derive and interpret the **Du Bois-Raymond equation**, to explain and apply **corner conditions**, and to analyse problems with **variable end-points** using transversality conditions.

We shall commence this Series by deriving the Du Bois-Raymond equation and examining its role in variational problems. Next, we will study corner conditions, which arise when extremals have discontinuities in their derivatives. Following that, we will explore variable end-points and transversality conditions, which generalise the Euler–Lagrange framework. Finally, we will conclude with applications of these concepts to practical variational problems.

Series learning outcomes

Upon completing this Series, you should be able to:

SLO 3.1 derive the Du Bois-Raymond equation and explain its significance in variational calculus,

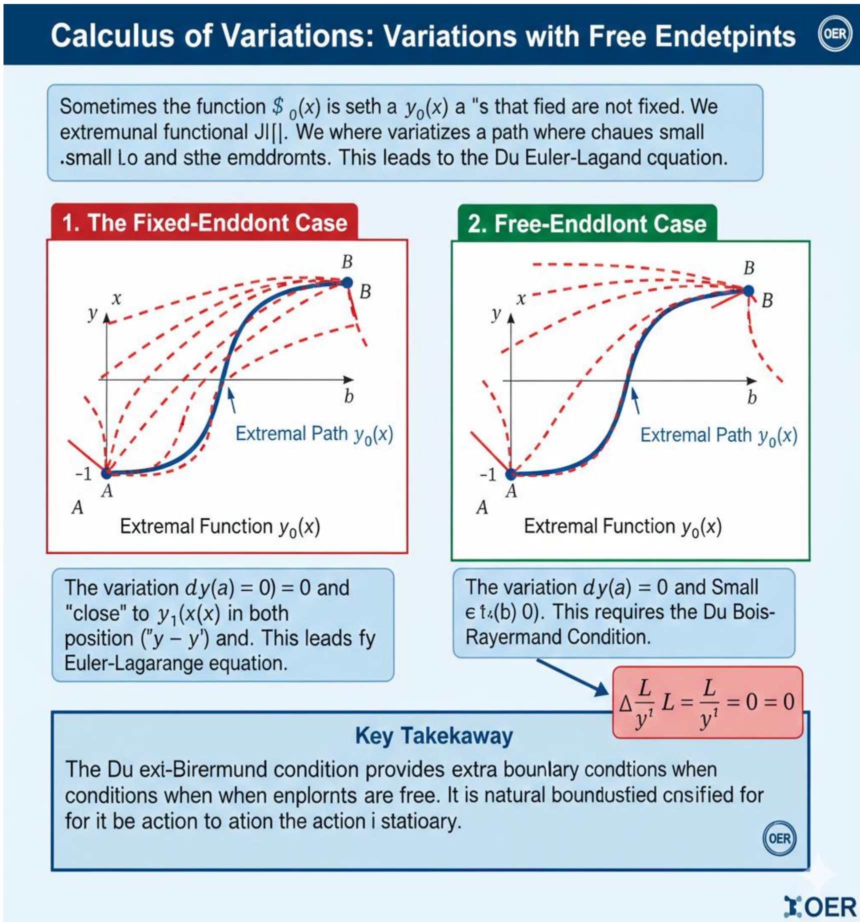
SLO 3.2 describe and apply corner conditions in problems with discontinuities,
SLO 3.3 analyse variable end-point problems using transversality conditions, and
SLO 3.4 apply the Du Bois-Raymond equation and corner conditions to solve practical variational problems.

3.1 Du Bois-Raymond Equation And Its Derivation

3.1.1 Background to the Du Bois-Raymond condition

In variational calculus, the **Du Bois-Raymond equation** is an alternative necessary condition for an extremum of a functional. While the **Euler–Lagrange equation** arises from the integral term of the first variation, the Du Bois-Raymond condition emerges from the boundary term. It becomes relevant when variations at the endpoints are not fixed, making it especially useful in problems with free or variable boundaries.

Fill in the blank: The Du Bois-Raymond equation arises when variations at the __________ are not fixed.



Endpoint variations leading to the Du Bois-Raymond condition.

3.1.2 Derivation of the Du Bois-Raymond equation

Consider the functional

$$J[y]=abL(x,y,y')\ dx.$$

The first variation is

$$\delta J=ab(Ly\eta+Ly'')dx.$$

Integrating the second term by parts gives

$$\delta J=ab(Ly-ddxLy')\ dx+[Ly'\eta]ab.$$

The first integral leads to the Euler–Lagrange equation. The boundary term, however, produces the **Du Bois-Raymond condition**:

$$Ly'=\text{constant}.$$

Fill in the blank: The Du Bois-Raymond condition comes directly from the __________ term in the first variation.

[Insert box here] Caption: Box 3.1: Du Bois-Raymond condition

$$L_{y'} = \text{constant}$$

3.1.3 Significance of the Du Bois-Raymond equation

The Du Bois-Raymond equation ensures consistency between extremals and boundary conditions. It complements the Euler–Lagrange equation by addressing cases where endpoints are free or variable. This makes it particularly important in physics and geometry, where boundaries often shift or are not predetermined.

Fill in the blank: The Du Bois-Raymond equation is especially important in problems with __________ boundaries.

: Comparison of Euler–Lagrange and Du Bois-Raymond conditions

Condition	Source	Application
Euler–Lagrange	Integral term	General extremals
Du Bois-Raymond	Boundary term	Free or variable boundaries

Pilot questions 3.1

The Du Bois-Raymond equation arises when variations at the: A. Endpoints are not fixed B. Interior points only C. Random values D. Undefined

The Du Bois-Raymond condition comes from the: A. Boundary term in the first variation B. Integral term only C. Random substitution D. Undefined

The Du Bois-Raymond condition is expressed as: A. $L_{y'}$=constantB. L_y=0C. y'=0D. Undefined

The Euler–Lagrange equation arises from the: A. Integral term in the first variation B. Boundary term only C. Random principle D. Undefined

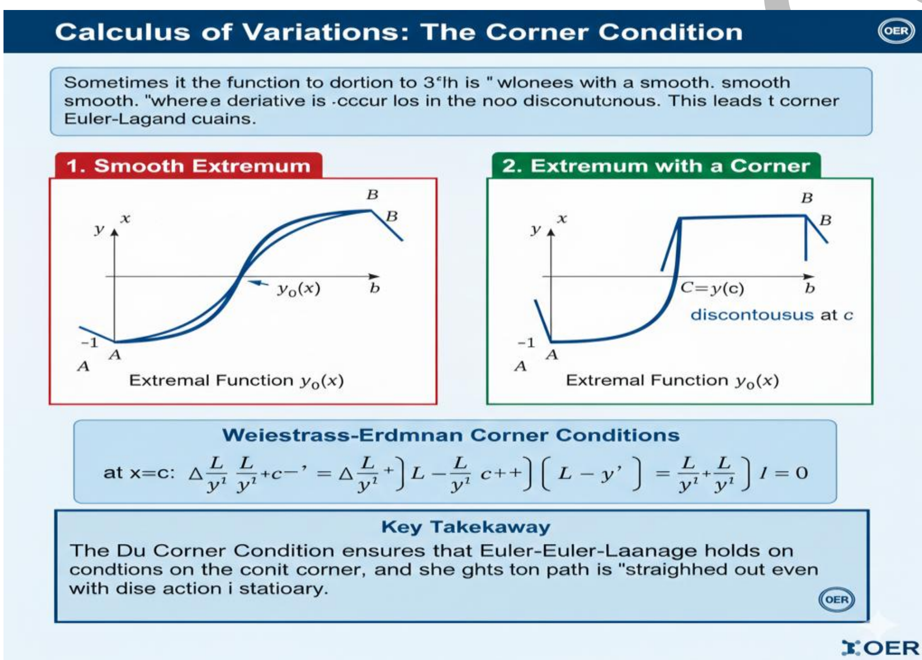
The Du Bois-Raymond equation is especially useful in problems with: A. Free or variable boundaries B. Fixed boundaries only C. Random paths D. Undefined

3.2 Corner Conditions In Variational Problems

3.2.1 Introduction to corner conditions

In variational problems, extremals are often assumed to be smooth functions. However, in some cases, the extremal may have points where its derivative is discontinuous. These points are called **corners**, and the conditions that must be satisfied at such points are known as **corner conditions**. They ensure that the extremal remains valid even when the path changes direction abruptly.

Fill in the blank: Corner conditions arise when extremals have discontinuities in their __________.



A corner in an extremal function. AI prompt:

3.2.2 Derivation of corner conditions

Consider a functional

$$J[y]=abL(x,y,y')\ dx,$$

where the extremal has a corner at $x=c$. At this point, the function $y(x)$is continuous, but its derivative $y'(x)$may have a jump. By analysing the first variation across the corner, we obtain the **Weierstrass–Erdmann corner conditions**:

$$[L_{y'}]_{c^-}^{c^+}=0,[L-y'L_{y'}]_{c^-}^{c^+}=0.$$

These conditions ensure that the extremal remains consistent across the discontinuity.

Fill in the blank: The Weierstrass–Erdmann corner conditions require continuity of both $L_{y'}$ and __________ across the corner.

Weierstrass–Erdmann corner conditions

$$[L_{y'}]_{c^-}^{c^+}=0,[L-y'L_{y'}]_{c^-}^{c^+}=0$$

3.2.3 Applications of corner conditions

Corner conditions are important in practical problems where extremals are not smooth:

- **Optics:** Light rays may change direction abruptly at boundaries, requiring corner conditions.
- **Mechanics:** Systems with piecewise-defined constraints often produce extremals with corners.
- **Geometry:** Piecewise geodesics must satisfy corner conditions at junction points.

Fill in the blank: In optics, corner conditions apply when light rays change __________ at boundaries.

Applications of corner conditions

Area	**Application**	**Outcome**
Optics	Light rays at boundaries	Direction change consistency
Mechanics	Piecewise constraints	Valid extremals
Geometry	Piecewise geodesics	Continuity at junctions

Pilot questions 3.2

- Corner conditions arise when extremals have discontinuities in their: A. Derivatives B. Values C. Random paths D. Undefined

The Weierstrass–Erdmann corner conditions require continuity of: A. Ly'and L-y'Ly'B. Only yC. Random terms D. Undefined

At a corner, the function y(x)is: A. Continuous but derivative may jump B. Discontinuous C. Random D. Undefined

In optics, corner conditions apply when light rays change: A. Direction at boundaries B. Colour randomly C. Undefined property D. Shape

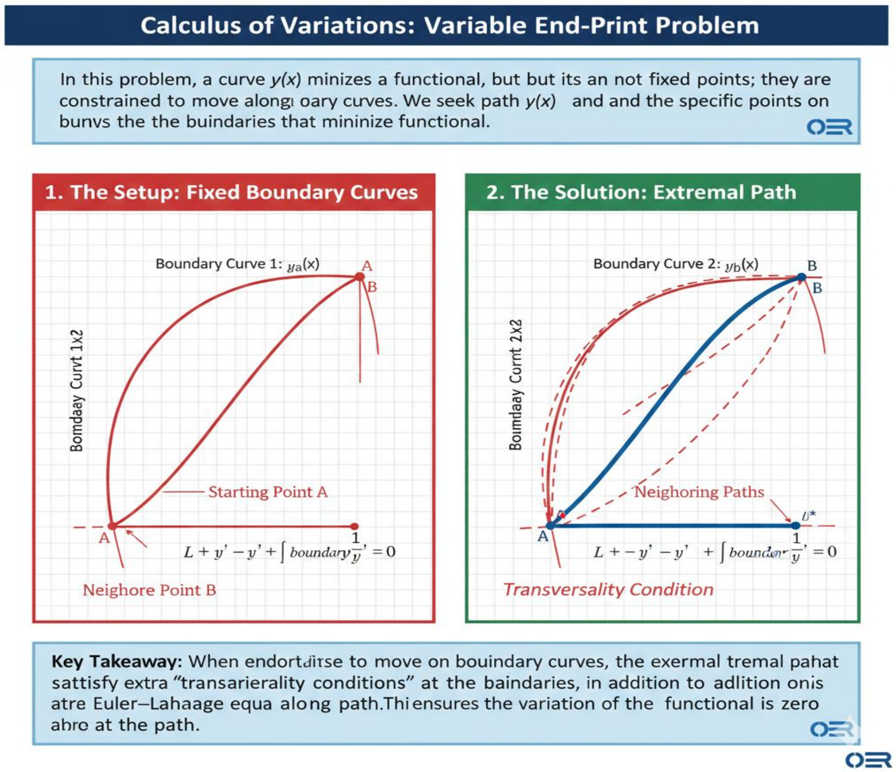
Corner conditions are useful in: A. Optics, mechanics, and geometry B. Random sets only C. Undefined problems D. Arithmetic only

3.3 Variable End-Points And Transversality Conditions

3.3.1 Introduction to variable end-points

In many variational problems, the end-points of the extremal curve are fixed. However, in some cases, the end-points are allowed to vary along given curves or surfaces. These are called **variable end-point problems**, and they require additional conditions beyond the Euler–Lagrange equation to determine valid extremals.

Fill in the blank: Variable end-point problems occur when the end-points are allowed to __________ along given curves or surfaces.



Variable end-points in variational problems. AI prompt: “

3.3.2 Transversality conditions

The additional conditions required for variable end-point problems are called **transversality conditions**. They ensure that the extremal meets the boundary curves or surfaces in a consistent way. Mathematically, if the end-point lies on a curve defined by $\phi(x,y)=0$, the transversality condition is:

$$(L-y'L_{y'})\delta x+L_{y'}\delta y=0.$$

This condition complements the Euler–Lagrange equation by accounting for variations at the boundaries.

Fill in the blank: Transversality conditions ensure that extremals meet the __________ curves or surfaces consistently.

Transversality condition for variable end-points

$$(L-y'L_{y'})\delta x+L_{y'}\delta y=0$$

3.3.3 Applications of transversality conditions

Transversality conditions are applied in several contexts:

- **Geometry:** Determining geodesics that meet boundary curves at specific angles.
- **Mechanics:** Analysing systems where constraints allow free movement of end-points.
- **Optimisation:** Problems in economics and engineering where boundaries are flexible.

Fill in the blank: In geometry, transversality conditions determine geodesics that meet boundary curves at specific __________.

Applications of transversality conditions

Area	Application	Outcome
Geometry	Geodesics with variable end-points	Boundary angle conditions
Mechanics	Systems with free boundaries	Valid extremals
Optimisation	Flexible constraints	Optimal solutions

Pilot questions 3.3

- Variable end-point problems occur when end-points are allowed to: A. Vary along given curves or surfaces B. Remain fixed only C. Randomly change D. Undefined
- Transversality conditions ensure that extremals meet: A. Boundary curves or surfaces consistently B. Random points only C. Undefined paths D. Arbitrary values
- The transversality condition is expressed as: A. $(L-y'L_{y'})\delta x+L_{y'}\delta y=0$B. $y'=0$C. $L=y$D. Undefined
- In geometry, transversality conditions determine geodesics that meet boundary curves at specific: A. Angles B. Random points C. Undefined values D. Circles
- Transversality conditions are useful in: A. Geometry, mechanics, and optimisation B. Random sets only C. Undefined problems D. Arithmetic only

3.4. Combined applications in physics and engineering

In practice, both the Du Bois-Raymond equation and corner conditions often appear together:

Structural engineering: Analysing beams or rods with variable endpoints and discontinuities.

Control theory: Optimising systems with flexible boundaries and abrupt changes in trajectories.

Physics: Variational principles in optics and mechanics frequently require both conditions for complete solutions.

Fill in the blank: In structural engineering, Du Bois-Raymond and corner conditions are applied to analyse __________ with variable endpoints.

applications of Du Bois-Raymond and corner conditions

Structural engineering: beams and rods with variable endpoints

Control theory: flexible trajectories

Physics: optics and mechanics with discontinuities

Pilot questions 3.4

The Du Bois-Raymond equation is especially useful in problems with: A. Free or variable boundaries B. Fixed boundaries only C. Random paths D. Undefined

Corner conditions ensure continuity when extremals have abrupt changes in their: A. Derivatives B. Values C. Random sets D. Undefined

In optics, corner conditions apply when light rays: A. Change direction at boundaries B. Change colour randomly C. Undefined behaviour D. Remain straight only

In structural engineering, Du Bois-Raymond and corner conditions are applied to analyse: A. Beams and rods with variable endpoints B. Random sets only C. Undefined structures D. Arithmetic only

Combined applications of Du Bois-Raymond and corner conditions are found in: A. Physics, engineering, and control theory B. Random sets only C. Undefined problems D. Arithmetic only

Series summary

In this Series, we extended the foundations of variational calculus to more advanced conditions. We began with the **Du Bois-Raymond equation**, derived from boundary terms in the first variation, which provides an alternative necessary condition for extremals when endpoints are free or variable. We then studied **corner conditions**, which ensure continuity of extremals at points where derivatives are discontinuous. Following that, we explored **variable end-points** and introduced **transversality conditions**, which generalise the Euler–Lagrange framework to problems where boundaries are flexible. Finally, we applied these concepts to practical problems in mechanics, geometry, optics, and engineering. Together, these tools equip you to handle variational problems with discontinuities and variable boundaries.

Glossary of terms

Du Bois-Raymond equation: A necessary condition for extremals derived from boundary terms in the first variation.

Boundary term: The part of the first variation that arises when variations at endpoints are not fixed.

Corner conditions: Conditions ensuring continuity of extremals at points where derivatives are discontinuous.

Weierstrass–Erdmann conditions: Specific corner conditions requiring continuity of Ly'and L-y'Ly'.

Variable end-points: Problems where the endpoints of extremals are allowed to move along given curves or surfaces.

Transversality conditions: Additional conditions ensuring extremals meet boundary curves or surfaces consistently.

Free boundary problems: Variational problems where endpoints or boundaries are not fixed.

Concept check questions 3

Upon completing this Series, you should be able to:

(SLO 3.1) Derive the Du Bois-Raymond equation and explain its role in variational calculus.

(SLO 3.2) State the Weierstrass–Erdmann corner conditions and describe their significance.

(SLO 3.3) Apply transversality conditions to a problem with variable end-points.

(SLO 3.4) Analyse a mechanical system with free boundaries using the Du Bois-Raymond equation.

(SLO 3.4) Solve a piecewise geodesic problem by applying corner conditions at junction points.

Answers to pilot questions 3

Pilot questions 3.1: 1A, 2A, 3A, 4A, 5A **Pilot questions 3.2:** 1A, 2A, 3A, 4A, 5A **Pilot questions 3.3:** 1A, 2A, 3A, 4A, 5A **Pilot questions 3.4:** 1A, 2A, 3A, 4A, 5A