La-Net

MTH405

GENERAL TOPOLOGY

MTH 405: General Topology

Series 1: Foundations of Topological Spaces

Introduction

When studying mathematics, one often begins with familiar structures such as sets and functions. However, as we move into the field of topology, we encounter a new way of thinking about spaces, one that focuses not on distances but on the relationships between points and the openness of sets. This perspective is fundamental to modern analysis and provides the language for many advanced areas of mathematics. By starting with the foundations of topological spaces, you will be equipped to understand the deeper concepts that follow in this course.

The aim of this Series is to introduce you to the basic building blocks of topology, including the definition of topological spaces, the role of open and closed sets, neighbourhoods, and the comparison of coarser and finer topologies. This will give you the essential framework needed to progress confidently through the rest of the course.

We shall commence this Series by defining what a topological space is and examining simple examples to build intuition. Then, we will study open and closed sets, clarifying how they interact and why they are central to topology. Following that, we will explore neighbourhoods, which provide a local perspective on points within a space. Finally, we will wrap up by comparing coarser and finer topologies, showing how different choices of topology can alter the structure of a space.

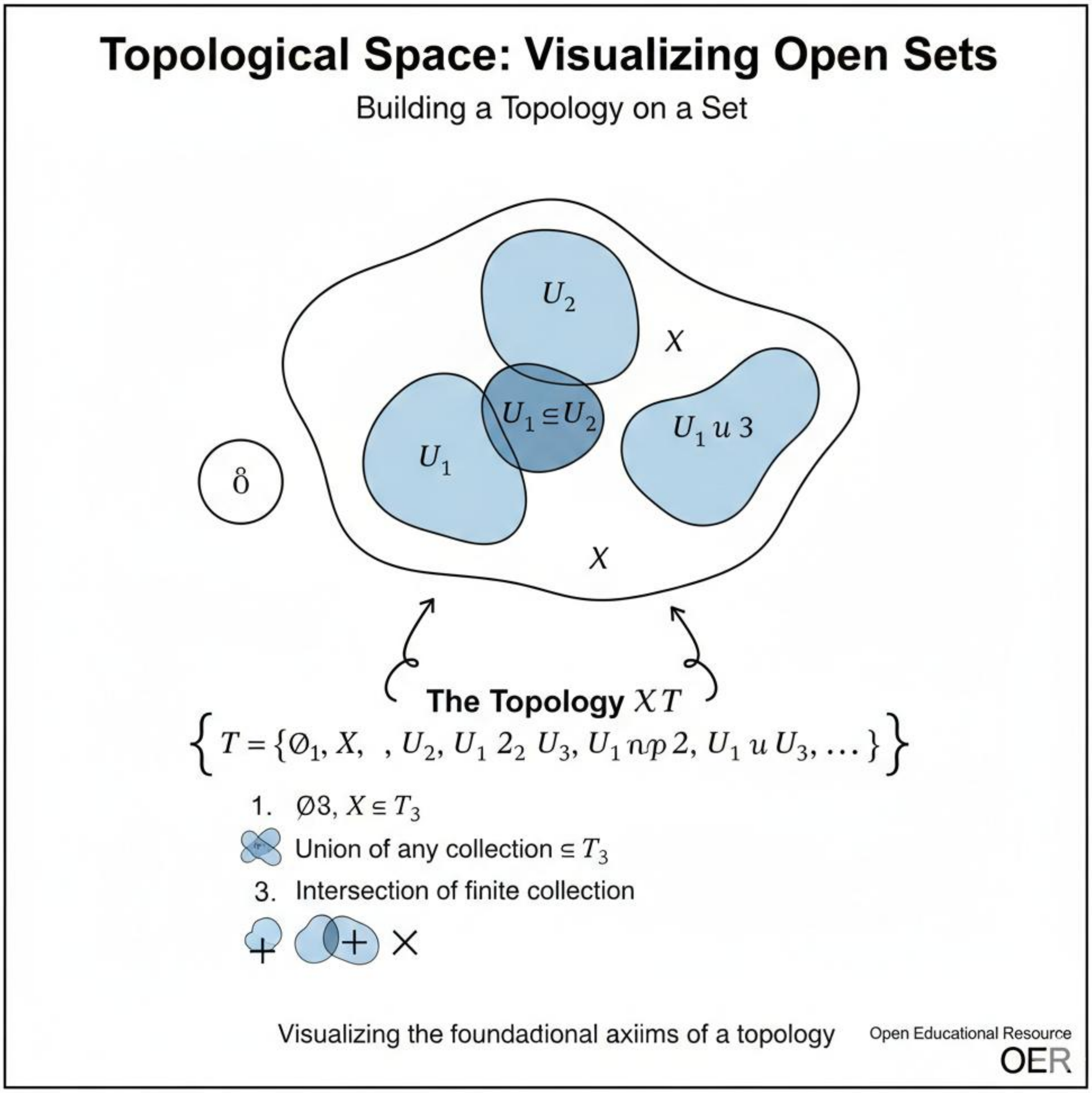
1.1 Definition Of Topological Spaces

Topology begins with the idea of a **topological space**, which is a set equipped with a collection of subsets that satisfy certain rules. A **topological space** is defined as a pair (X,T), where $X$is a set and $T$is a collection of subsets of Xcalled a **topology**. The subsets in $T$are known as **open sets**, and they must satisfy three conditions: the empty set and the whole set $X$are open, the union of any collection of open sets is open, and the intersection of any finite number of open sets is open.

Fill in the blank: A topological space is a pair (X,T), where $T$is a collection of __________ sets.

1.1.1 Concept of a set and topology

A **set** is simply a collection of distinct objects, often called elements. A **topology** on a set is a way of specifying which subsets of the set are considered open. This structure allows us to study continuity, convergence, and connectedness without relying on distances.



set with a topology defined by open subsets.

Fill in the blank: A topology on a set specifies which subsets are considered __________.

1.1.2 Examples of topological spaces

Discrete topology: Every subset of Xis open.

Trivial topology: Only the empty set and the whole set are open.

Standard topology on R: Open intervals form the basis of the topology.

table showing examples of discrete, trivial, and standard topologies

Topology Type	Formal Definition / Open Sets	Key Characteristic	Typical Example
Trivial (Indiscrete)	\tau = \{\emptyset, X\}	The "coarsest" possible topology; only the empty set and the entire space are open.	On X = \{1, 2, 3\}, the open sets are just \{\emptyset, \{1, 2, 3\}\}.
Discrete	\tau = \mathcal{P}(X) (The Power Set)	The "finest" possible topology; **every** subset of X is considered an open set.	On X = \{a, b\}$, the open sets are \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
Standard (Euclidean)	\tau = \{U \subseteq \mathbb{R} \mid U \text{ is a union of open intervals } (a, b)\}	Based on the concept of distance (metric); open sets are defined by intervals without endpoints.	On the real line \mathbb{R}, any interval like (0, 1) or (3, 5) \cup (10, 12) is open.

Fill in the blank: In the discrete topology, __________ subset of the set is open.

1.1.3 Motivation for studying topology

Topology provides a framework for generalising concepts such as continuity and convergence beyond metric spaces. It allows mathematicians to study spaces in terms of their structure rather than their geometry. This abstraction is essential in analysis, algebra, and geometry.

Fill in the blank: Topology studies spaces in terms of their __________ rather than their geometry.

Pilot questions 1.1

A topological space is defined as: A. A set with a collection of open sets satisfying certain rules B. A set with only closed sets C. A set with random subsets D. A set with a metric only

In the discrete topology: A. Every subset is open B. Only the empty set is open C. Only the whole set is open D. No set is open

The trivial topology consists of: A. The empty set and the whole set B. All subsets of the set C. Only singletons D. Open intervals

The standard topology on Ris generated by: A. Open intervals B. Closed intervals C. Singletons D. Random subsets

Topology studies spaces in terms of: A. Structure B. Geometry only C. Randomness D. Distance only

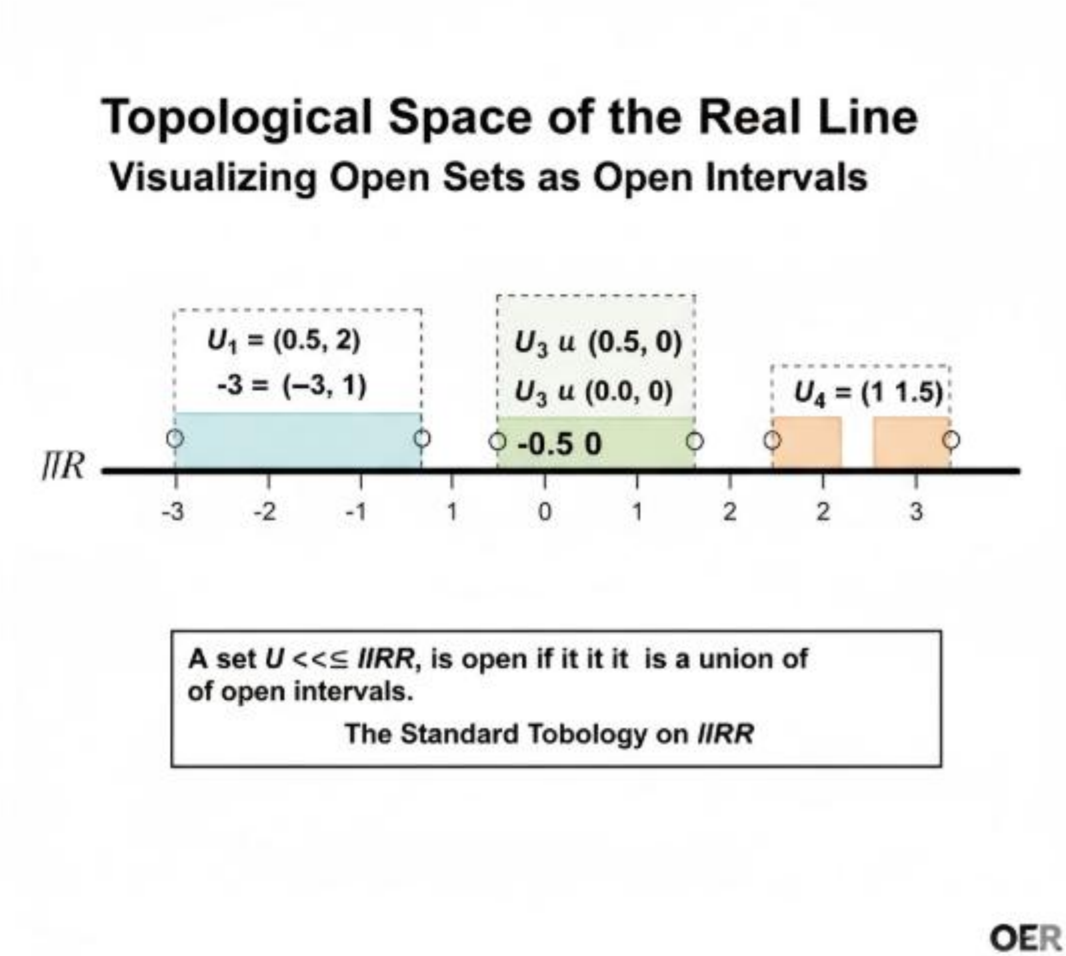
1.2 Open And Closed Sets

In topology, the concepts of **open sets** and **closed sets** are fundamental. An **open set** is a subset of a topological space that belongs to the topology, meaning it satisfies the rules defined for openness. A **closed set** is the complement of an open set within the space. These two notions are not opposites in the everyday sense, but rather complementary: a set can be both open and closed, or neither, depending on the topology.

Fill in the blank: A closed set in a topological space is defined as the __________ of an open set.

1.2.1 Definition of open sets

An **open set** is any subset of a topological space that is included in the topology. For example, in the standard topology on R, open intervals such as (a,b)are open sets. Open sets are used to define continuity and convergence in topology.

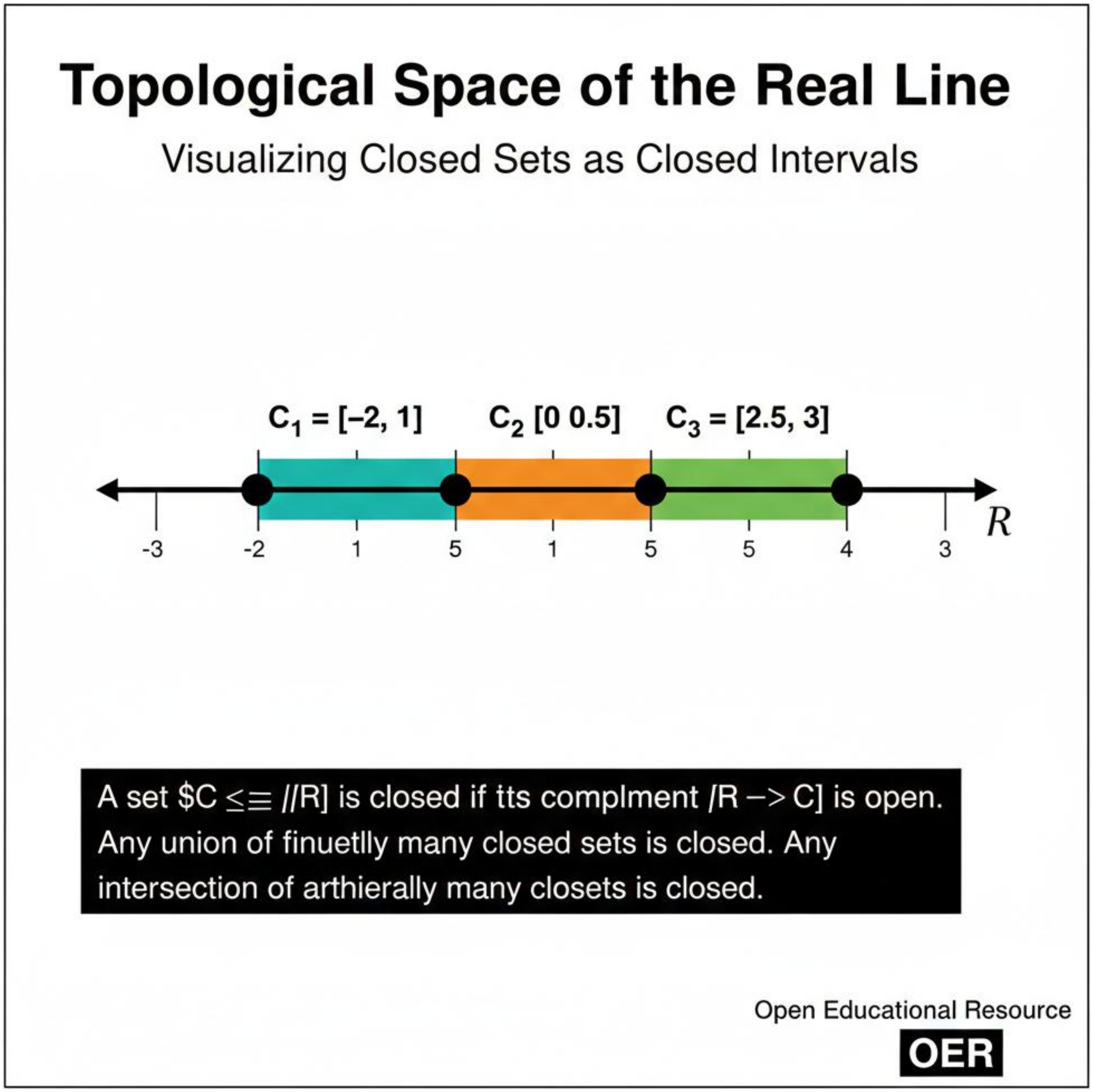


Open intervals as open sets in the standard topology on R**.**

Fill in the blank: In the standard topology on R, sets of the form (a,b)are __________ sets.

1.2.2 Definition of closed sets

A **closed set** is the complement of an open set. For example, in the standard topology on R, closed intervals such as [a,b]are closed sets. Closed sets are important in defining concepts such as compactness and closure.



Closed intervals as closed sets in the standard topology on R**.**

Fill in the blank: In the standard topology on R, sets of the form [a,b]are ___________ sets.

1.2.3 Relationship between open and closed sets

Open and closed sets are related through complementation. If a set is open, its complement is closed, and vice versa. Some sets can be both open and closed, called **clopen sets**. For example, in the discrete topology, every set is both open and closed.

[Insert box here] Caption: Box 1.1: Relationship between open and closed sets

Open sets: belong to the topology

Closed sets: complements of open sets

Clopen sets: both open and closed

Set Type	Characteristic	Example (in $\mathbb{R}$ with standard topology)
Open	Does not contain its boundary; every point has "breathing room."	$(0,1)$
Closed	Contains its boundary; it contains all its limit points.	$[0,1]$
Clopen	Both open and closed simultaneously.	$\emptyset$ and the entire space $\mathbb{R}$
Neither	Contains some boundary points but not others.	$[0,1)$

Fill in the blank: A set that is both open and closed is called a __________ set.

Pilot questions 1.2

An open set in a topological space is: A. A subset included in the topology B. Always finite C. Always closed D. Random

A closed set in a topological space is: A. The complement of an open set B. Always finite C. Always open D. Random

In the standard topology on R, sets of the form (a,b)are: A. Open sets B. Closed sets C. Clopen sets D. Random sets

In the standard topology on R, sets of the form [a,b]are: A. Closed sets B. Open sets C. Clopen sets D. Random sets

A set that is both open and closed is called: A. Clopen set B. Random set C. Empty set only D. Infinite set

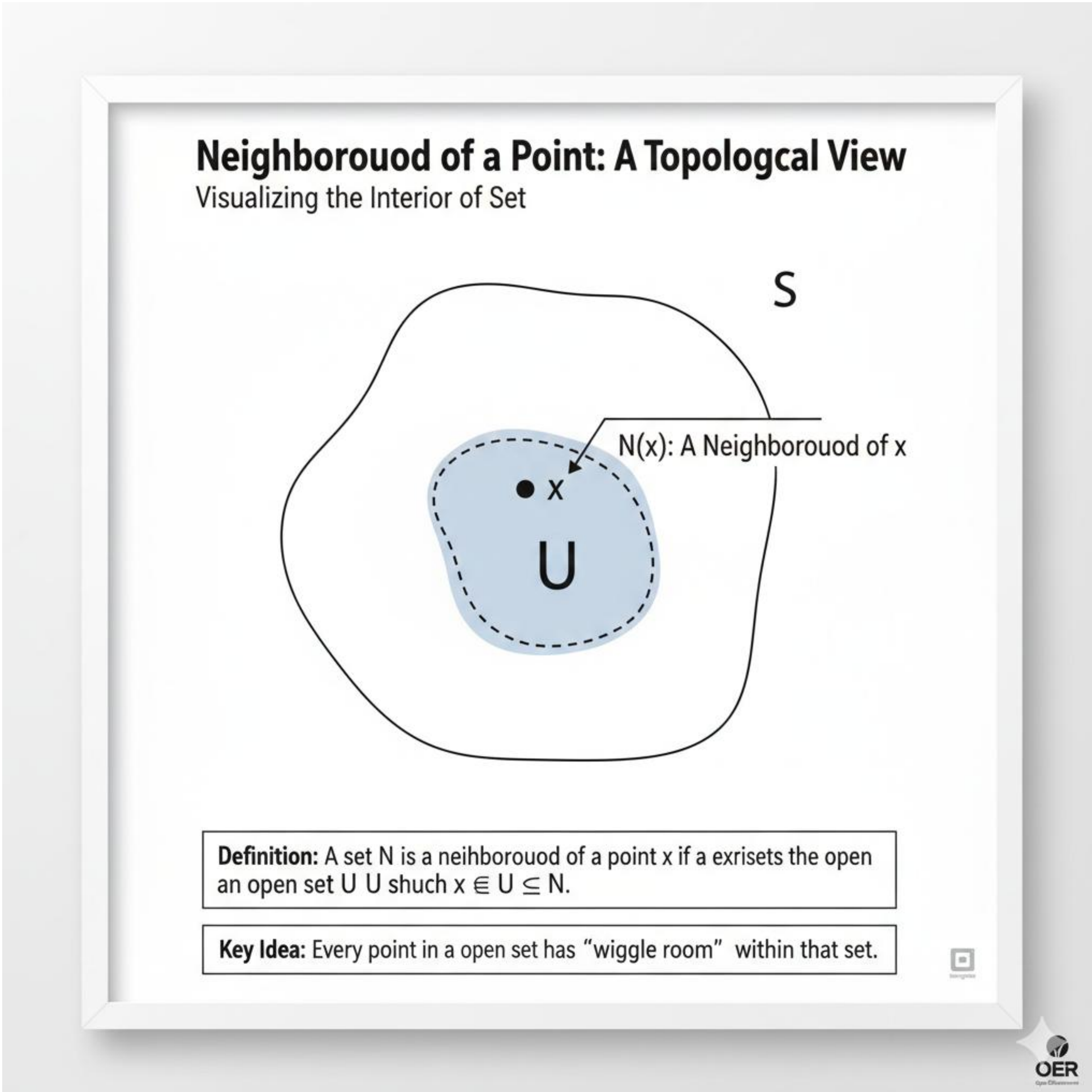
1.3 Neighbourhoods

In topology, the concept of a **neighbourhood** provides a local view of a point within a space. A **neighbourhood** of a point $x \in X$ is any set that contains an open set in which x lies. Neighbourhoods help us describe local properties such as continuity and convergence without relying on distance.

Fill in the blank: A neighbourhood of a point is any set that contains an __________ set in which the point lies.

1.3.1 Definition of neighbourhood of a point

Formally, if (X,T) is a topological space and $x \in X$, then a set N is a **neighbourhood** of x if there exists an open set $U \in T$ such that $x \in U \subseteq N$. This definition ensures that neighbourhoods always contain some open set around the ***point.***



A neighbourhood of a point in a topological space.

Fill in the blank: A set Nis a neighbourhood of xif it contains an __________ set around x.

1.3.2 Neighbourhood systems

The collection of all neighbourhoods of a point xis called the **neighbourhood system** of x. This system provides a way to study the local behaviour of functions and sequences at that point. For example, continuity at a point can be defined using neighbourhood systems.

Neighbourhood systemf a poin

- Collection of all neighbourhoods of a point
- Used to study local behaviour
- Basis for defining continuity and convergence

Fill in the blank: The collection of all neighbourhoods of a point is called its __________ system.

1.3.3 Role of neighbourhoods in topology

Neighbourhoods are central to defining continuity, convergence, and limits in topology. For instance, a function $f:X\to Y$is continuous at a point $x\in X$if for every neighbourhood $N$of $f(x)$, there exists a neighbourhood $M$of $x$such that $f(M)\subseteq N$. Neighbourhoods thus provide the local framework for global properties of spaces.

Fill in the blank: A function is continuous at a point if the image of every neighbourhood of the point maps into a neighbourhood of __________.

Pilot questions 1.3

- A neighbourhood of a point is defined as: A. A set containing an open set around the point B. Any random subset of the space C. Only the point itself D. A closed set only
- The collection of all neighbourhoods of a point is called: A. Neighbourhood system B. Open set system C. Closed set system D. Random system
- Neighbourhoods are used to define: A. Continuity and convergence B. Randomness C. Geometry only D. Distance only
- A function is continuous at a point if: A. The image of every neighbourhood of the point maps into a neighbourhood of its image B. The function is constant C. The function is random D. The function is discontinuous
- In topology, neighbourhoods provide a framework for: A. Local properties of spaces B. Random subsets C. Geometry only D. Distance only

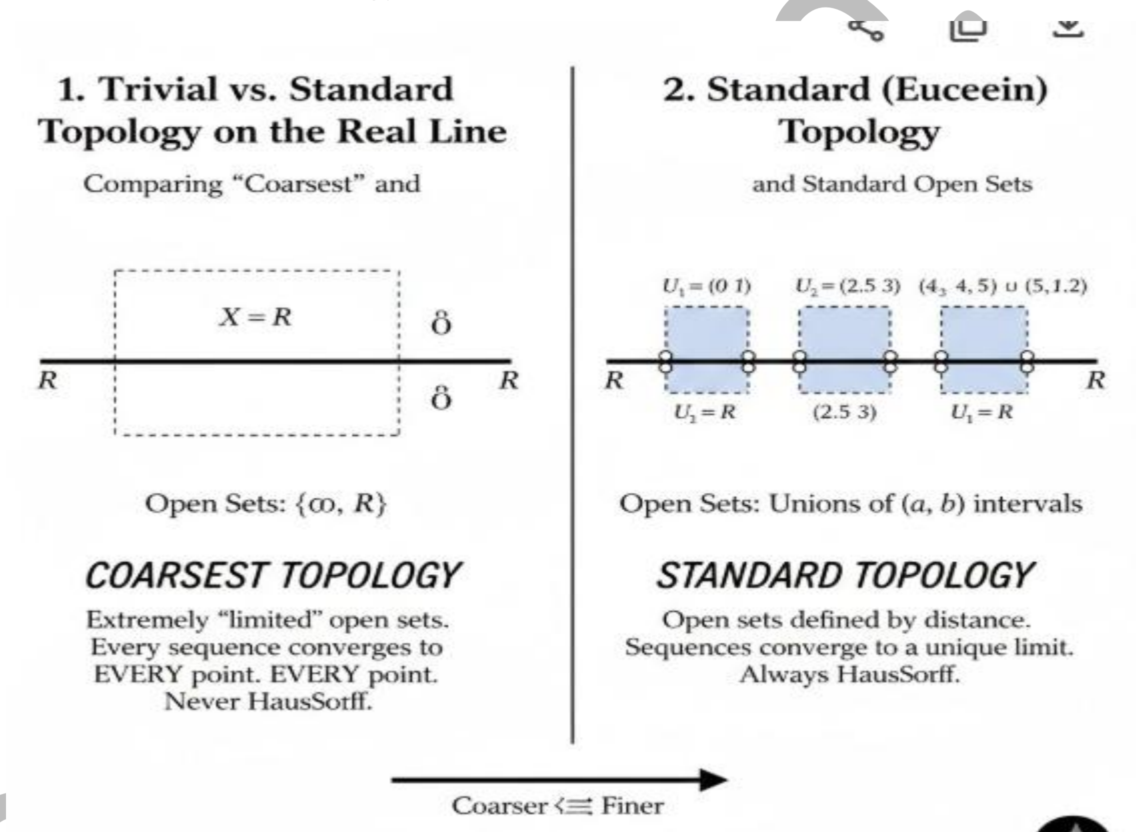
1.4 Coarser And Finer Topologies

In topology, different choices of open sets can lead to different structures on the same underlying set. A coarser topology is one that has fewer open sets, while a finer topology has more open sets. These terms allow us to compare topologies and understand how changing the collection of open sets affects the properties of the space.

Fill in the blank: A topology with fewer open sets is called a __________ topology.

1.4.1 Definition of coarser topology

A **coarser topology** on a set Xis a topology that contains fewer open sets than another topology on the same set. For example, the trivial topology {∅,X}is coarser than the standard topology on R.

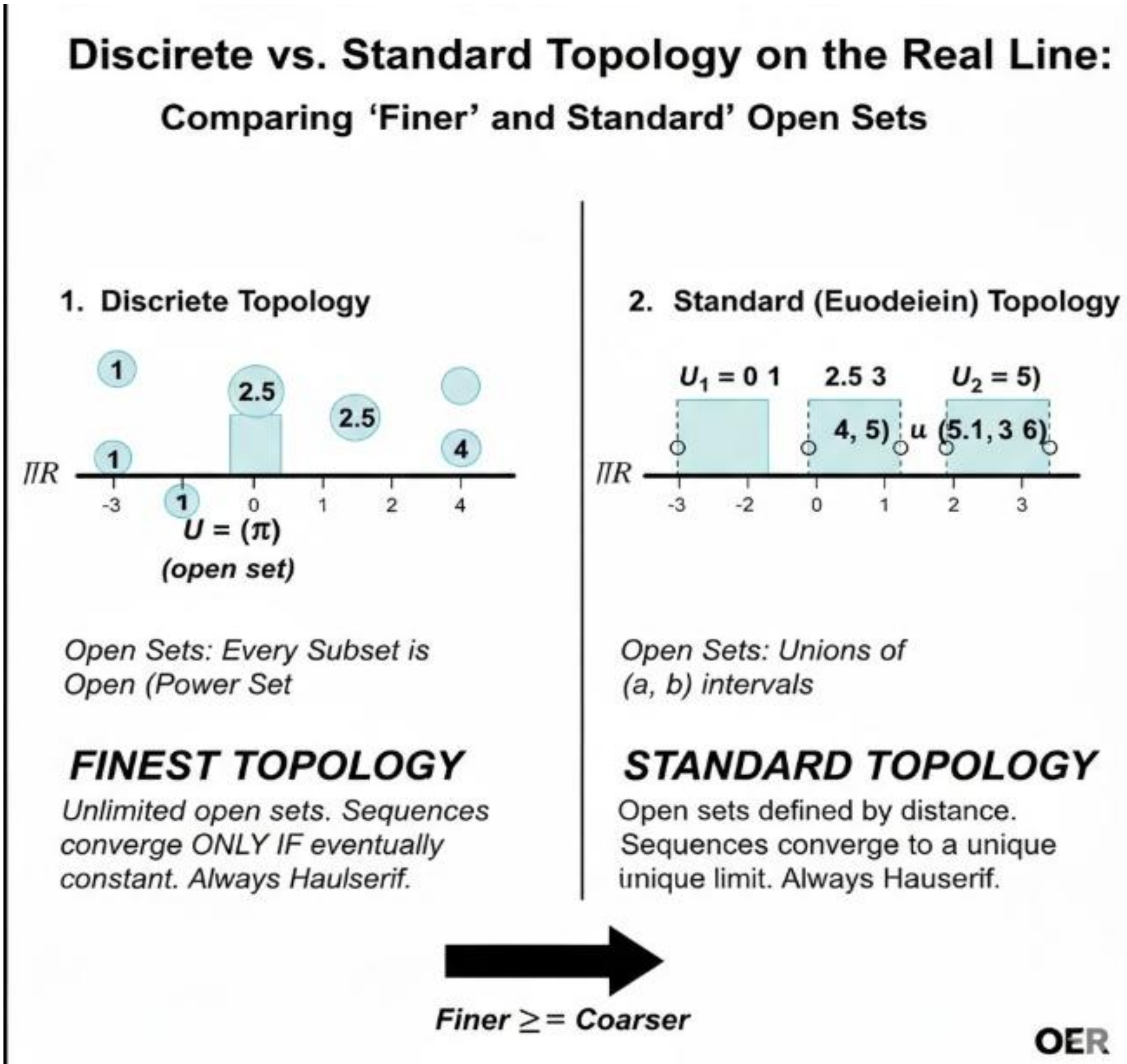


The trivial topology is coarser than the standard topology on* R*. AI prompt:

Fill in the blank: The trivial topology is an example of a __________ topology

1.4.2 Definition of finer topology

A **finer topology** on a set Xis a topology that contains more open sets than another topology on the same set. For example, the discrete topology, where every subset is open, is finer than the standard topology on R.



The discrete topology is finer than the standard topology on R***. AI prompt:***

Fill in the blank: The discrete topology is an example of a __________ topology.

1.4.3 Examples and comparisons

The trivial topology is coarser than the standard topology on R.

The discrete topology is finer than the standard topology on R.

Between two topologies, one may be neither coarser nor finer if they are incomparable.

Comparing coarser and finer topologies

Feature	Coarser Topology ($\tau1 \subseteq \tau2$)	Finer Topology ($\tau2 \supseteq \tau1$)
Number of Open Sets	Fewer open sets.	More open sets.
"Granularity"	Blurry/Rough; harder to separate points.	Detailed/Sharp; easier to separate points.
Convergence	Easier for sequences to converge.	Harder for sequences to converge.
Continuity	Identity map f: (X, \tau_2) \to (X, \tau_1)$is continuous.	Identity map f: (X, \tau_1) \to (X, \tau_2) is **not** necessarily continuous.

Coarser topology: fewer open sets

Finer topology: more open sets

Examples: trivial (coarser), discrete (finer)

Fill in the blank: If one topology has more open sets than another, it is said to be __________.

Pilot questions 1.4

A coarser topology has: A. Fewer open sets B. More open sets C. Random subsets D. No open sets

A finer topology has: A. More open sets B. Fewer open sets C. Random subsets D. No open sets

The trivial topology is: A. Coarser than the standard topology B. Finer than the standard topology C. Equal to the standard topology D. Random

The discrete topology is: A. Finer than the standard topology B. Coarser than the standard topology C. Equal to the standard topology D. Random

Two topologies may be incomparable if: A. Neither is coarser nor finer than the other B. One has no open sets C. Both are trivial D. Both are discrete

Series summary

This Series introduced the fundamental ideas that underpin the study of topology. We began by defining topological spaces and examining examples such as the discrete, trivial, and standard topologies. Then we explored open and closed sets, clarifying their relationship and the special case of clopen sets. Following that, we studied neighbourhoods, which provide a local perspective on points and are central to defining continuity and convergence. Finally, we compared coarser and finer topologies, showing how different choices of open sets affect the structure of a space. By completing this Series, you should now be able to define topological spaces, explain open and closed sets, describe neighbourhoods, and evaluate coarser and finer topologies, thereby achieving the Series Learning Outcomes.

Glossary of terms

Topological space: A set equipped with a collection of subsets called a topology, satisfying specific rules.

Topology: A collection of subsets of a set that are designated as open, satisfying union and intersection properties.

Open set: A subset of a topological space that belongs to the topology.

Closed set: The complement of an open set in a topological space.

Clopen set: A set that is both open and closed.

Neighbourhood: A set containing an open set around a point.

Neighbourhood system: The collection of all neighbourhoods of a point.

Coarser topology: A topology with fewer open sets compared to another.

Finer topology: A topology with more open sets compared to another.

Concept check questions 1

(SLO 1.1) Define a topological space and state the conditions for a topology.

(SLO 1.1) Give one example each of a discrete topology and a trivial topology.

(SLO 1.2) Explain the difference between open and closed sets, and give an example of each in R.

(SLO 1.2) What is a clopen set? Provide an example.

(SLO 1.3) Define a neighbourhood of a point and explain the role of neighbourhood systems.

(SLO 1.3) How can continuity at a point be defined using neighbourhoods?

(SLO 1.4) Distinguish between coarser and finer topologies, with examples.

(SLO 1.4) Can two topologies on the same set be incomparable? Explain with an example.

Concept Check Questions 1

Objective questions

A topological space is a set equipped with a collection of __________ satisfying certain axioms. (Linked to SLO 1.1)

Open sets in a topology must include: A. The empty set and the whole set B. Only the empty set C. Only the whole set D. None of the above (Linked to SLO 1.2)

A closed set is defined as the __________ of an open set. (Linked to SLO 1.2)

A neighbourhood of a point must contain an __________ set that includes the point. (Linked to SLO 1.3)

A finer topology has __________ open sets than a coarser topology. (Linked to SLO 1.4)

Short-answer questions

Define a topological space. (Linked to SLO 1.1)

Give one example of an open set in the standard topology on R. (Linked to SLO 1.2)

Explain the relationship between open and closed sets. (Linked to SLO 1.2)

State one role of neighbourhoods in topology. (Linked to SLO 1.3)

Provide an example of a finer topology compared to the standard topology on R. (Linked to SLO 1.4)

Long-answer questions

Analyse the concept of a set and topology, and explain why topology generalises geometry. (Linked to SLO 1.1)

Evaluate the importance of open and closed sets in defining continuity. (Linked to SLO 1.2)

Discuss neighbourhood systems and their role in convergence and continuity. (Linked to SLO 1.3)

Examine coarser and finer topologies with examples, highlighting their applications. (Linked to SLO 1.4)

Answers to Concept Check Questions 1

Objective questions

Open sets

A

Complement

Open

More

Short-answer questions

A topological space is a set Xtogether with a collection of subsets of X(called open sets) such that satisfies the topology axioms.

An open interval bin R.

A set is closed if its complement is open; thus open and closed sets are dual concepts.

Neighbourhoods help define local properties of points, such as continuity and convergence.

The discrete topology on Ris finer than the standard topology.

Long-answer questions

Basic response: A topology is a collection of subsets satisfying union and intersection properties. *Value-added response*: Topology generalises geometry by focusing on properties preserved under continuous transformations, such as connectedness and compactness.

Basic response: Open and closed sets are fundamental in defining continuity. *Value-added response*: Continuity of a function is defined by the preimage of open sets being open, and equivalently by the preimage of closed sets being closed, linking topology with analysis.

Basic response: Neighbourhood systems describe local surroundings of points. *Value-added response*: They are crucial in defining convergence of sequences and continuity of functions, providing a local perspective in topology.

Basic response: Coarser topologies have fewer open sets, finer topologies have more. *Value-added response*: For example, the trivial topology is coarser than the standard topology, while the discrete topology is finer. These distinctions are important in comparing structures and studying continuity.

Answers to Pilot Questions 1

Part 1.1: 1-Concept of a set and topology, 2-Examples of topological spaces, 3-Motivation for studying topology Part 1.2: 1-Definition of open sets, 2-Definition of closed sets, 3-Relationship between open and closed sets Part 1.3: 1-Definition of neighbourhood of a point, 2-Neighbourhood systems, 3-Role of neighbourhoods in topology Part 1.4: 1-Definition of coarser topology, 2-Definition of finer topology, 3-Examples and comparisons

Series 2: Basis, Sub-basis And Topological Structures

Series outline

Since this is a 2-credit course, each Series should contain 3 to 5 Parts. Here is the proposed outline for Series 2:

2.1 Basis of a topology

2.1.1 Definition and properties of a basis

2.1.2 Examples of bases

2.1.3 Generating a topology from a basis

2.2 Sub-basis of a topology

2.2.1 Definition and properties of a sub-basis

2.2.2 Examples of sub-bases

Introduction

In the previous Series, we explored the foundations of topological spaces, including open and closed sets, neighbourhoods, and comparisons of coarser and finer topologies. Building on that foundation, we now turn to the tools used to construct and describe topologies more efficiently.

The aim of this Series is to introduce you to the concepts of bases and sub-bases, and to show how they are used to generate topologies. This will help you understand how topological structures are built and compared, and will prepare you for more advanced topics such as separation axioms and compactness.

We shall commence this Series by examining the definition and properties of a basis of a topology. Next, we will study sub-bases and how they differ from bases. Following that, we will explore how topologies can be generated from bases and sub-bases. Finally, we will conclude with examples of topological structures, illustrating how these concepts are applied in practice.

Series learning outcomes

By the end of this Series, you should be able to:

SLO 2.1: Define a basis of a topology and explain its properties.

SLO 2.2: Define a sub-basis of a topology and distinguish it from a basis.

SLO 2.3: Demonstrate how to generate a topology from a basis or sub-basis.

SLO 2.4: Analyse examples of topological structures constructed using bases and sub-bases.

2.1 Basis Of A Topology

A **basis** is a fundamental tool for constructing a topology. A **basis of a topology** on a set $X$is a collection $B$of subsets of Xsuch that every open set in the topology can be expressed as a union of elements of B. This means that instead of listing all open sets directly, we can describe them more efficiently using a smaller generating collection.

Fill in the blank: A basis of a topology is a collection of subsets such that every open set is a __________ of them.

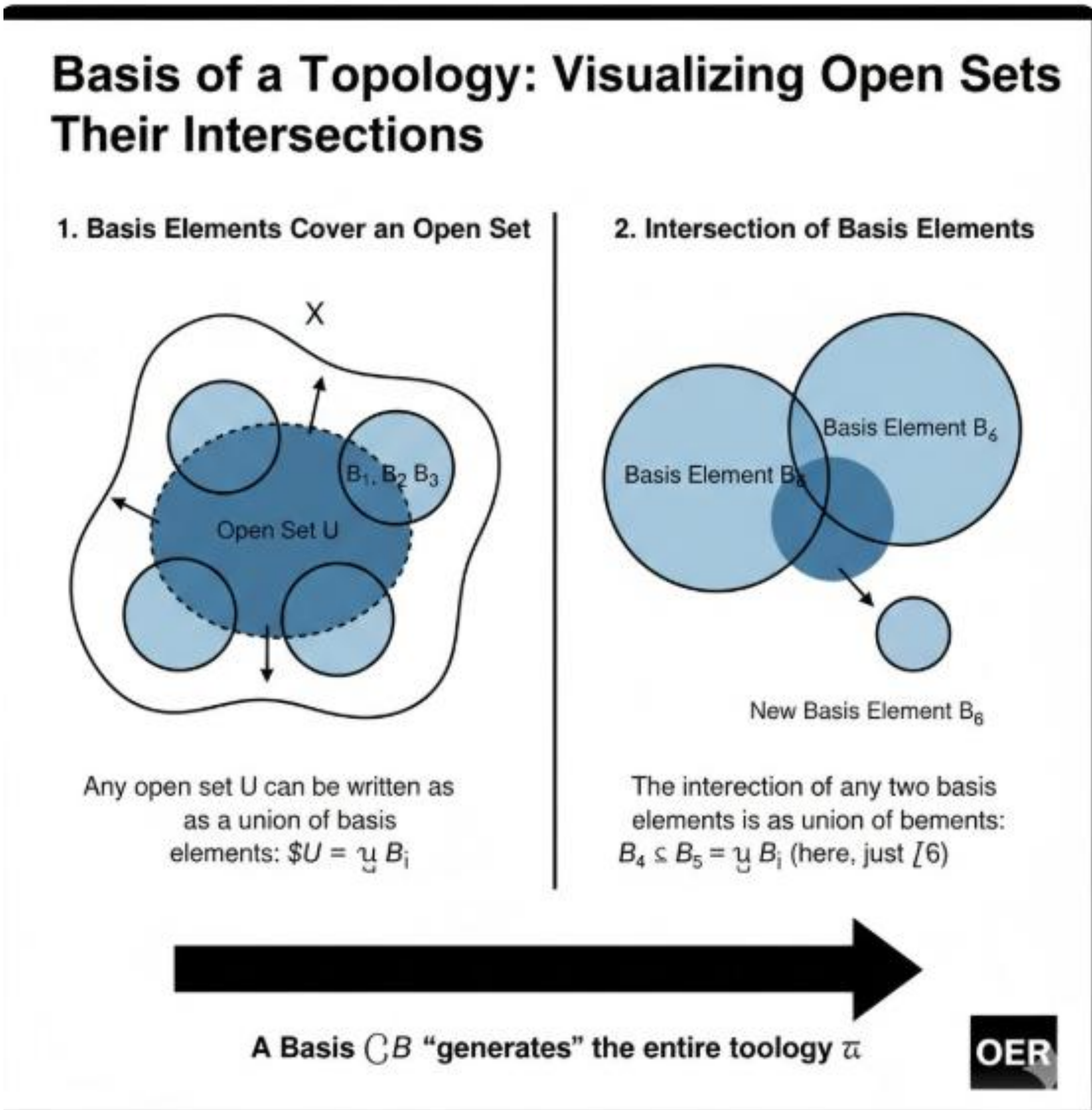
2.1.1 Definition and properties of a basis

Formally, a collection $B$of subsets of $X$is a basis if:

For each $x \in X$, there exists at least one basis element $B \in B$such that $x \in B$.

If $x \in B_1 \cap B_2$for $B_1, B_2 \in B$, then there exists a basis element $B_3 \in B$such that $x \in B_3 \subseteq B_1 \cap B_2$.

These conditions ensure that the basis generates a valid topology.



Basis elements covering a set and their intersections.

Fill in the blank: If a point lies in the intersection of two basis elements, there must be another basis element contained in the __________.

2.1.2 Examples of bases

Standard topology on R**:** The collection of all open intervals (a,b)forms a basis.

Discrete topology: The collection of all singletons $\{x\}$forms a basis.

Trivial topology: The collection $\{X\}$forms a basis.

Examples of bases of topologies

Space	**Basis**	**Resulting topology**
R	Open intervals (a,b)	Standard topology
Discrete	Singletons {x}	Discrete topology
Trivial	Whole set {X}	Trivial topology

Fill in the blank: The collection of all open intervals (a,b)in Rforms the __________ of the standard topology.

2.1.3 Generating a topology from a basis

Given a basis B, the topology generated by Bis the collection of all unions of elements of B. This provides a systematic way to construct topologies without listing every open set individually.

Generating a topology from a basis

Start with a basis B

Form all unions of basis elements

The resulting collection is the topology T

Fill in the blank: The topology generated by a basis is the collection of all __________ of basis elements.

Pilot questions 2.1

A basis of a topology is: A. A collection of subsets whose unions form open sets B. A random collection of subsets C. Only closed sets D. A metric

The standard topology on Rhas as its basis: A. Open intervals (a,b)B. Closed intervals [a,b]C. Singletons {x}D. Random subsets

The discrete topology has as its basis: A. Singletons {x}B. Open intervals (a,b)C. Whole set {X}D. Empty set only

The trivial topology has as its basis: A. Whole set {X}B. Singletons {x}C. Open intervals (a,b)D. Random subsets

The topology generated by a basis is formed by: A. All unions of basis elements B. All intersections of basis elements C. Only complements of basis elements D. Random subsets

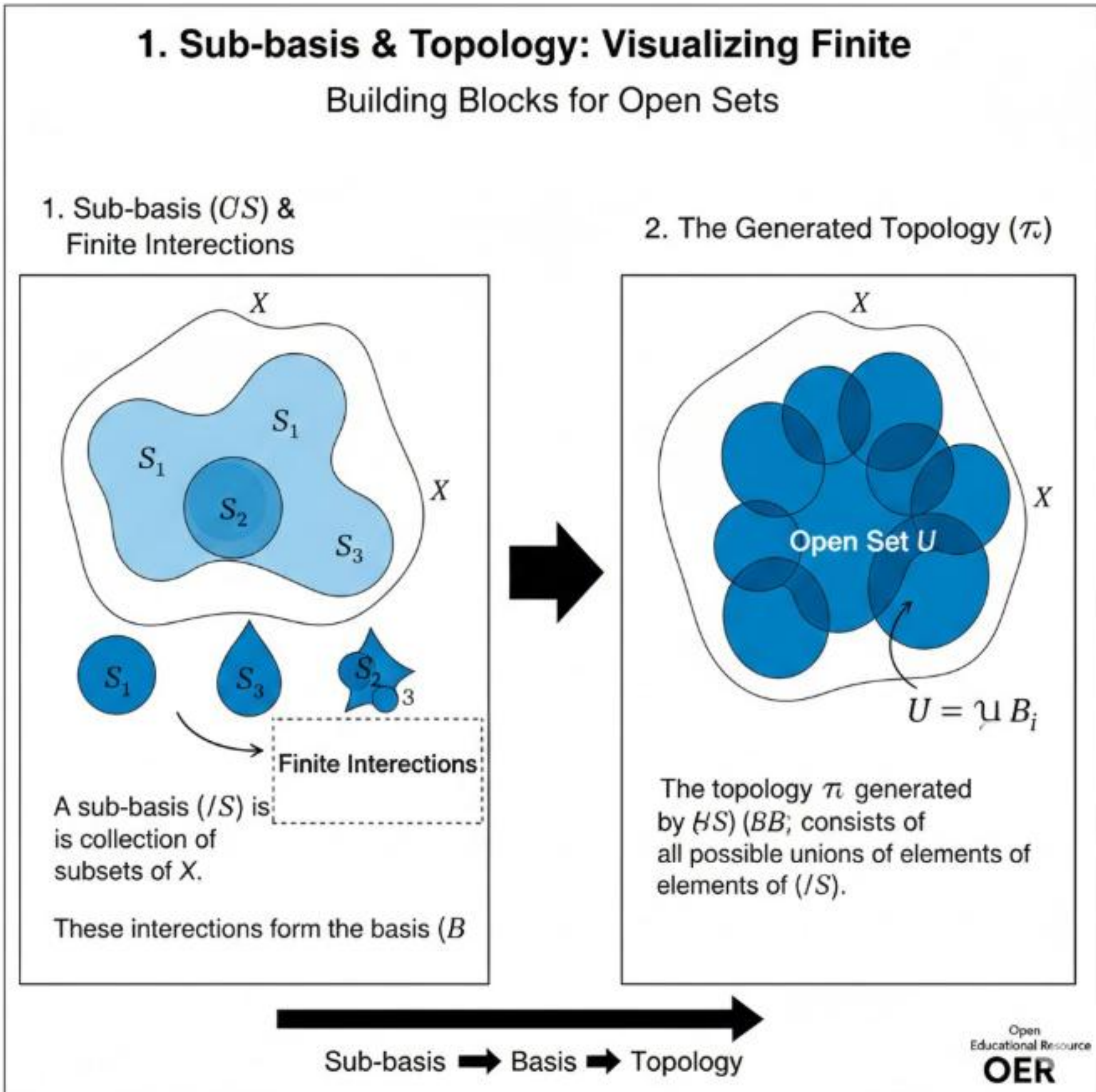
2.2 Sub-basis Of A Topology

A **sub-basis** is another way of generating a topology, often more flexible than a basis. A **sub-basis of a topology** on a set Xis a collection Sof subsets of Xsuch that the finite intersections of elements of Sform a basis for the topology. In other words, the topology generated by a sub-basis consists of all unions of finite intersections of elements of S.

Fill in the blank: A sub-basis generates a topology by taking unions of finite __________ of its elements.

2.2.1 Definition and properties of a sub-basis

Formally, if Sis a collection of subsets of X, then the topology generated by Sis the collection of all unions of finite intersections of elements of S. This ensures that the resulting structure satisfies the axioms of a topology.



Sub-basis generating a topology through finite intersections.

Fill in the blank: The topology generated by a sub-basis is formed by unions of finite __________ of its subsets.

2.2.2 Examples of sub-bases

Standard topology on R: The collection of all open rays (a,)and (-,b)forms a sub-basis. Their finite intersections give open intervals (a,b), which form a basis.

Product topology: The collection of sets of the form U×V, where Uand Vare open sets in their respective spaces, forms a sub-basis for the product topology.

Examples of sub-bases of topologies

Space	**Sub-basis**	**Resulting basis**
R	Open rays (a,),(-,b)	Open intervals (a,b)
Product space	Sets of form U×V	Basis for product topology

Fill in the blank: In the standard topology on R, the collection of open rays forms a __________.

2.2.3 Distinction between basis and sub-basis

A **basis** directly generates a topology by unions of its elements, while a **sub-basis** generates a topology by unions of finite intersections of its elements. Thus, every basis is sufficient on its own, but a sub-basis requires intersections to produce a basis first.

Distinction between basis and sub-basis

Feature	**Basis (B)**	**Sub-basis (S)**
Generation Power	Generates a topology τ via **arbitrary unions**.	Generates a topology τ via **finite intersections**, then **arbitrary unions**.
Internal Requirements	Must satisfy specific intersection and covering axioms.	**Any** collection of subsets of X whose union is X can be a sub-basis.
Complexity	"More finished": Basis elements are already open sets in the topology.	"More raw": Sub-basis elements are simpler, but you must work harder to get the open sets.

Feature	Basis (B)	Sub-basis (S)
Relationship	Every basis is a sub-basis.	A sub-basis is used to *create* a basis.

Basis: generates topology by unions of basis elements

Sub-basis: generates topology by unions of finite intersections of its elements

Sub-basis is often simpler to specify than a full basis

Fill in the blank: A basis generates a topology by unions, while a sub-basis requires finite __________ first.

Pilot questions 2.2

A sub-basis of a topology generates open sets by: A. Unions of finite intersections of its elements B. Random unions of subsets C. Complements of subsets only D. Metrics

In the standard topology on R, the collection of open rays (a,)and (-,b)forms: A. A sub-basis B. A basis directly C. A trivial topology D. A discrete topology

In the product topology, the collection of sets of the form U×Vis: A. A sub-basis B. A basis directly C. A trivial topology D. A discrete topology

The distinction between basis and sub-basis is that: A. Basis uses unions directly, sub-basis requires finite intersections first B. Basis uses complements, sub-basis uses unions C. Basis uses metrics, sub-basis uses random subsets D. Basis and sub-basis are identical

Every basis is sufficient on its own, but a sub-basis must first generate a __________. A. Basis B. Metric C. Random set D. Closed set

2.3 Generating Topologies From Bases And Sub-Bases

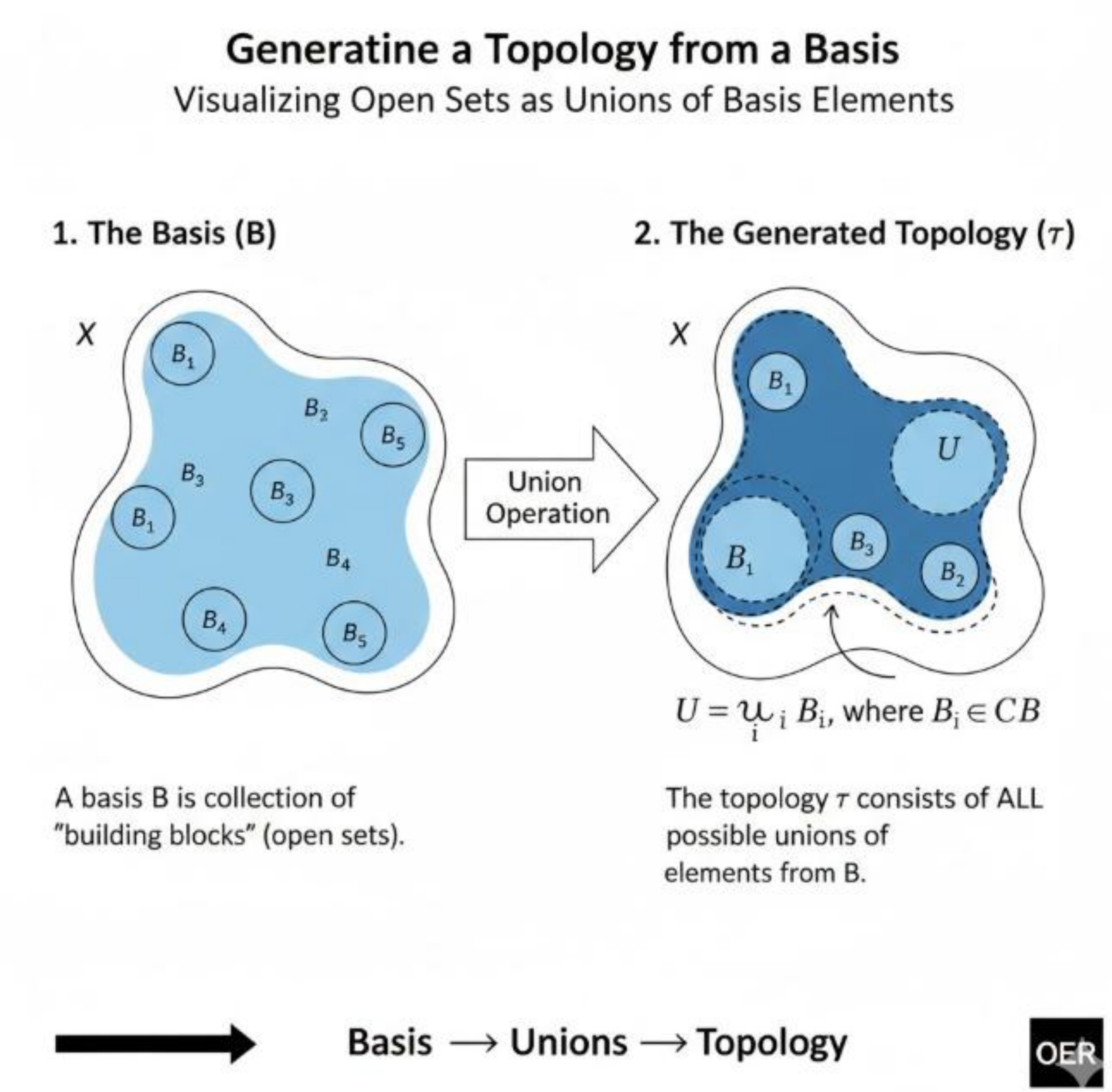
One of the most powerful aspects of topology is that we do not need to list every open set individually. Instead, we can generate a topology systematically from a

basis or a **sub-basis**. This approach makes it easier to construct and compare topologies, especially in complex spaces.

Fill in the blank: A topology can be generated systematically from a __________ or a sub-basis.

2.3.1 Generating from a basis

If Bis a basis for a set X, then the topology generated by Bis the collection of all unions of elements of B. This ensures that the topology satisfies the axioms of openness.

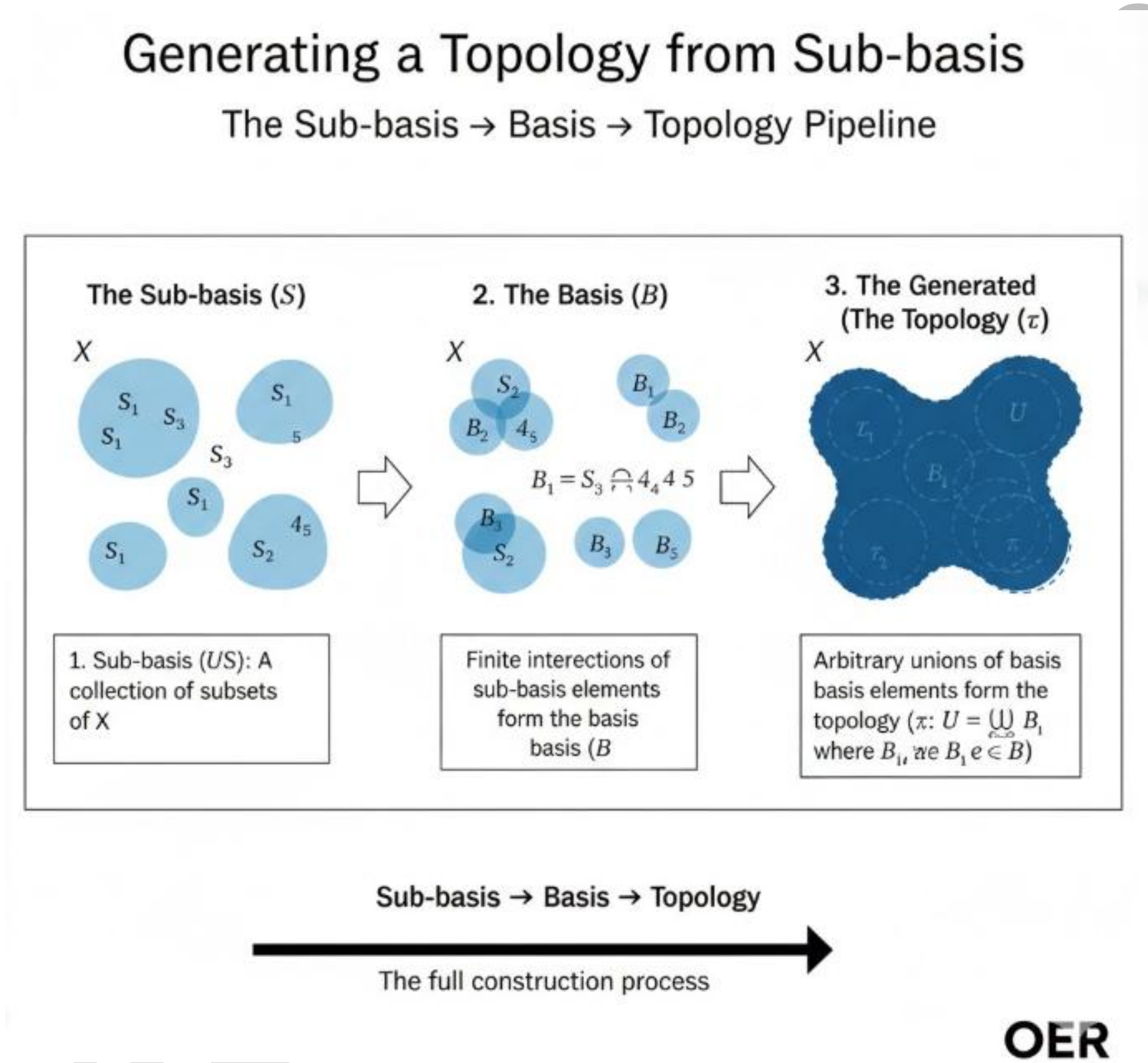


Generating a topology from a basis by unions of basis elements.

: Fill in the blank: The topology generated by a basis is the collection of all __________ of basis elements.

2.3.2 Generating from a sub-basis

If Sis a sub-basis for a set X, then the topology generated by Sis the collection of all unions of finite intersections of elements of S. This process first produces a basis, and then generates the topology from that basis.



Generating a topology from a sub-basis.

Fill in the blank: The topology generated by a sub-basis is formed by unions of finite __________ of its elements.

2.3.3 Examples of generated topologies

Standard topology on R**:** Generated from the basis of open intervals (a,b).

Standard topology on R**:** Also generated from the sub-basis of open rays (a,)and (-,b).

Product topology: Generated from the sub-basis of sets of the form U×V.

Examples of topologies generated from bases and sub-bases

Space	**Basis/Sub-basis**	**Generated topology**
R	Basis: open intervals (a,b)	Standard topology
R	Sub-basis: open rays (a,),(-,b)	Standard topology
Product space	Sub-basis: sets of form U×V	Product topology

Fill in the blank: The product topology is generated from a sub-basis of sets of the form __________.

Pilot questions 2.3

The topology generated by a basis is formed by: A. All unions of basis elements B. All intersections of basis elements C. Complements of basis elements only D. Random subsets

The topology generated by a sub-basis is formed by: A. Unions of finite intersections of its elements B. Complements of subsets only C. Random unions of subsets D. Metrics

The standard topology on Rcan be generated from: A. Basis of open intervals B. Sub-basis of open rays C. Both A and B D. Neither

The product topology is generated from: A. Sub-basis of sets of the form U×VB. Basis of singletons C. Trivial topology D. Random subsets

A sub-basis first generates a __________ before generating the topology. A. Basis B. Metric C. Random set D. Closed set

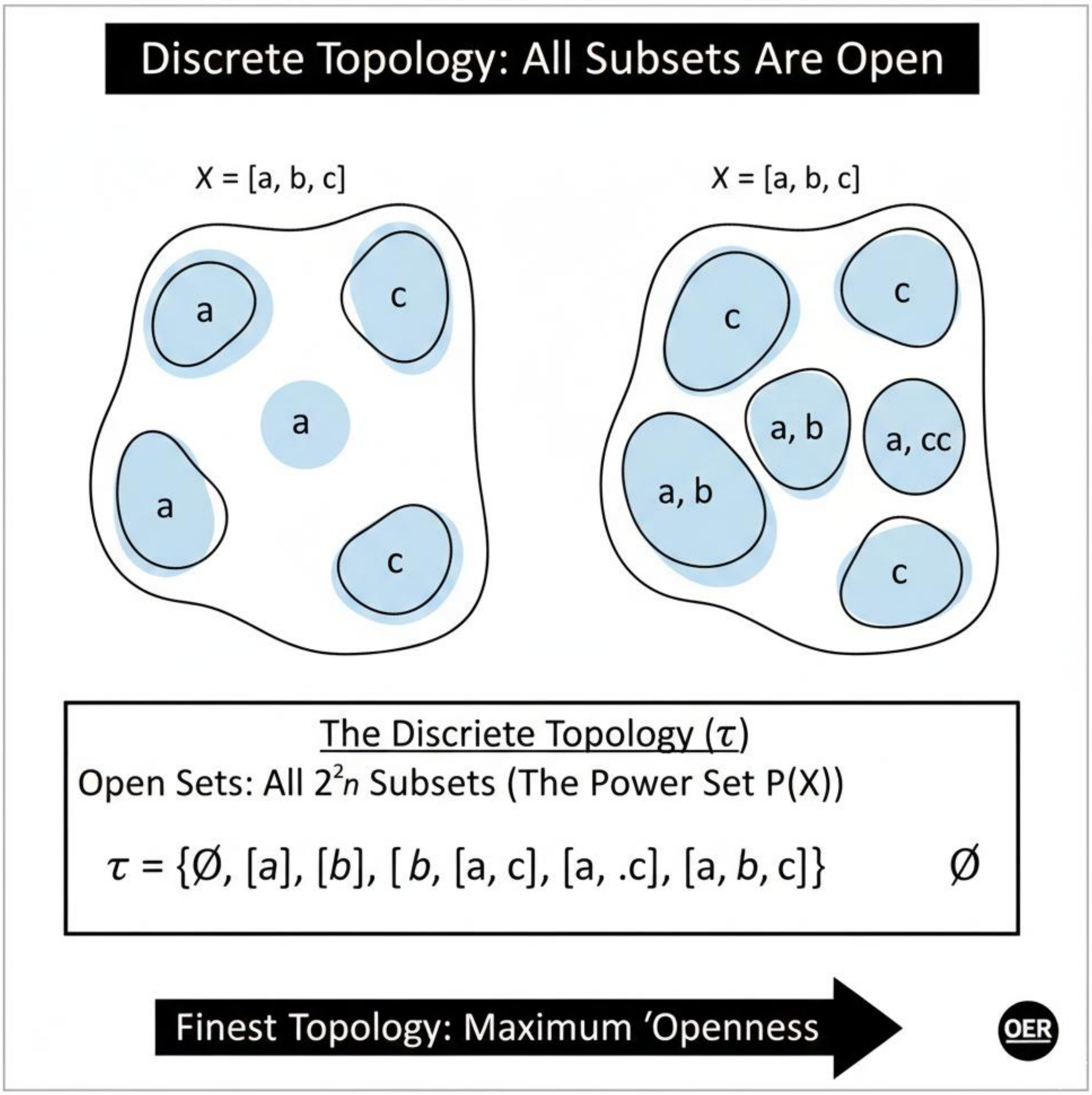
2.4 Examples Of Topological Structures

Topological structures arise when we apply bases and sub-bases to generate topologies on sets. These examples help us see how abstract definitions translate into concrete spaces. By studying them, you will gain intuition about how different choices of bases or sub-bases affect the resulting topology.

Fill in the blank: Topological structures are created by applying __________ or sub-bases to generate topologies.

2.4.1 Discrete topology

The **discrete topology** on a set Xis generated by the basis consisting of all singletons {x}. In this topology, every subset of Xis open. This structure is useful for illustrating extreme cases where openness is maximised.

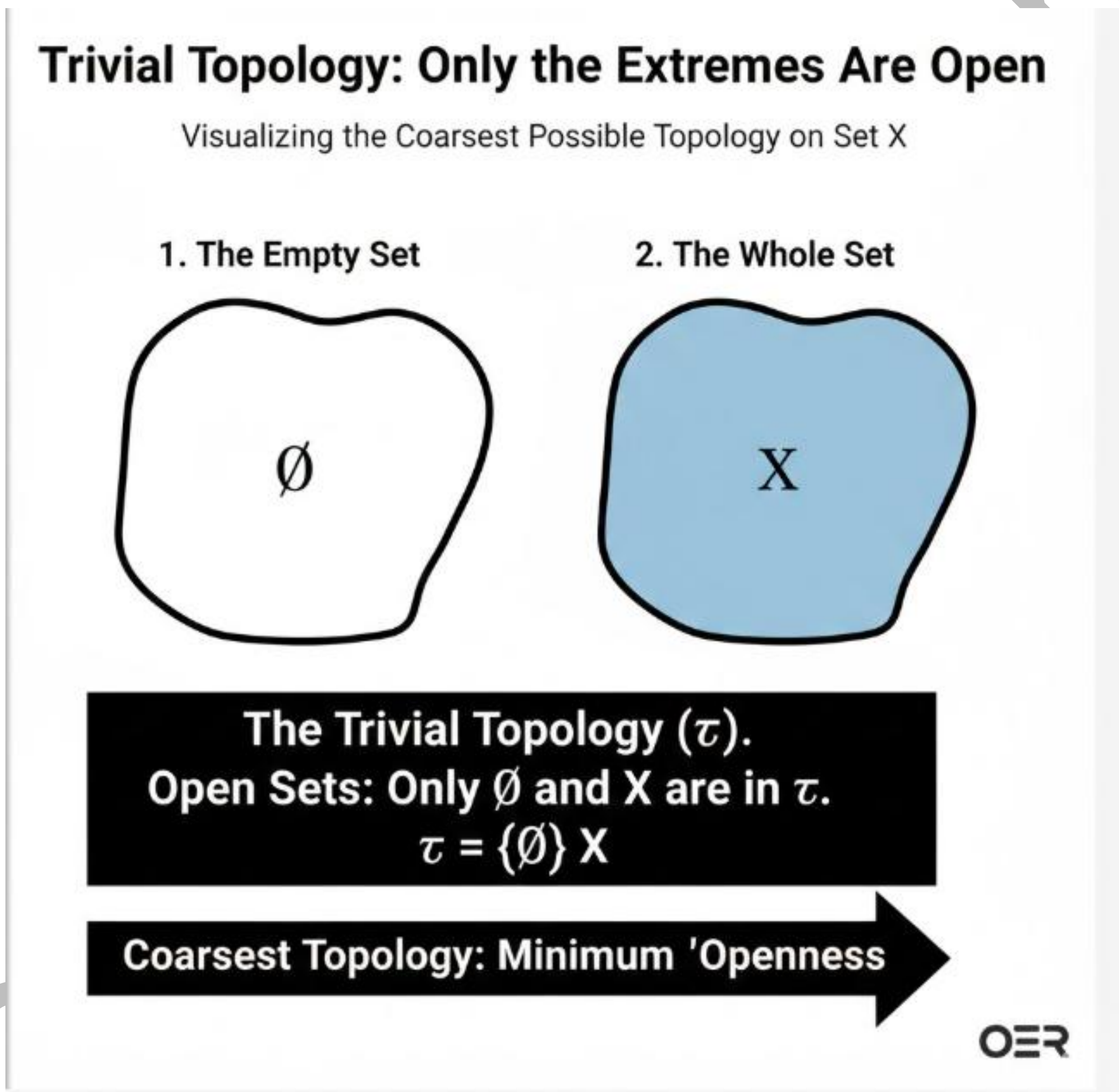


Discrete topology where every subset is open.

Fill in the blank: In the discrete topology, __________ subset of the set is open.

2.4.2 Trivial topology

The **trivial topology** on a set Xis generated by the basis {X}. In this topology, only the empty set and the whole set are open. This structure illustrates the opposite extreme, where openness is minimised.

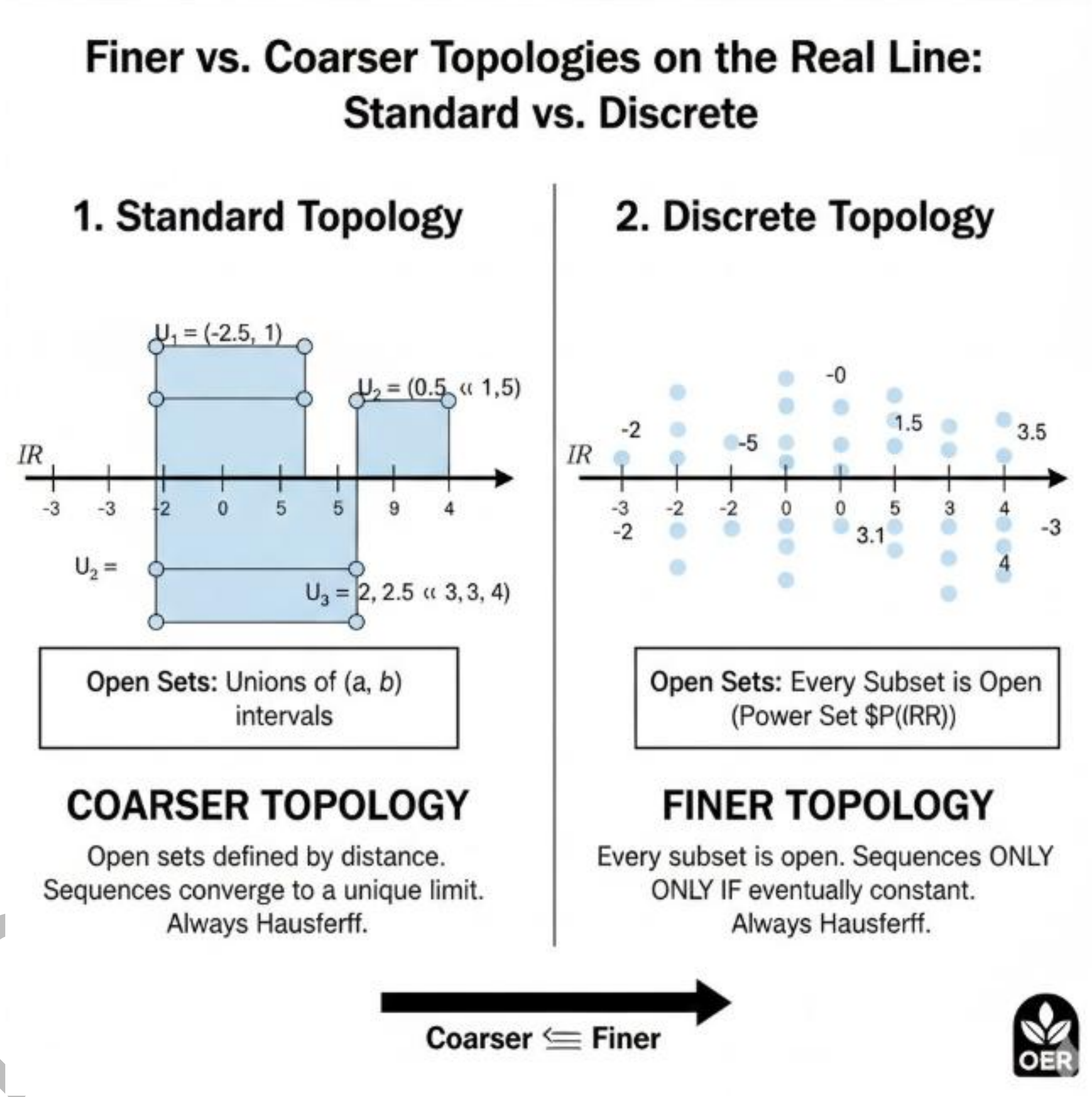


Trivial topology where only the empty set and whole set are open.

Fill in the blank: In the trivial topology, only the empty set and the __________ set are open.

2.4.3 Standard topology on ℝ

The **standard topology** on ℝis generated by the basis of open intervals (a,b). It is the most familiar topology, used in analysis and calculus. This structure allows us to study continuity and convergence in the real line.

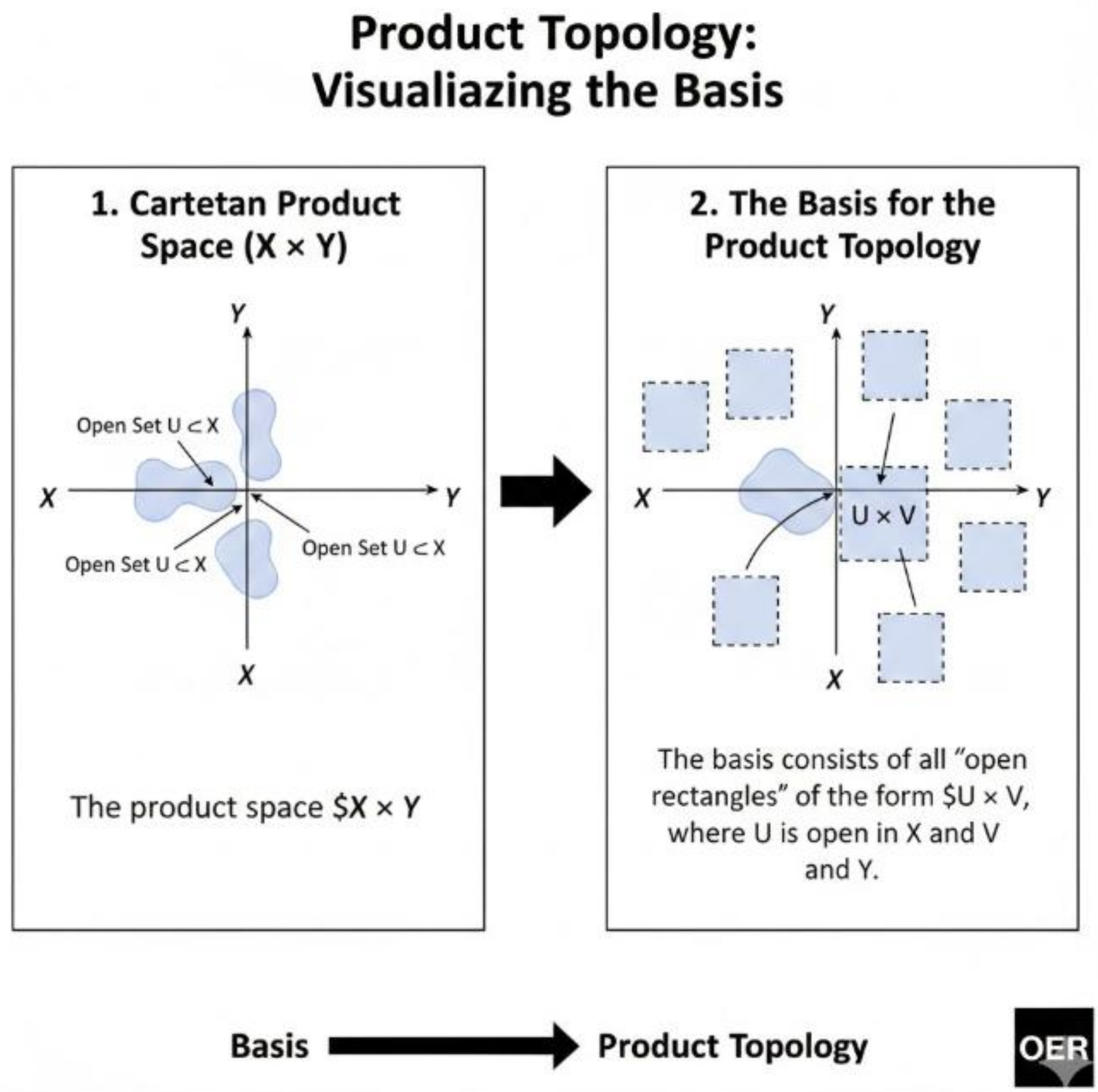


Standard topology on ℝgenerated by open intervals.

Fill in the blank: The standard topology on Ris generated by open __________.

2.4.4 Product topology

The **product topology** is generated from the sub-basis consisting of sets of the form U×V, where Uand Vare open sets in their respective spaces. This structure is essential in multivariable analysis and functional spaces.



Product topology generated by sets of the form U×V**.**

Fill in the blank: The product topology is generated from sets of the form __________.

Pilot questions 2.4

The discrete topology is generated by: A. Singletons {x}B. Open intervals (a,b)C. Whole set {X}D. Random subsets

The trivial topology has as its open sets: A. Empty set and whole set only B. All subsets C. Open intervals D. Random subsets

The standard topology on Ris generated by: A. Open intervals (a,b)B. Closed intervals [a,b]C. Singletons {x}D. Random subsets

The product topology is generated from: A. Sets of the form U×VB. Singletons {x}C. Whole set {X}D. Random subsets

The trivial topology illustrates the extreme case where openness is: A. Minimised B. Maximised C. Randomised D. Undefined

Series summary

This Series has shown how bases and sub-bases provide efficient ways of constructing and describing topologies. We began by defining a basis of a topology and examining its properties, then moved on to sub-bases and their role in generating topologies through finite intersections. Following that, we studied how topologies can be systematically generated from bases and sub-bases, ensuring that the axioms of openness are satisfied. Finally, we explored examples of topological structures such as the discrete, trivial, standard, and product topologies, which illustrate how different generating sets lead to different spaces. By completing this Series, you should now be able to define bases and sub-bases, distinguish between them, generate topologies from either, and analyse examples of topological structures, thereby achieving the Series Learning Outcomes.

Glossary of terms

Basis: A collection of subsets of a set such that every open set is a union of basis elements.

Sub-basis: A collection of subsets of a set whose finite intersections form a basis for a topology.

Generated topology: The topology formed by unions of basis elements or unions of finite intersections of sub-basis elements.

Discrete topology: A topology where every subset is open.

Trivial topology: A topology where only the empty set and the whole set are open.

Standard topology on R: The topology generated by open intervals (a,b).

Product topology: The topology generated by sets of the form U×V, where Uand Vare open sets in their respective spaces.

Concept check questions 2

(SLO 2.1) Define a basis of a topology and state its two properties.

(SLO 2.1) Give an example of a basis for the standard topology on R.

(SLO 2.2) Explain how a sub-basis generates a topology.

(SLO 2.2) Provide an example of a sub-basis for the standard topology on .

(SLO 2.3) Describe the process of generating a topology from a basis.

(SLO 2.3) Describe the process of generating a topology from a sub-basis.

(SLO 2.4) Compare the discrete and trivial topologies in terms of openness.

(SLO 2.4) Explain how the product topology is generated and why it is important.

Concept Check Questions 2

Objective questions

A basis of a topology is a collection of __________ sets whose unions form the topology. (Linked to SLO 2.1)

A sub basis is defined as: A. A collection of sets whose finite intersections form a basis B. A collection of sets whose unions form a topology directly C. A discrete topology D. None of the above (Linked to SLO 2.2)

The standard topology on Ris generated by: A. Open intervals B. Closed intervals C. Singletons D. None of the above (Linked to SLO 2.4)

The discrete topology assigns every subset of a set as __________. (Linked to SLO 2.4)

The trivial topology on a set consists of only two sets: the empty set and the __________ set. (Linked to SLO 2.4)

Short-answer questions

Define a basis of a topology. (Linked to SLO 2.1)

Give one example of a sub basis. (Linked to SLO 2.2)

Explain the distinction between a basis and a sub basis. (Linked to SLO 2.2)

State one example of a generated topology from a basis. (Linked to SLO 2.3)

Provide one application of product topology. (Linked to SLO 2.4)

Long-answer questions

Analyse the properties of a basis and explain how it generates a topology. (Linked to SLO 2.1)

Evaluate the role of sub bases in constructing topologies, with examples. (Linked to SLO 2.2)

Discuss the process of generating topologies from bases and sub bases, highlighting similarities and differences. (Linked to SLO 2.3)

Examine examples of topological structures such as discrete, trivial, standard, and product topologies, and explain their importance. (Linked to SLO 2.4)

Answers to Concept Check Questions 2

Objective questions

Open

A

A

Open

Whole

Short-answer questions

A basis is a collection of open sets such that every open set in the topology can be expressed as a union of basis elements.

The collection of all open half-intervals in Rforms a sub basis.

A basis directly generates a topology through unions, while a sub basis generates a basis through finite intersections, which then generates the topology.

The standard topology on Rgenerated by open intervals.

Product topology is used in analysis to study convergence in spaces like Rn.

Long-answer questions

Basic response: A basis must cover the space and allow intersections to be expressed as unions of basis elements. *Value-added response*: This ensures that the topology generated is consistent and allows construction of familiar structures like Euclidean spaces.

Basic response: A sub basis is a collection of sets whose finite intersections form a basis. *Value-added response*: Sub bases are useful in defining complex topologies, such as the initial topology in functional analysis, where open sets are generated from sub bases.

Basic response: Bases generate topologies directly, while sub bases generate bases first. *Value-added response*: Both approaches are equivalent in the sense that they lead to the same topology, but sub bases provide flexibility in construction, especially in product and quotient topologies.

Basic response: Discrete topology makes every subset open, trivial topology has only two open sets, standard topology on Ruses open intervals, and product topology combines topologies on component spaces. *Value-added response*: These examples illustrate the diversity of topological structures, from the simplest (trivial) to more complex (product), and show how topology adapts to different mathematical contexts.

Answers to Pilot Questions 2

Part 2.1: 1-Definition of basis, 2-Examples, 3-Generating topology from basis **Part 2.2**: 1-Definition of sub basis, 2-Examples, 3-Distinction between basis and sub basis **Part 2.3**: 1-Generating from basis, 2-Generating from sub basis, 3-Examples of generated topologies **Part 2.4**: 1-Discrete topology, 2-Trivial topology, 3-Standard topology on R, 4-Product topology

Series 3: Separation Axioms, Compactness And Connectedness

Series outline

3.1 Separation axioms

3.1.1 T0 (Kolmogorov) spaces

3.1.2 T1 (Fréchet) spaces

3.1.3 T2 (Hausdorff) spaces

Introduction

In the previous Series, we studied how bases and sub-bases can be used to generate topologies and explored examples of topological structures. Building on that knowledge, we now turn to some of the most important properties of topological spaces: separation, compactness, and connectedness. These properties allow us to classify spaces, understand their behaviour, and apply them in analysis and geometry.

The aim of this Series is to introduce you to the separation axioms, compactness, and connectedness, and to show how these concepts are used to distinguish and analyse different topological spaces. This will provide you with the tools to evaluate spaces more critically and to apply topology in broader mathematical contexts.

We shall commence this Series by examining the separation axioms, which describe how distinct points and sets can be distinguished using open sets. Next, we will study compactness, a property that generalises the idea of boundedness and closedness in analysis. Following that, we will explore connectedness, which captures the idea of a space being “in one piece”. Finally, we will conclude with applications and examples, showing how these properties are used in practice.

Series learning outcomes

By the end of this Series, you should be able to:

- **SLO 3.1:** Define and explain the separation axioms and classify spaces accordingly.
- **SLO 3.2:** Analyse the concept of compactness and apply it to topological spaces.
- **SLO 3.3:** Evaluate connectedness in topological spaces and explain its significance.
- **SLO 3.4:** Apply separation axioms, compactness, and connectedness to examples and problems in topology.

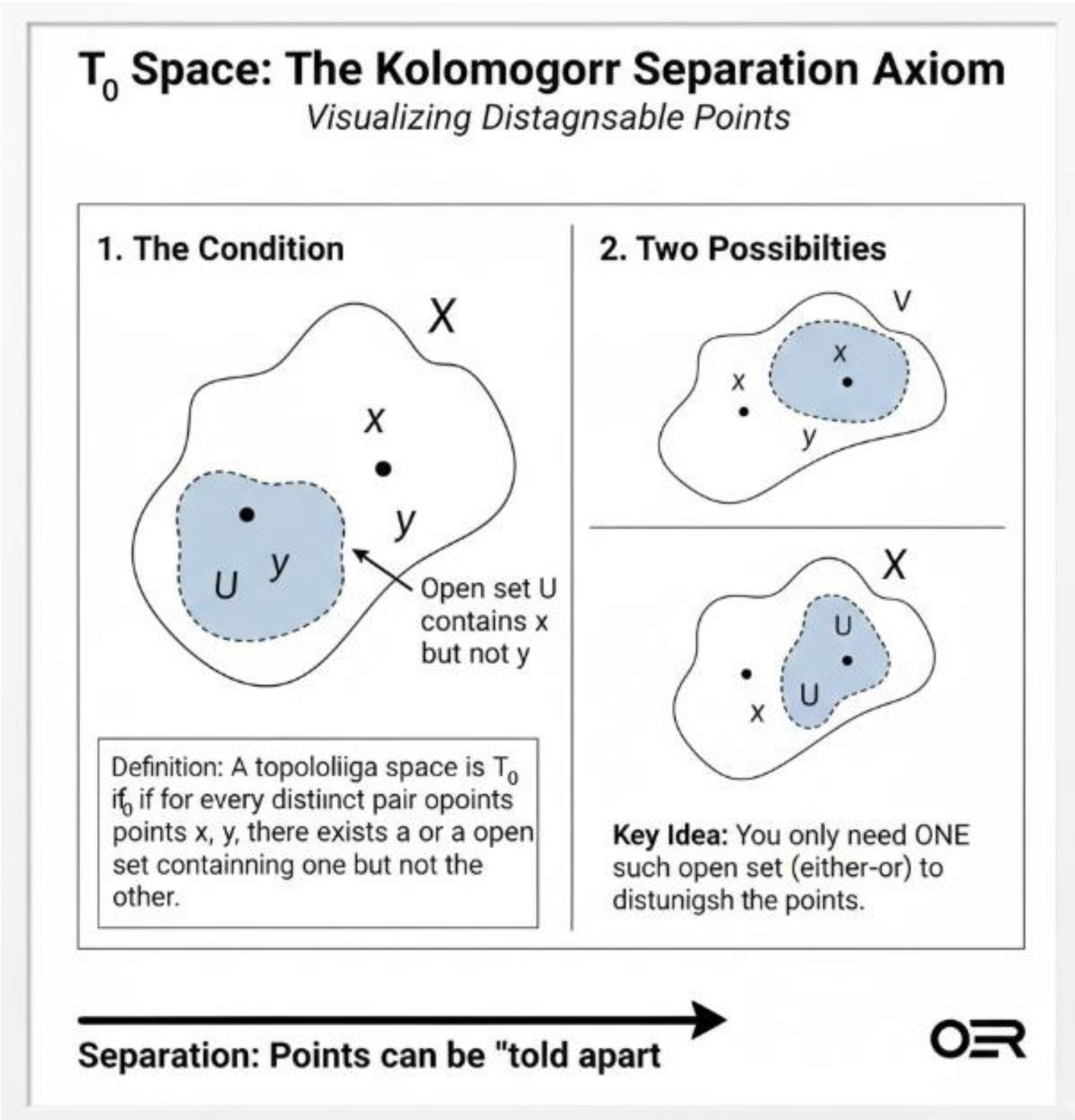
3.1 Separation Axioms

The **separation axioms** are conditions that describe how distinct points or sets in a topological space can be distinguished using open sets. These axioms provide a hierarchy of increasingly stronger properties, ranging from the weakest (T0) to stronger conditions such as Hausdorff (T2). They are essential for classifying spaces and understanding their behaviour in analysis and geometry.

Fill in the blank: Separation axioms describe how distinct points or sets can be distinguished using __________ sets.

3.1.1 T0 (Kolmogorov) spaces

A **T0 space** is a topological space where for any two distinct points, there exists an open set containing one but not the other. This ensures that points are topologically distinguishable.

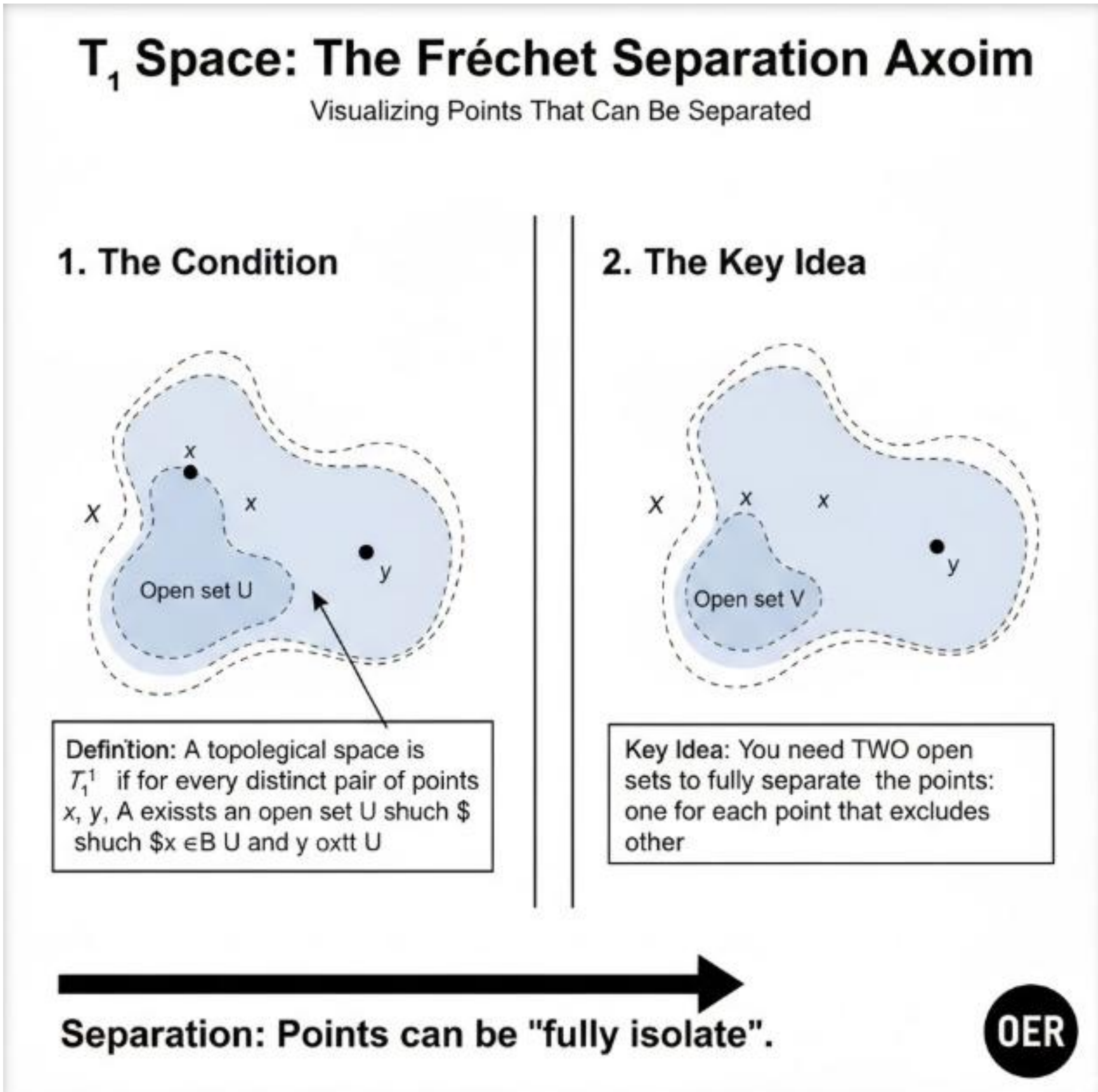


T0 space where points are distinguishable by open sets

Fill in the blank: A T0 space ensures that distinct points are __________ by open sets.

3.1.2 T1 (Fréchet) spaces

A **T1 space** is a topological space where for any two distinct points, each has an open set containing it but not the other. This means that singletons are closed sets in T1 spaces.

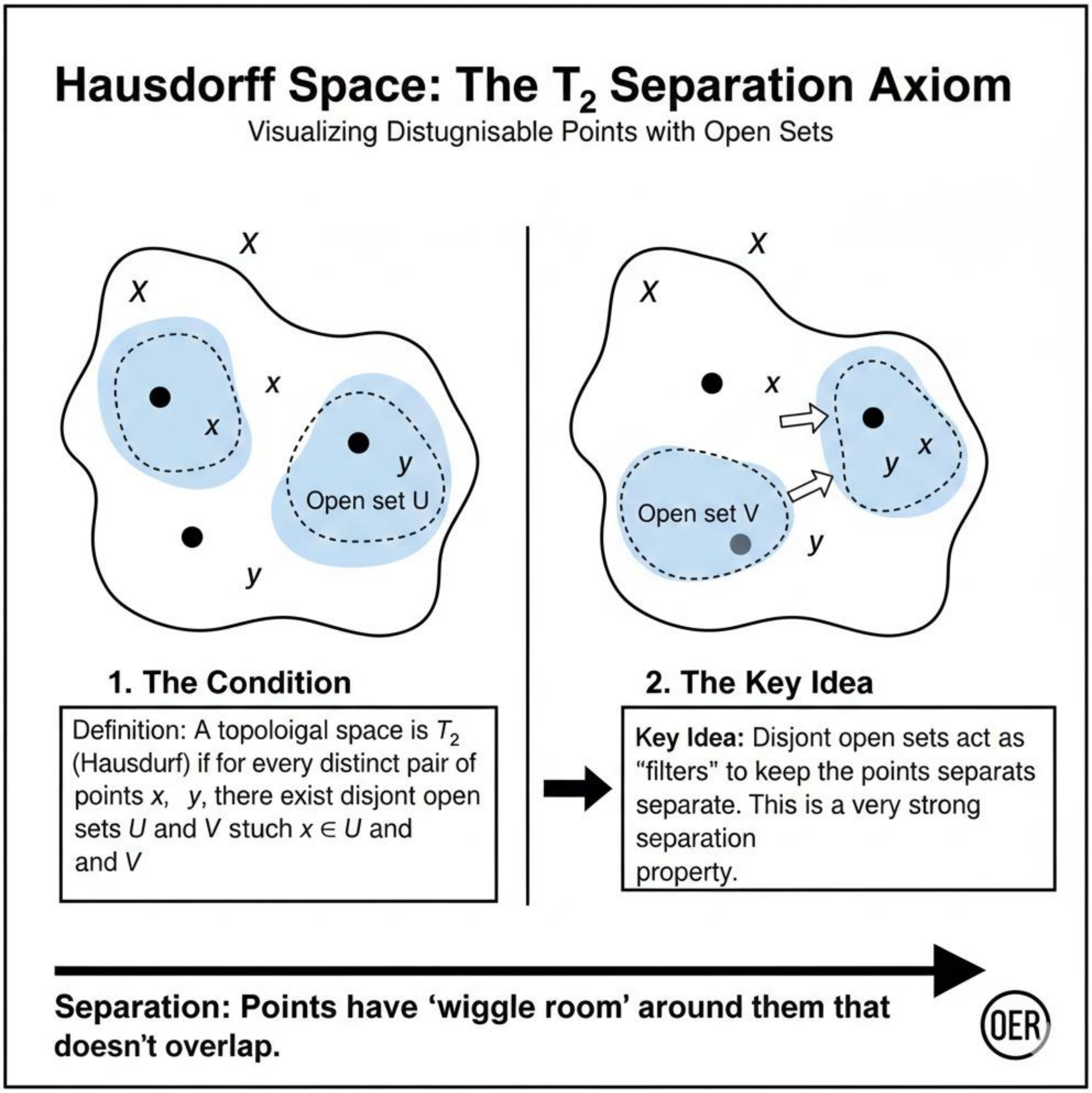


T1 space where singletons are closed. AI prompt: "

Fill in the blank: In a T1 space, singletons are always __________ sets.

3.1.3 T2 (Hausdorff) spaces

A **Hausdorff space** (T2) is a topological space where for any two distinct points, there exist disjoint open sets containing each point. This property is crucial in analysis, as it guarantees uniqueness of limits.



Hausdorff space where distinct points have disjoint neighbourhoods.

Fill in the blank: In a Hausdorff space, distinct points have __________ open sets around them.

3.1.4 Higher separation axioms

Beyond T2, there are stronger separation axioms:

- **T3 (Regular spaces):** Points and closed sets can be separated by disjoint open sets.

T4 (Normal spaces): Disjoint closed sets can be separated by disjoint open sets. These axioms are important in advanced analysis and topology, particularly in the study of continuous functions and compactness.

Higher separation axioms

T3: Points and closed sets separated by disjoint open sets

T4: Disjoint closed sets separated by disjoint open sets

Fill in the blank: In a T4 space, disjoint __________ sets can be separated by disjoint open sets.

Pilot questions 3.1

A T0 space ensures that: A. Distinct points are distinguishable by open sets B. Singletons are closed C. Distinct points have disjoint open sets D. Random subsets exist

A T1 space ensures that: A. Singletons are closed B. Distinct points are indistinguishable C. Distinct points have disjoint open sets D. Random subsets exist

A Hausdorff space ensures that: A. Distinct points have disjoint open sets B. Singletons are closed C. Distinct points are indistinguishable D. Random subsets exist

In a T3 space: A. Points and closed sets can be separated by disjoint open sets B. Disjoint closed sets can be separated by disjoint open sets C. Distinct points are indistinguishable D. Random subsets exist

In a T4 space: A. Disjoint closed sets can be separated by disjoint open sets B. Points and closed sets can be separated by disjoint open sets C. Distinct points are indistinguishable D. Random subsets exist

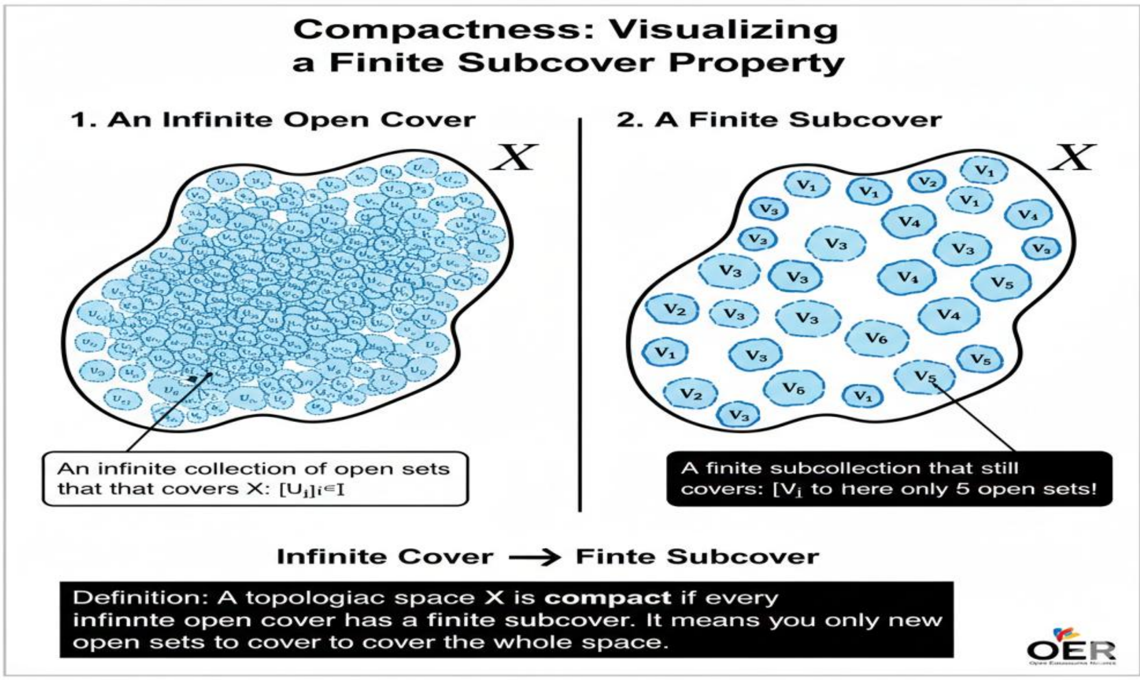
3.2 Compactness

The concept of **compactness** generalises the idea of closed and bounded sets in analysis. A topological space is **compact** if every open cover of the space has a finite subcover. This property is fundamental in topology and analysis because it ensures that certain behaviours, such as convergence and continuity, are well-controlled.

Fill in the blank: A space is compact if every open cover has a finite __________.

3.2.1 Definition of compactness

Formally, a topological space Xis compact if for every collection of open sets $\{U\}$such that $X=\bigcup U$, there exists a finite subcollection $\{U1,U2,\ldots,Un\}$that still covers X. Compactness is often described as a way of ensuring that infinite coverings can be reduced to finite ones.



Compactness illustrated by reducing an infinite cover to a finite subcover.

Fill in the blank: Compactness ensures that infinite coverings can be reduced to __________ coverings.

3.2.2 Examples of compact spaces

Closed interval [0,1]**in** R**:** Compact in the standard topology.

Finite sets: Always compact, since any open cover has a finite subcover.

Unit circle S1**:** Compact in the standard topology on R2.

Fill in the blank: The closed interval [0,1]in R is compact because it is __________ and bounded.

3.2.3 Importance of compactness

Compactness is crucial in analysis and topology because:

- Continuous functions on compact spaces are bounded and attain their maximum and minimum values.
- Compactness ensures convergence properties, such as every sequence having a cluster point in compact metric spaces.
- Many theorems, such as the Extreme Value Theorem, rely on compactness.

[Insert box here] Caption: Box 3.2: Importance of compactness

- Continuous functions on compact spaces are bounded and attain extrema
- Compactness ensures convergence properties
- Essential for theorems in analysis and topology

Fill in the blank: Continuous functions on compact spaces are always __________ and attain their extrema.

Pilot questions 3.2

- A space is compact if: A. Every open cover has a finite subcover B. Every closed cover has a finite subcover C. It is infinite D. It is discrete
- The closed interval [0,1]in Ris compact because: A. It is closed and bounded B. It is open and bounded C. It is infinite D. It is discrete
- Finite sets are always: A. Compact B. Non-compact C. Hausdorff D. Random
- Continuous functions on compact spaces are: A. Bounded and attain extrema B. Random C. Undefined D. Always discontinuous
- Compactness ensures that infinite coverings can be reduced to: A. Finite coverings B. Random coverings C. Closed coverings D. Metric coverings

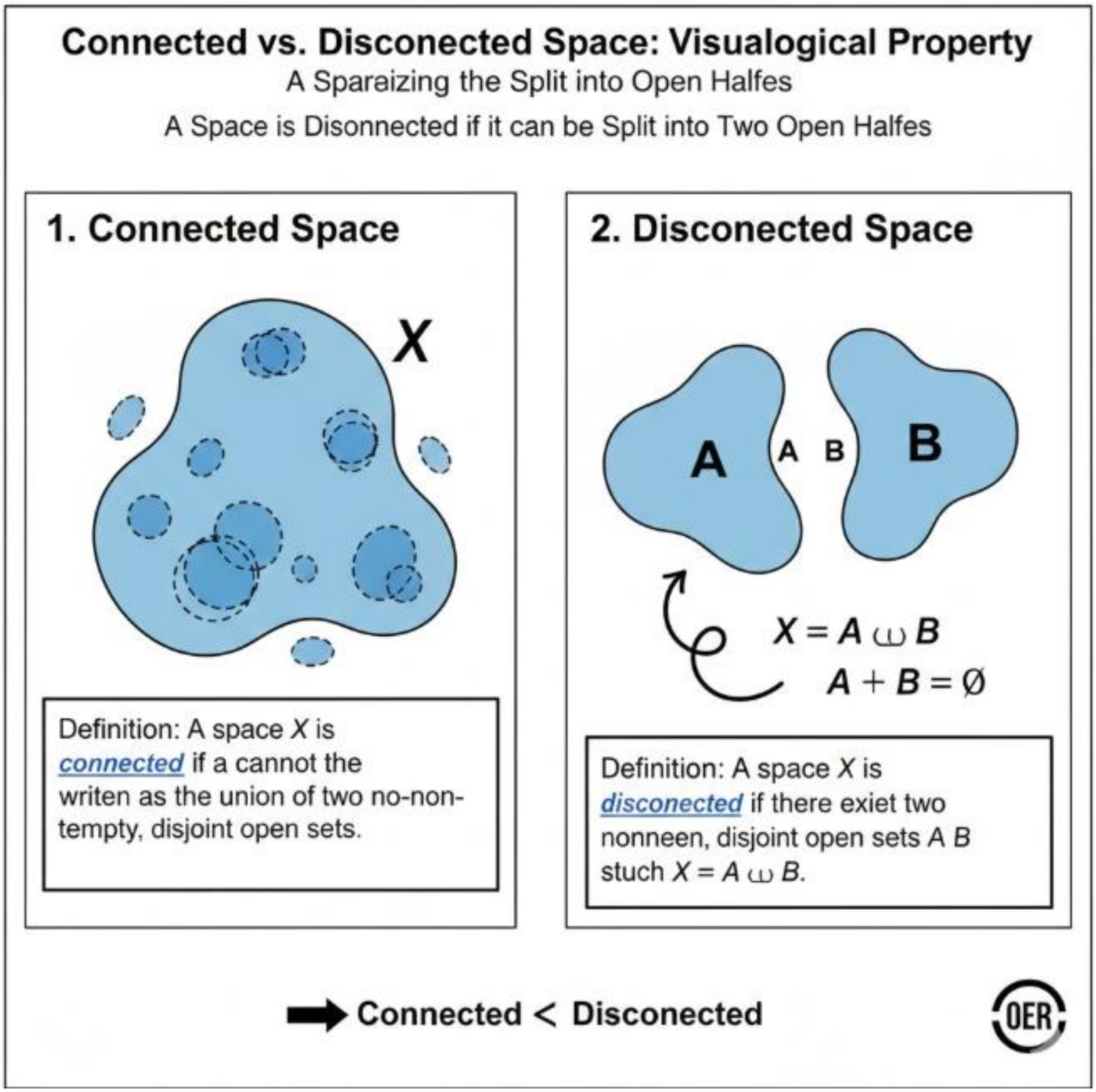
3.3 Connectedness

The concept of **connectedness** captures the idea of a space being “in one piece.” A topological space is **connected** if it cannot be divided into two disjoint non-empty open sets. This property is fundamental in topology because it distinguishes spaces that are whole from those that can be split apart.

Fill in the blank: A space is connected if it cannot be divided into two disjoint non-empty __________ sets.

3.3.1 Definition of connectedness

Formally, a topological space Xis connected if the only subsets of Xthat are both open and closed (clopen sets) are the empty set and Xitself. This definition ensures that the space cannot be partitioned into separate open pieces.



Connected and disconnected spaces illustrated.

Fill in the blank: In a connected space, the only clopen sets are the empty set and the __________ set.

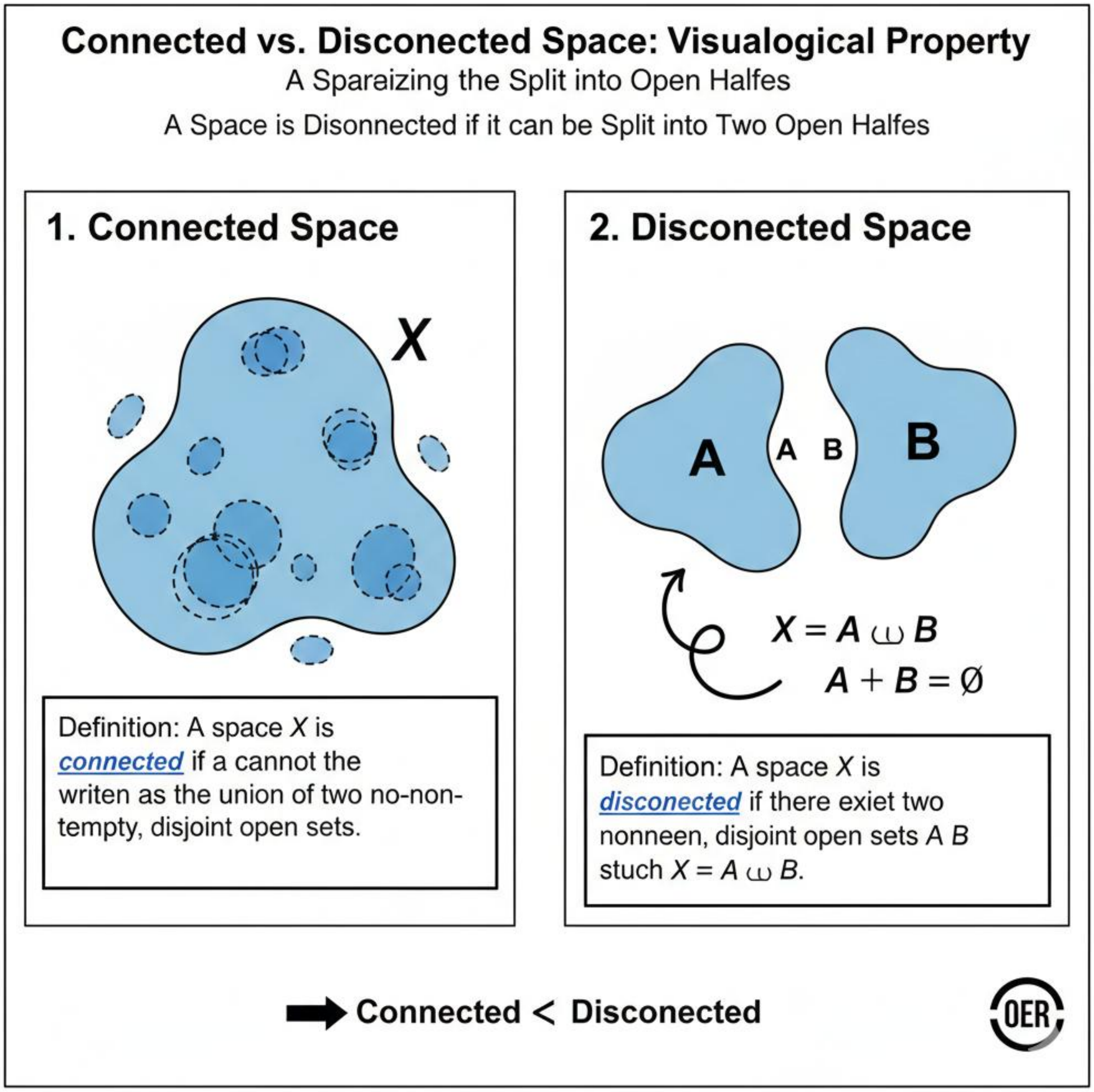
3.3.2 Examples of connected and disconnected spaces

Connected: The interval [0,1]in Ris connected.

Disconnected: The union of two disjoint intervals [0,1]∪[2,3]in Ris disconnected.

Connected: The unit circle S1in R2is connected.

Examples of connected and disconnected



Fill in the blank: The union of two disjoint intervals $[0,1]\cup[2,3]$in Ris __________.

3.3.3 Path connectedness

A stronger notion is **path connectedness**, where a space is path connected if any two points can be joined by a continuous path lying entirely within the space. Path connectedness implies connectedness, but the converse is not always true.

Path connectedness

Path connected: any two points joined by a continuous path

Path connected ⇒ connected

Connected does not always ⇒ path connected

Feature	Description
The Definition	A space X is path-connected if for every pair of points x, y \in X, there exists a continuous map f: [0, 1] \to X such that f(0) = x and f(1) = y
The "Path"	The map f is the "path." You can think of t \in [0, 1] as time, where at t=0 you are at the start and at t=1 you have reached the destination.
Relationship to Connectedness	**Path-connected \implies Connected.** If you can walk everywhere, the space is necessarily "all one piece."
The "Catch"	**Connected does not implies Path-connected.** Some spaces are technically one piece but are so "jagged" or "wiggly" that you can't draw a continuous line between certain points.

Fill in the blank: A space is path connected if any two points can be joined by a continuous __________.

Pilot questions 3.3

A space is connected if: A. It cannot be divided into two disjoint non-empty open sets B. It can be divided into two disjoint open sets C. It is finite D. It is discrete

In a connected space, the only clopen sets are: A. Empty set and whole set B. Singletons C. Open intervals D. Random subsets

The interval [0,1]in Ris: A. Connected B. Disconnected C. Random D. Undefined

The union [0,1]∪[2,3]in Ris: A. Disconnected B. Connected C. Random D. Undefined

A space is path connected if: A. Any two points can be joined by a continuous path B. It cannot be divided into open sets C. It is finite D. It is discrete

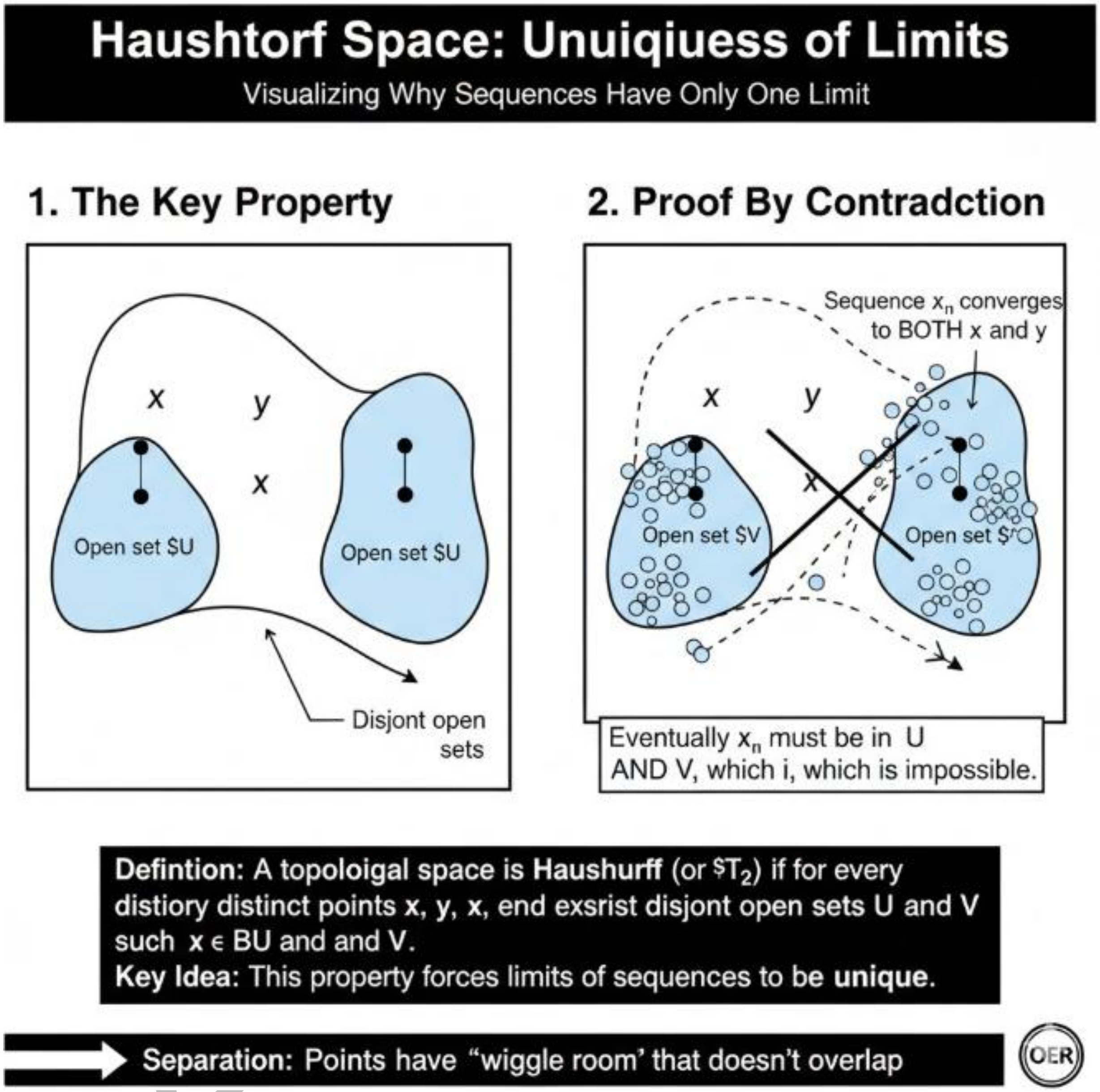
3.4 Applications And Examples

The concepts of separation axioms, compactness, and connectedness are not just abstract ideas. They have powerful applications in analysis, geometry, and beyond. By examining examples, you will see how these properties help classify spaces and ensure important results in mathematics.

Fill in the blank: Separation axioms, compactness, and connectedness are used to classify and __________ spaces in mathematics.

3.4.1 Applications of separation axioms

Separation axioms are crucial in analysis because they guarantee uniqueness of limits and allow us to separate points and sets. For example, Hausdorff spaces ensure that sequences and functions behave predictably, making them essential in calculus and functional analysis.



Hausdorff spaces guarantee uniqueness of limits. “

Fill in the blank: Hausdorff spaces are important because they guarantee uniqueness of __________.

3.4.2 Applications of compactness

Compactness ensures that continuous functions attain maximum and minimum values, which is vital in optimisation problems. It also guarantees convergence properties in metric spaces. For example, the Extreme Value Theorem in calculus relies on compactness.

[Insert box here] Caption: Box 3.4: Applications of compactness

Continuous functions attain extrema on compact spaces

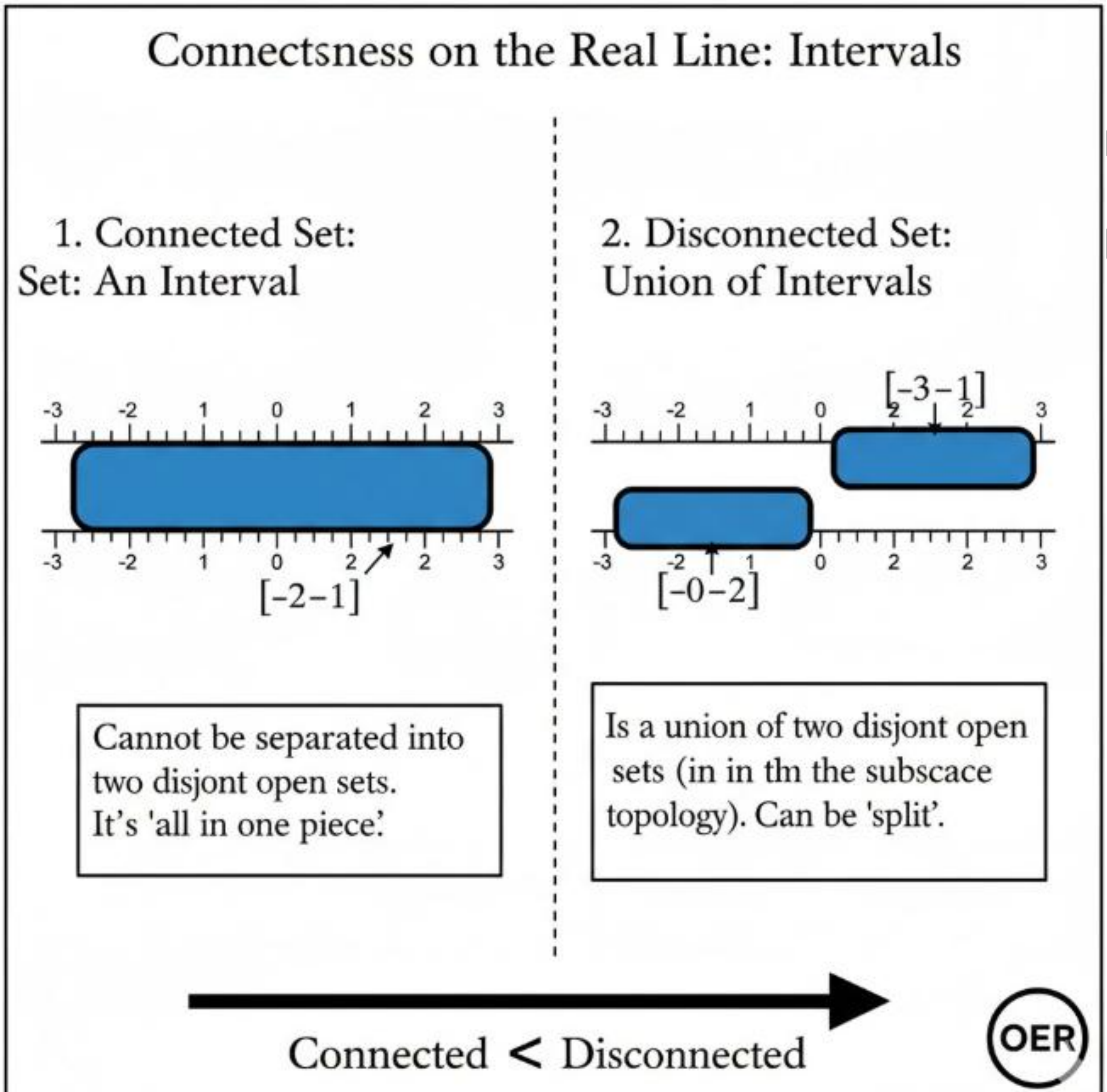
Compactness ensures convergence in metric spaces

Essential in optimisation and analysis

Fill in the blank: The Extreme Value Theorem relies on the property of __________.

3.4.3 Applications of connectedness

Connectedness helps us understand whether a space is “in one piece.” In analysis, connected subsets of Rare intervals. In geometry, connectedness ensures that paths exist between points in spaces like circles or surfaces. Path connectedness is particularly important in studying continuous functions and manifolds.



Connectedness illustrated with intervals in ℝ. **:**

Fill in the blank: Connected subsets of Rare always __________.

3.4.4 Combined applications

When combined, separation axioms, compactness, and connectedness provide a framework for studying continuity, convergence, and classification of spaces. For example, compact Hausdorff spaces play a central role in functional analysis, while connected compact spaces are fundamental in geometry.

[Insert table here] Caption: Table 3.3: Combined applications of topological properties

Application	Core Properties Used	How They Work Together
Existence of Solutions (ODEs)	Compactness + Connectedness	The **Peano Existence Theorem** uses compactness to find a convergent subsequence of approximate solutions, while connectedness ensures the solution exists over a continuous interval.
Optimization (Extreme Value Theorem)	Compactness + T2	In a **Compact Hausdorff** space, every continuous real-valued function is bounded and attains its maximum and minimum. Hausdorff ensures the "uniqueness" of points being tested.
Rigidity of Maps	Compactness + T2	A continuous bijection from a **Compact** space to a **Hausdorff** space is automatically a **homeomorphism**. This is a massive shortcut in proving two spaces are topologically identical.
Intermediate Value Theorem	Connectedness	Ensures that if a process moves from state A to state B continuously, it must pass through every state in between. In analysis, this is used to find roots of equations.
Data Manifolds (ML)	Connectedness + T2	Manifold learning assumes high-dimensional data lies on a **Connected Hausdorff** manifold. This allows

Application	Core Properties Used	How They Work Together
		algorithms to "unroll" data without it "shattering" into unrelated pieces.
Metrization (Urysohn)	T3 (Regular) + T2	The **Urysohn Metrization Theorem** tells us that if a space is T3 and has a countable base, we can define a distance metric for it, effectively turning topology back into geometry.

Fill in the blank: Compact Hausdorff spaces are central in __________ analysis.

Pilot questions 3.4

Hausdorff spaces are important because they guarantee: A. Uniqueness of limits B. Randomness C. Disconnectedness D. Undefined sets

Compactness ensures that continuous functions: A. Attain extrema B. Are always discontinuous C. Are random D. Are undefined

Connected subsets of Rare always: A. Intervals B. Disjoint unions C. Random sets D. Undefined

The Extreme Value Theorem relies on: A. Compactness B. Connectedness C. Separation axioms D. Randomness

Compact Hausdorff spaces are central in: A. Functional analysis B. Randomness C. Geometry only D. Undefined sets

Series summary

This Series has introduced three of the most important properties used to classify and analyse topological spaces. We began with the separation axioms, which describe how points and sets can be distinguished using open sets, progressing from T0 to Hausdorff (T2) and beyond. We then studied compactness, a property that generalises closed and bounded sets and ensures that infinite coverings can be reduced to finite ones. Following that, we explored connectedness, which captures the idea of a space being "in one piece" and introduced path connectedness as a stronger notion. Finally, we examined applications and examples, showing how these properties are used in analysis, optimisation, and geometry. By completing this Series, you should now be able to classify spaces using separation axioms, evaluate compactness, determine connectedness, and apply these concepts in practical contexts, thereby achieving the Series Learning Outcomes.

Glossary of terms

Separation axioms (T0–T4): Conditions describing how points and sets can be distinguished using open sets.

T0 (Kolmogorov space): Distinct points are topologically distinguishable.

T1 (Fréchet space): Singletons are closed.

T2 (Hausdorff space): Distinct points have disjoint neighbourhoods.

T3 (Regular space): Points and closed sets can be separated by disjoint open sets.

T4 (Normal space): Disjoint closed sets can be separated by disjoint open sets.

Compactness: Property where every open cover has a finite subcover.

Connectedness: Property where a space cannot be divided into two disjoint non-empty open sets.

Path connectedness: Stronger property where any two points can be joined by a continuous path.

Compact Hausdorff space: A space that is both compact and Hausdorff, central in functional analysis.

Concept check questions 3

(SLO 3.1) Define a T0 space and explain how it differs from a T1 space.

(SLO 3.1) What property makes Hausdorff spaces essential in analysis?

(SLO 3.2) State the definition of compactness and give an example of a compact space.

(SLO 3.2) Why are continuous functions on compact spaces guaranteed to attain extrema?

(SLO 3.3) Define connectedness and explain the role of clopen sets.

(SLO 3.3) Distinguish between connectedness and path connectedness with examples.

(SLO 3.4) Explain why compact Hausdorff spaces are important in functional analysis.

(SLO 3.4) Give an example of how connectedness is applied in geometry

Concept Check Questions 3

Objective questions

A T0space is also called a __________ space. (Linked to SLO 3.1)

In a Hausdorff (T2) space, any two distinct points have: A. Disjoint neighbourhoods B. The same neighbourhood C. No neighbourhoods D. None of the above (Linked to SLO 3.1)

Compactness means that every open cover has a __________ subcover. (Linked to SLO 3.2)

A space is connected if it cannot be written as the union of two __________ open sets. (Linked to SLO 3.3)

Path connectedness implies that any two points can be joined by a __________. (Linked to SLO 3.3)

Short-answer questions

Define a T1(Fréchet) space. (Linked to SLO 3.1)

Give one example of a compact space. (Linked to SLO 3.2)

Explain why compactness is important in analysis. (Linked to SLO 3.2)

Provide an example of a connected space. (Linked to SLO 3.3)

State one application of separation axioms in topology. (Linked to SLO 3.4)

Long-answer questions

Analyse the differences between T0, T1, and T2spaces with examples. (Linked to SLO 3.1)

Evaluate the role of compactness in ensuring convergence and continuity. (Linked to SLO 3.2)

Discuss the concept of connectedness and path connectedness, providing examples of each. (Linked to SLO 3.3)

Examine combined applications of separation axioms, compactness, and connectedness in topology. (Linked to SLO 3.4)

Answers to Concept Check Questions 3

Objective questions

Kolmogorov

A

Finite

Disjoint

Path

Short-answer questions

A T1space is one where for any two distinct points, each has a neighbourhood not containing the other.

The closed interval 1in Ris compact.

Compactness ensures that continuous functions on compact spaces are bounded and attain maxima and minima.

The real line Ris connected.

Separation axioms are used to distinguish spaces and ensure uniqueness of limits.

Long-answer questions

Basic response: T0spaces distinguish points by open sets, T1spaces allow each point to be separated, and T2spaces separate points with disjoint neighbourhoods. *Value-added response*: Examples include the trivial topology (often T0), the cofinite topology (T1), and the Euclidean line (T2). These distinctions are crucial for understanding separation and uniqueness of limits.

Basic response: Compactness ensures every open cover has a finite subcover. *Value-added response*: This property guarantees that continuous functions on compact spaces are bounded and achieve extrema, making compactness vital in analysis and optimisation.

Basic response: A space is connected if it cannot be split into two disjoint open sets. Path connectedness means any two points can be joined by a continuous path.

Value-added response: For example, Ris connected and path connected, while the topologist's sine curve is connected but not path connected. These distinctions highlight subtle differences in topology.

Basic response: Separation axioms, compactness, and connectedness each have applications in topology. *Value-added response*: Together, they provide a framework for classifying spaces, proving theorems such as the Heine–Borel theorem, and constructing spaces with desired properties in analysis and geometry.

Answers to Pilot Questions 3

Part 3.1: 1-Kolmogorov, 2-Fréchet, 3-Hausdorff, 4-Higher separation axioms **Part 3.2**: 1-Definition of compactness, 2-Examples, 3-Importance **Part 3.3**: 1-Definition of connectedness, 2-Examples, 3-Path connectedness **Part 3.4**: 1-Applications of separation axioms, 2-Applications of compactness, 3-Combined applications

Series 4: Continuity, Homeomorphisms And Quotient Topologies

Series outline

4.1 Continuity in topological spaces
- 4.1.1 Definition of continuity
- 4.1.3 Examples of continuous functions
- 4.1.2 Equivalent characterisations

4.2 Homeomorphisms
- 4.2.1 Definition of homeomorphism
- 4.2.2 Examples of homeomorphisms
- 4.2.3 Importance of homeomorphisms

4.3 Quotient topologies
- 4.3.1 Definition of quotient topology
- 4.3.2 Examples of quotient topologies
- 4.3.3 Importance of quotient topologies

4.4 Applications and examples
- 4.4.4 Combined applications
- 4.4.1 Applications of continuity
- 4.4.2 Applications of homeomorphisms
- 4.4.3 Applications of quotient topologies

Introduction

In the previous Series, we studied separation axioms, compactness, and connectedness, which are fundamental properties used to classify and analyse topological spaces. Building on that foundation, we now turn to the study of functions between topological spaces and how they preserve or transform structure. The aim of this Series is to introduce you to continuity, homeomorphisms, and quotient topologies. These concepts are central to understanding how spaces relate to one another and how topological structures can be transformed or simplified. We shall commence this Series by examining continuity in topological spaces, which generalises the familiar notion of continuity from calculus. Next, we will study homeomorphisms, which are structure-preserving maps that identify when two spaces are "the same" topologically. Following that, we will explore quotient topologies, which allow us to construct new spaces by identifying points. Finally, we will conclude with applications and examples, showing how these concepts are used in practice.

Series learning outcomes

By the end of this Series, you should be able to:

- SLO 4.1: Define continuity in topological spaces and explain its properties.
- SLO 4.2: Define homeomorphisms and explain their role in classifying spaces.
- SLO 4.3: Construct quotient topologies and analyse their properties.
- SLO 4.4: Apply continuity, homeomorphisms, and quotient topologies to examples and problems in topology.

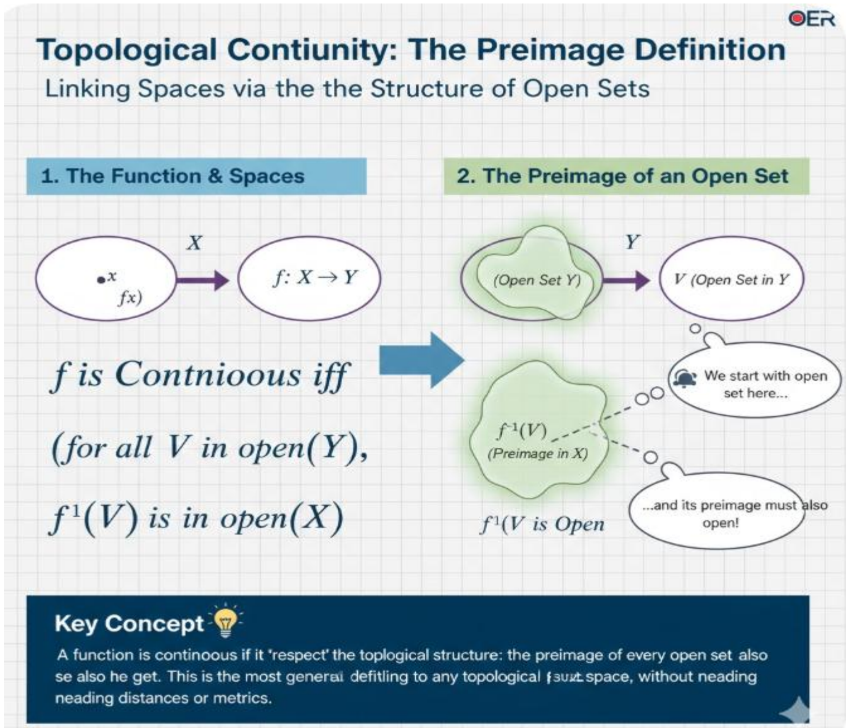
4.1 Continuity in Topological Spaces

Continuity in topology generalises the familiar notion of continuity from calculus. Instead of relying on distances, continuity is defined using open sets. A function between two topological spaces is continuous if the preimage of every open set is open.

Fill in the blank: A function is continuous if the preimage of every __________ set is open.

4.1.1 Definition of continuity

Let $f:X\to Y$be a function between topological spaces. The function $f$is continuous if for every open set $U\subseteq Y$, the preimage $f\text{-}1(U)$is open in X. This definition generalises the – definition from analysis.



Continuity defined by preimages of open sets.

Fill in the blank: Continuity in topology is defined using __________ sets rather than distances.

4.1.2 Equivalent characterisations

Continuity can be characterised in several equivalent ways:

- A function is continuous if the preimage of every closed set is closed.
- A function is continuous if it preserves neighbourhoods: for every neighbourhood of f(x), there exists a neighbourhood of xmapping into it.
- In metric spaces, this definition coincides with the usual – definition.

Equivalent characterisations of continuity

- Preimage of open sets is open
- Preimage of closed sets is closed
- Preserves neighbourhoods
- Fill in the blank: A function is continuous if the preimage of every __________ set is closed.

4.1.3 Examples of continuous functions

- Identity function: Always continuous, since preimages are identical to sets themselves.
- Polynomial functions on R: Continuous in the standard topology.

Projection maps in product spaces: Continuous by definition of the product topology.

Examples of continuous functions

Function	Space	Reason for continuity
Identity f(x)=x	Any space	Preimage equals the set itself
Polynomial f(x)=x2	R	Preimages of open sets are open
Projection π(x,y)=x	Product space	Defined by product topology

Fill in the blank: The identity function is always __________ in any topological space.

Pilot questions 4.1

A function is continuous if: A. The preimage of every open set is open B. The image of every open set is open C. The preimage of every closed set is open D. Random subsets exist

Continuity in topology is defined using: A. Open sets B. Distances only C. Random subsets D. Metrics only

A function is continuous if the preimage of every: A. Closed set is closed B. Open set is closed C. Random set is open D. Singleton is open

The identity function is: A. Always continuous B. Never continuous C. Random D. Undefined

Projection maps in product spaces are: A. Continuous B. Discontinuous C. Random D. Undefined

4.2 Homeomorphisms

A homeomorphism is a function between two topological spaces that shows they are "the same" from a topological perspective. It is a bijective continuous function whose inverse is also continuous. If such a function exists, the two spaces are said to be homeomorphic.

Fill in the blank: Two spaces are homeomorphic if there exists a bijective continuous function with a continuous __________.

4.2.1 Definition of homeomorphism

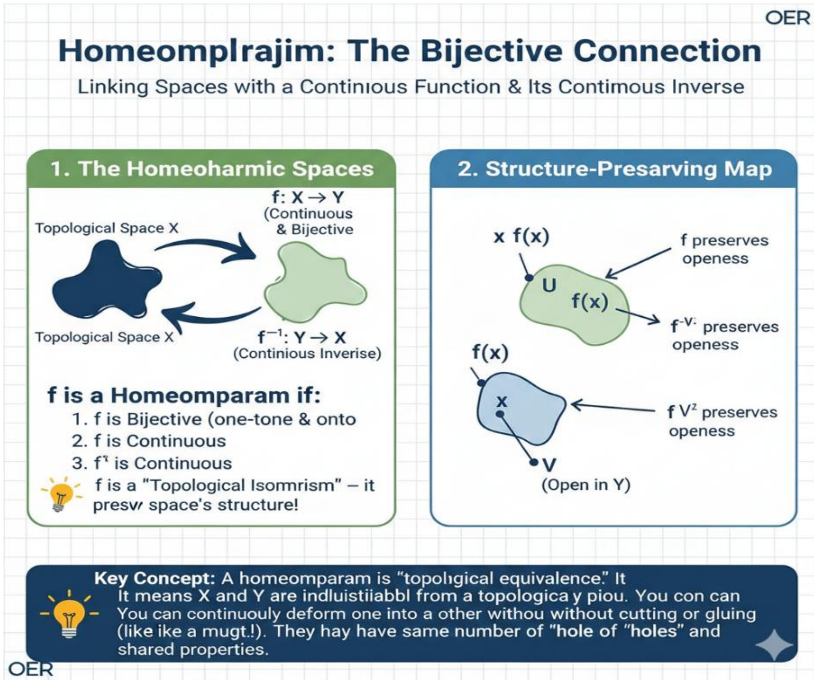
Formally, a function f:X→Yis a homeomorphism if:

fis bijective (one-to-one and onto).

fis continuous.

f-1is continuous.

This definition ensures that the spaces Xand Yhave identical topological structure.



Homeomorphism between two spaces.

Fill in the blank: A homeomorphism preserves the __________ structure of spaces.

4.2.2 Examples of homeomorphisms

Open interval (0,1)and R: Homeomorphic via the function $f(x)=\tan(\pi(x-1/2))$.

Unit circle S1and boundary of unit square: Homeomorphic because both are continuous closed curves.

Identity map: Always a homeomorphism between a space and itself.

Fill in the blank: The unit circle and the boundary of the unit square are __________.

4.2.3 Importance of homeomorphisms

Homeomorphisms are central in topology because they classify spaces up to topological equivalence. If two spaces are homeomorphic, they share all topological properties such as connectedness, compactness, and separation axioms. This allows mathematicians to study spaces by transforming them into simpler or more familiar ones.

Importance of homeomorphisms

Classify spaces up to topological equivalence

Preserve properties like connectedness and compactness

Allow transformation into simpler spaces for study

1. The Concept of "Topological Equivalence"

Homeomorphisms allow us to ignore properties that are "brittle"—like exact distance, angle, or curvature—and focus on properties that are "rubbery." Because a homeomorphism is a continuous map with a continuous inverse, it behaves like a **perfectly elastic deformation**:

- You can **stretch, twist, and bend** the space.
- You **cannot** tear it, puncture it, or glue separate parts together.

2. Identifying Topological Invariants

The true power of homeomorphisms is that they preserve **Topological Invariants**. If space X is homeomorphic to space Y, they must share:

- **Connectedness:** If you can get from any point to any other in X, you can do the same in Y.
- **Compactness:** If X is "small and closed" in a specific sense, Y must be too.
- **Dimension:** A 1D line can never be homeomorphic to a 2D plane.
- **The Number of Holes:** A doughnut (genus 1) is not homeomorphic to a sphere (genus 0).

3. Simplification of Problems

Homeomorphisms allow mathematicians to move a problem from a "difficult" space to an "easy" one.

- **Example:** If you need to prove a property about a complex, jagged loop, and you can prove that the loop is homeomorphic to a perfect circle, you can often prove the property for the circle and know it applies to the jagged loop automatically.

Pilot questions 4.2

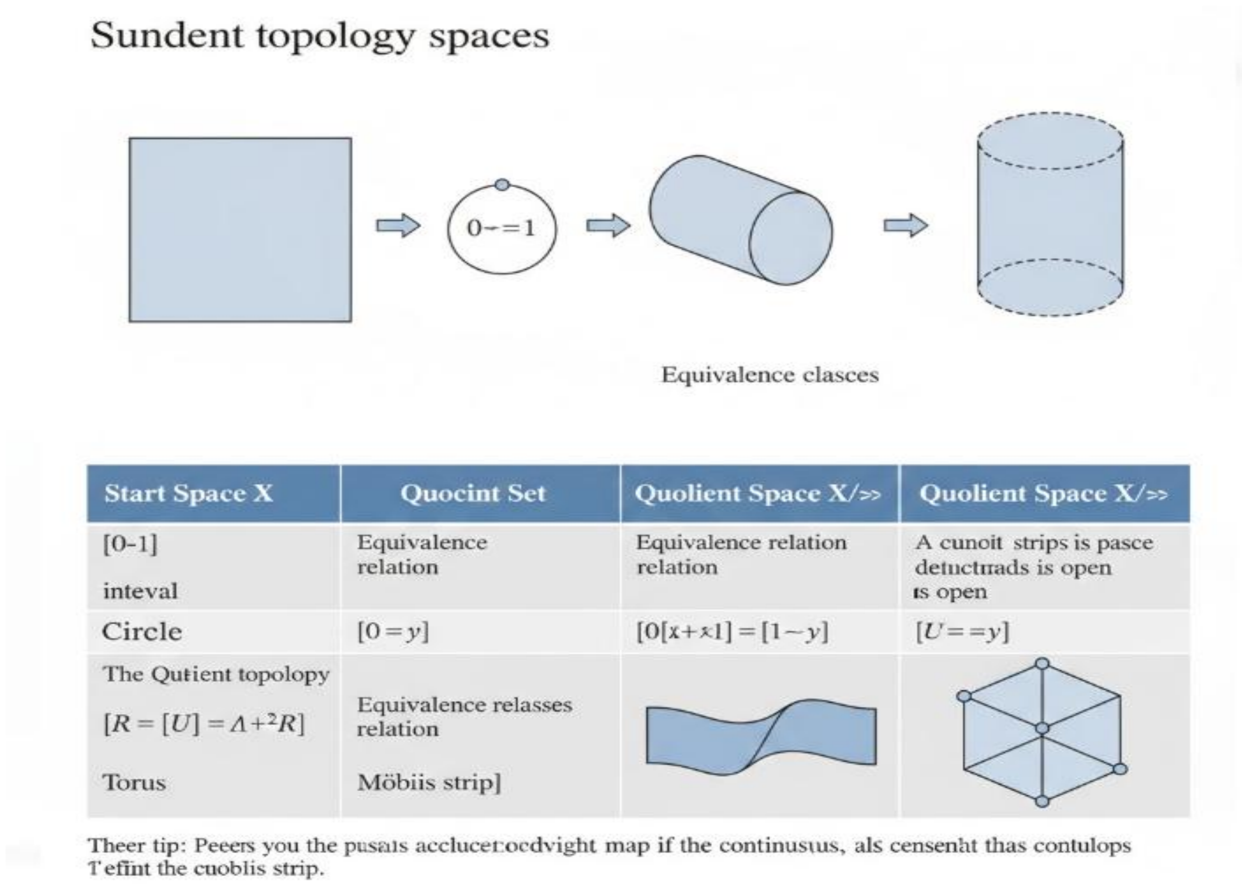
A homeomorphism is: A. A bijective continuous function with continuous inverse B. Any bijective function C. Any continuous function D. Random subset

Two spaces are homeomorphic if: A. There exists a bijective continuous function with continuous inverse B. They are equal as sets C. They are both finite D. They are random

The identity map is: A. Always a homeomorphism B. Never a homeomorphism C. Random D. Undefined
The unit circle and boundary of the unit square are: A. Homeomorphic B. Not homeomorphic C. Random D. Undefined
Homeomorphisms classify spaces up to: A. Topological equivalence B. Metric equivalence C. Randomness D. Undefinedness

4.3 Quotient Topologies
A quotient topology is a way of constructing new topological spaces by identifying points of an existing space according to an equivalence relation. This process allows us to "glue" points together and study the resulting structure.
Fill in the blank: A quotient topology is formed by identifying points according to an __________ relation.

4.3.1 Definition of quotient topology
Let Xbe a topological space and an equivalence relation on X. The set of equivalence classes is denoted X/. The quotient topology on X/is defined such that a set U⊆X/is open if and only if its preimage under the natural projection map π:X→X/is open in X.



Constructing a quotient topology by identifying points.

__________ under the projection map is open.

4.3.2 Examples of quotient topologies

- Circle from interval: The unit circle S1can be obtained by taking the interval [0,1]and identifying the endpoints 0∼1.
- Real projective plane: Constructed by identifying opposite points on the unit sphere in R3.
- Cylinder and Möbius strip: Obtained by identifying edges of a square in different ways.

Examples of quotient topologies

Space	Construction	Identification
Circle S1	Interval [0,1]	Endpoints 0∼1
Real projective plane	Sphere in R3	Opposite points
Cylinder	Square	Opposite vertical edges

Möbius strip	Square	Opposite vertical edges with twist

Fill in the blank: The unit circle can be obtained by identifying the __________ of the interval [0,1].

4.3.3 Importance of quotient topologies

Quotient topologies are important because they allow construction of new spaces from familiar ones. They are widely used in geometry, algebraic topology, and manifold theory. Many complex spaces, such as projective spaces and tori, are defined using quotient topologies.

Importance of quotient topologies

Construct new spaces by identifying points

Used in geometry, algebraic topology, and manifold theory

Basis for defining projective spaces and tori

Benefit	*Explanation*
Dimensionality	*Allows us to study n-dimensional manifolds as quotients of \mathbb{R}^n.*
Simplification	*Reduces a space by removing redundant information (symmetry).*
Abstract Modeling	*Defines "Projective Spaces," which are critical in computer vision and algebraic geometry.*

Fill in the blank: Quotient topologies are widely used in geometry and __________ topology.

Pilot questions 4.3

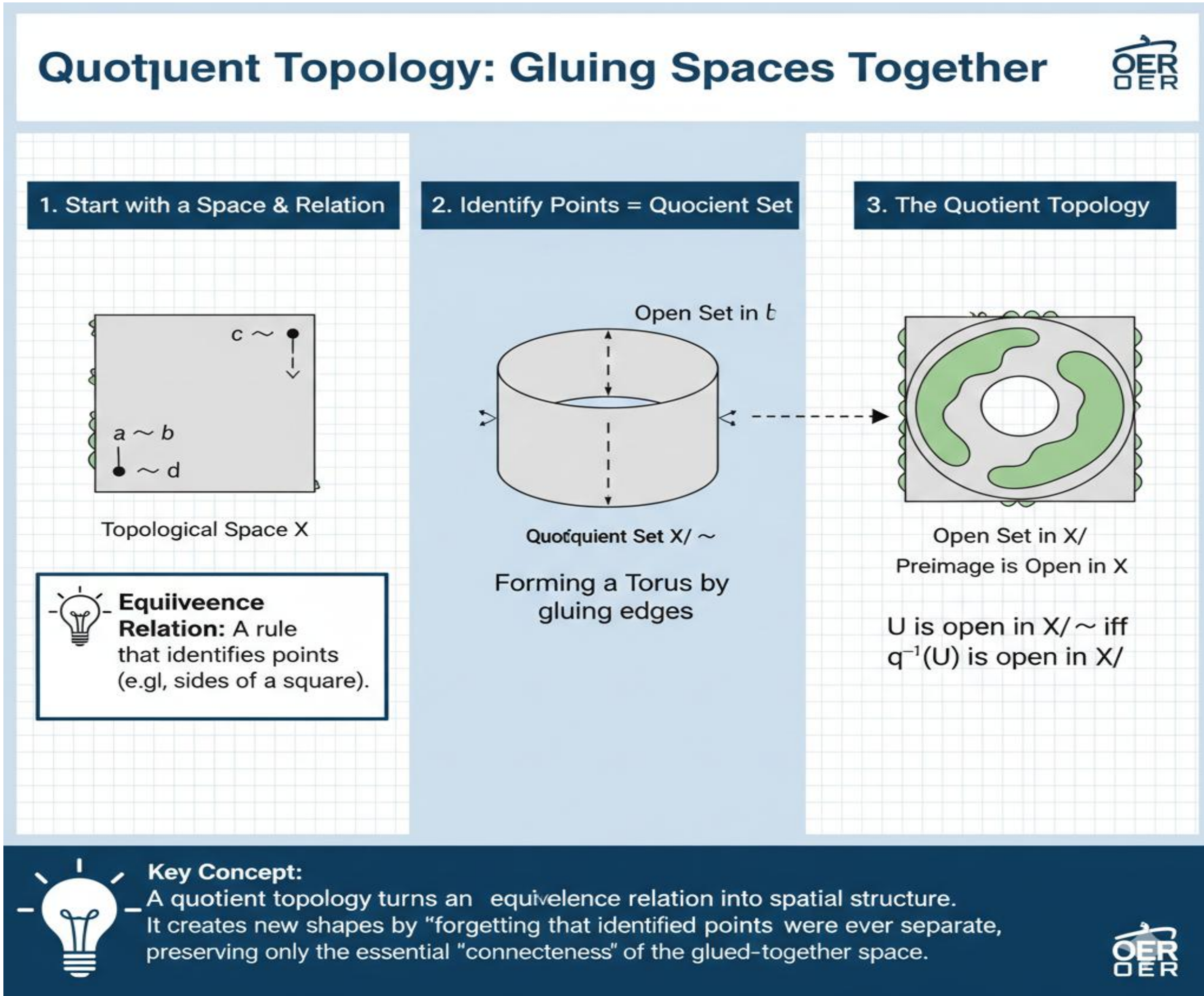
- A quotient topology is formed by: A. Identifying points according to an equivalence relation B. Randomly deleting points C. Adding new points arbitrarily D. Using metrics only
- In a quotient topology, a set is open if: A. Its preimage under the projection map is open B. Its image is closed C. It is finite D. It is random
- The unit circle can be obtained by: A. Identifying endpoints of the interval [0,1]B. Identifying midpoints of the interval [0,1]C. Identifying random points D. Adding new points
- The real projective plane is constructed by: A. Identifying opposite points on the sphere B. Identifying endpoints of an interval C. Identifying edges of a square with twist D. Randomly deleting points

Quotient topologies are important because they: A. Construct new spaces by identifying points B. Delete points randomly C. Add points arbitrarily D. Are undefined

4.3 Quotient Topologies
A quotient topology is a way of constructing new topological spaces by identifying points of an existing space according to an equivalence relation. This process allows us to "glue" points together and study the resulting structure.

Fill in the blank: A quotient topology is formed by identifying points according to an __________ relation.

4.3.1 Definition of quotient topology
Let Xbe a topological space and an equivalence relation on X. The set of equivalence classes is denoted X/. The quotient topology on X/is defined such that a set U⊆X/is open if and only if its preimage under the natural projection map π:X→X/is open in X.



Constructing a quotient topology by identifying points.

Fill in the blank: In a quotient topology, a set is open if its __________ under the projection map is open.

4.3.2 Examples of quotient topologies

- Circle from interval: The unit circle S1can be obtained by taking the interval [0,1]and identifying the endpoints 0∼1.
- Real projective plane: Constructed by identifying opposite points on the unit sphere in R3.
- Cylinder and Möbius strip: Obtained by identifying edges of a square in different ways.

Examples of quotient topologies

Space	Construction	Identification
Circle S1	Interval [0,1]	Endpoints 0∼1

Real projective plane	Sphere in R3	Opposite points
Cylinder	Square	Opposite vertical edges
Möbius strip	Square	Opposite vertical edges with twist

Fill in the blank: The unit circle can be obtained by identifying the __________ of the interval [0,1].

4.3.3 Importance of quotient topologies

Quotient topologies are important because they allow construction of new spaces from familiar ones. They are widely used in geometry, algebraic topology, and manifold theory. Many complex spaces, such as projective spaces and tori, are defined using quotient topologies.

Importance of quotient topologies

- Construct new spaces by identifying points
- Used in geometry, algebraic topology, and manifold theory
- Basis for defining projective spaces

Key Properties of the Quotient Map

A map $q : X \to Y$ is a **quotient map** if it is onto (surjective) and a set $U \subseteq Y$ is open if and only if its preimage $q^{-1}(U)$ is open in X.

Benefit	Explanation
Dimensionality	Allows us to study n-dimensional manifolds as quotients of $\mathbb{R}^n$.
Symmetry Reduction	Collapses redundant information in systems with high symmetry.
Abstract Modeling	Defines "Moduli Spaces," where each point represents an entire geometric object.

Fill in the blank: Quotient topologies are widely used in geometry and __________ topology.

Pilot questions 4.3

- A quotient topology is formed by: A. Identifying points according to an equivalence relation B. Randomly deleting points C. Adding new points arbitrarily D. Using metrics only
- In a quotient topology, a set is open if: A. Its preimage under the projection map is open B. Its image is closed C. It is finite D. It is random

The unit circle can be obtained by: A. Identifying endpoints of the interval [0,1]B. Identifying midpoints of the interval [0,1]C. Identifying random points D. Adding new points

The real projective plane is constructed by: A. Identifying opposite points on the sphere B. Identifying endpoints of an interval C. Identifying edges of a square with twist D. Randomly deleting points

Quotient topologies are important because they: A. Construct new spaces by identifying points B. Delete points randomly C. Add points arbitrarily D. Are undefined

4.4 Applications and Examples

The concepts of continuity, homeomorphisms, and quotient topologies are central to how topology interacts with analysis, geometry, and algebra. By examining applications and examples, we see how these abstract ideas have concrete consequences.

Fill in the blank: Continuity, homeomorphisms, and quotient topologies are used to relate and __________ spaces in mathematics.

4.4.1 Applications of continuity

Continuity ensures that functions behave predictably with respect to open sets. In analysis, continuous functions preserve limits and are essential for calculus. In topology, continuity allows us to compare spaces and study maps between them.

Comparison: What is Preserved?

Feature	Continuous Function	Homeomorphism (Stronger)
Open Sets	Preimages are open	Both preimages and images are open
Convergence	Preserved ($x_n \to L \implies f(x_n) \to f(L)$)	Preserved in both directions
Connectedness	If A is connected, $f(A)$ is connected	Topology is identical
Compactness	If K is compact, $f(K)$ is compact	Compactness is identical

Continuity preserves structure between spaces.

Fill in the blank: In topology, continuity allows us to compare and study __________ between spaces.

4.4.2 Applications of homeomorphisms

Homeomorphisms classify spaces up to topological equivalence. For example, a circle and the boundary of a square are homeomorphic, meaning they share the same topological properties. This allows mathematicians to study complex spaces by transforming them into simpler ones.

Applications of homeomorphisms

- *Classify spaces up to equivalence*
- *Preserve properties like connectedness and compactness*
- *Simplify study of complex spaces*

Common Examples of Topological Equivalence

Space A	Space B	Relationship
Open Interval $(0,1)$	**Real Line** $\mathbb{R}$	Homeomorphic via tangent function scaling.
Square (Boundary)	**Circle** (S^1)	Homeomorphic (bending without tearing).
Sphere (S^2)	**Punctured Plane**	Homeomorphic via Stereographic Projection.

Fill in the blank: Homeomorphisms preserve properties such as connectedness and __________.

4.4.3 Applications of quotient topologies

Quotient topologies allow construction of new spaces by identifying points. They are widely used in geometry and algebraic topology. For example:

- The circle S1is obtained by identifying endpoints of an interval.
- The cylinder and Möbius strip are obtained by identifying edges of a square.
- Projective spaces are defined using quotient topologies.

Applications of quotient topologies

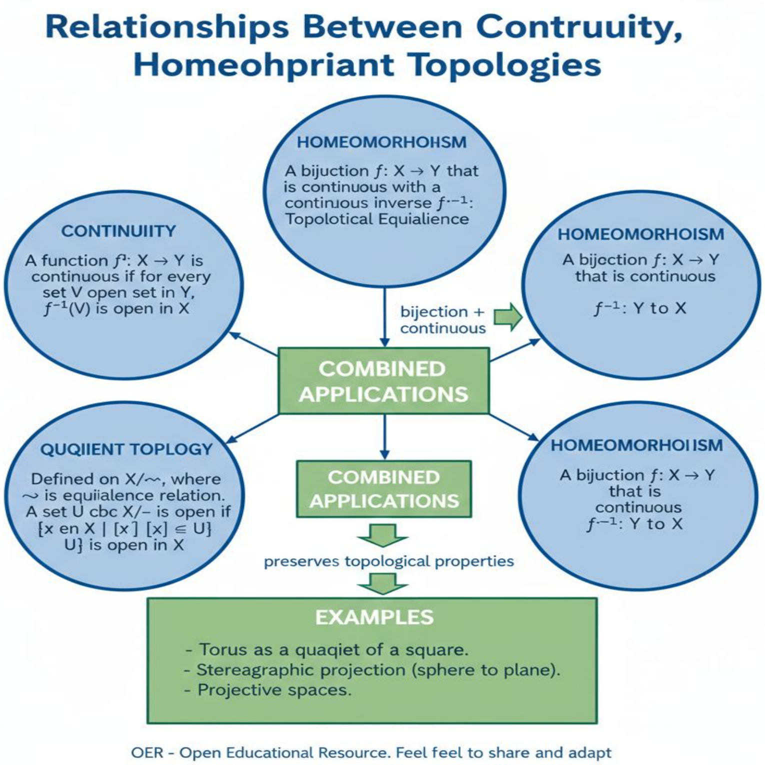
Space	Construction	Application
Circle S1	Interval with endpoints identified	Geometry, trigonometry
Cylinder	Square with opposite edges identified	Geometry, surfaces

Möbius strip	Square with twisted edge identification	Non-orientable surfaces
Projective plane	Sphere with opposite points identified	Algebraic topology

Fill in the blank: The Möbius strip is obtained by identifying edges of a square with a __________.

4.4.4 Combined applications

Together, continuity, homeomorphisms, and quotient topologies provide a framework for understanding transformations and classifications of spaces. For example, continuous maps between quotient spaces, or homeomorphisms between compact spaces, are central in functional analysis and geometry.



Combined applications of continuity, homeomorphisms, and quotient topologies.

Fill in the blank: Compact spaces related by homeomorphisms are central in __________ analysis.

Pilot questions 4.4

Continuity in topology allows us to study: A. Maps between spaces B. Random subsets C. Distances only D. Undefined sets

Homeomorphisms classify spaces up to: A. Topological equivalence B. Metric equivalence C. Randomness D. Undefinedness

The circle S1can be obtained by: A. Identifying endpoints of an interval B. Identifying midpoints of an interval C. Identifying random points D. Adding new points

The Möbius strip is obtained by: A. Identifying edges of a square with a twist B. Identifying endpoints of an interval C. Identifying opposite points on a sphere D. Randomly deleting points

Compact spaces related by homeomorphisms are central in: A. Functional analysis B. Randomness C. Geometry only D. Undefined sets

Series summary

This Series introduced three fundamental concepts that describe how topological spaces relate to one another. We began with continuity, which generalises the familiar notion from calculus by requiring that preimages of open sets are open. We then studied homeomorphisms, bijective continuous functions with continuous inverses, which classify spaces as topologically equivalent. Next, we explored quotient topologies, which construct new spaces by identifying points under an equivalence relation. Finally, we examined applications and examples, showing how these concepts are used in analysis, geometry, and algebraic topology. By completing this Series, you should now be able to define continuity, recognise homeomorphisms, construct quotient topologies, and apply these ideas to classify and transform spaces, thereby achieving the Series Learning Outcomes.

Glossary of terms

Continuity: A function is continuous if the preimage of every open set is open.

Homeomorphism: A bijective continuous function with continuous inverse, showing two spaces are topologically equivalent.

Topological equivalence: When two spaces are homeomorphic, they share all topological properties.

Quotient topology: A topology formed by identifying points under an equivalence relation, with openness defined via preimages.

Projection map: The natural map from a space to its quotient space.

Examples of quotient spaces: Circle, cylinder, Möbius strip, projective plane.

Concept check questions 4

(SLO 4.1) Define continuity in topological spaces using open sets.

(SLO 4.1) Give an example of a continuous function in topology.

(SLO 4.2) State the definition of a homeomorphism.

(SLO 4.2) Explain why homeomorphisms classify spaces up to topological equivalence.

(SLO 4.3) Define a quotient topology and explain how openness is determined.

(SLO 4.3) Give an example of a space constructed using a quotient topology.
(SLO 4.4) Explain how continuity, homeomorphisms, and quotient topologies are applied together in functional analysis.
(SLO 4.4) Describe how the Möbius strip is constructed using quotient topology.

Concept Check Questions 4
Objective questions

Continuity in topological spaces means that the preimage of every __________ set is open. (Linked to SLO 4.1)

A homeomorphism is a function that is: A. Continuous and bijective with a continuous inverse B. Only bijective C. Only continuous D. None of the above (Linked to SLO 4.2)

The quotient topology is defined by: A. The finest topology making the projection map continuous B. The coarsest topology making the projection map continuous C. The discrete topology D. None of the above (Linked to SLO 4.3)

Which of the following is an application of continuity? A. Preserving limits of sequences B. Defining quotient spaces C. Establishing homeomorphisms D. None of the above (Linked to SLO 4.4)

Short-answer questions

Define continuity in the context of topological spaces. (Linked to SLO 4.1)
Give one example of a homeomorphism between two spaces. (Linked to SLO 4.2)
What is the importance of quotient topologies? (Linked to SLO 4.3)
State one application of homeomorphisms in topology. (Linked to SLO 4.2)
Explain why continuity is essential in analysis. (Linked to SLO 4.1)

Long-answer questions

Analyse the equivalent characterisations of continuity in topological spaces. (Linked to SLO 4.1)
Evaluate the importance of homeomorphisms in classifying topological spaces. (Linked to SLO 4.2)
Discuss the construction of quotient topologies and provide an example. (Linked to SLO 4.3)
Examine combined applications of continuity, homeomorphisms, and quotient topologies in topology. (Linked to SLO 4.4)

Answers to Concept Check Questions 4
Objective questions

Open
A
B

A

Short-answer questions

A function f:X→Yis continuous if the preimage of every open set in Yis open in X.
The function f:RR, f(x)=x3, is a homeomorphism.
Quotient topologies allow the construction of new spaces by identifying points, preserving continuity of the projection map.
Homeomorphisms are used to show that two spaces are topologically equivalent.
Continuity ensures that limits and convergence are preserved under mappings.

Long-answer questions

Basic response: Continuity can be characterised by open sets, closed sets, or convergence of sequences. *Value-added response*: These characterisations are equivalent and provide flexibility in proofs and applications, linking topology with analysis.
Basic response: Homeomorphisms classify spaces by showing they are topologically the same. *Value-added response*: They preserve all topological properties, making them central to the study of equivalence classes of spaces.
Basic response: Quotient topologies are constructed by defining open sets via the projection map. *Value-added response*: For example, identifying opposite edges of a square produces a torus, illustrating how quotient topologies create new spaces.
Basic response: Continuity, homeomorphisms, and quotient topologies each have applications in topology. *Value-added response*: Together, they provide a framework for analysing spaces, proving equivalence, and constructing new topological structures, such as manifolds and compactifications.

Answers to Pilot Questions 4

Part 4.1: 1-Open, 2-Equivalent characterisations, 3-Examples of continuous functions **Part 4.2**: 1-Definition of homeomorphism, 2-Examples, 3-Importance **Part 4.3**: 1-Definition of quotient topology, 2-Examples, 3-Importance **Part 4.4**: 1-Applications of continuity, 2-Applications of homeomorphisms, 3-Applications of quotient topologies, 4-Combined applications