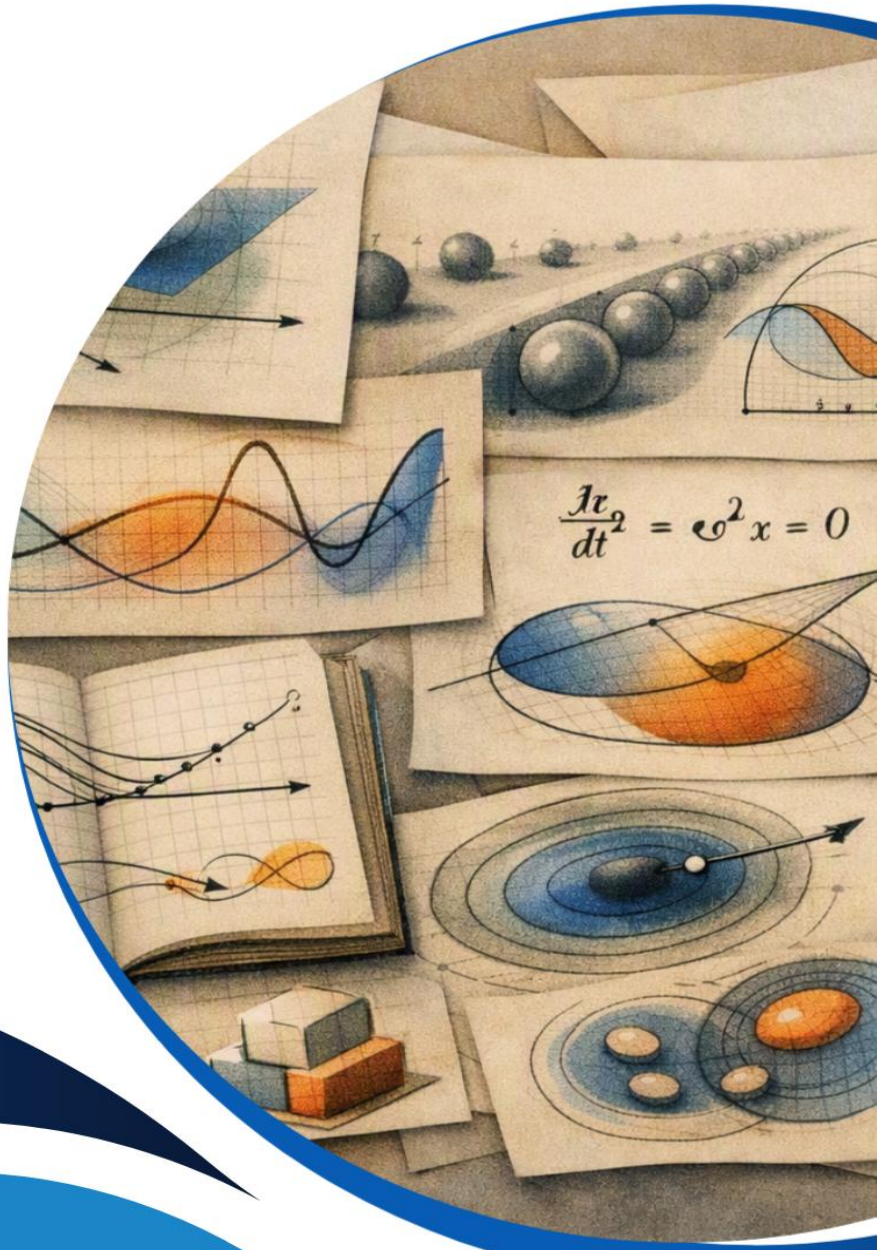
La-Net

MTH403

FUNCTIONAL ANALYSIS



Series 1: Vector Spaces and Topological Vector Spaces. (MTH 403: Function Analysis)

1.1 Definition and Axioms of Vector Spaces

1.1.1 Basic definition of a vector space

1.1.2 Axioms of vector spaces (closure, associativity, distributivity, identity, inverse)

1.1.3 Examples of vector spaces (Rn, polynomial spaces, function spaces)

1.2 Subspaces, Linear Independence, and Basis

1.2.1 Definition of subspaces

1.2.2 Linear independence and dependence

1.2.3 Basis and dimension of a vector space

1.3 Normed Vector Spaces

1.3.1 Definition of a norm

1.3.2 Properties of norms (positivity, homogeneity, triangle inequality)

1.3.3 Examples of normed spaces (Lp spaces, Euclidean norm)

1.4 Topological Vector Spaces

1.4.1 Definition of topology in vector spaces

1.4.2 Continuity of addition and scalar multiplication

1.4.3 Examples and applications in functional analysis

Introduction

Mathematics often begins with the study of familiar spaces such as R2or R3, where vectors can be added and scaled with ease. As analysis progresses, however, we require more general structures that can accommodate infinite dimensions, abstract functions, and notions of continuity. This is where the concept of vector spaces and their extension into topological vector spaces becomes essential. They provide the foundation upon which Hilbert spaces, Banach spaces, and Banach algebras are built, making this Series a crucial starting point for the entire course in Functional Analysis.

The aim of this Series is to introduce you to the fundamental ideas of vector spaces and topological vector spaces, to explain their defining properties, and to highlight their importance as the groundwork for more advanced structures studied later in the course.

We shall commence this Series by defining vector spaces and examining the axioms that govern them, supported with clear examples. Next, we will move on to subspaces, linear independence, and the concept of basis and dimension, which are central to understanding the structure of these spaces. Subsequently, we will explore normed vector spaces, focusing on the role of norms and their properties. Finally, we will wrap up by considering topological vector spaces, where continuity and convergence are introduced, rounding off the Series with applications that connect these abstract ideas to functional analysis.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 1.1 define the concept of a vector space and state its axioms,

SLO 1.2 illustrate examples of vector spaces drawn from Rn, polynomial spaces, and function spaces,

SLO 1.3 explain the notions of subspaces, linear independence, basis, and dimension,

SLO 1.4 define a norm and analyse the properties that characterise normed vector spaces,

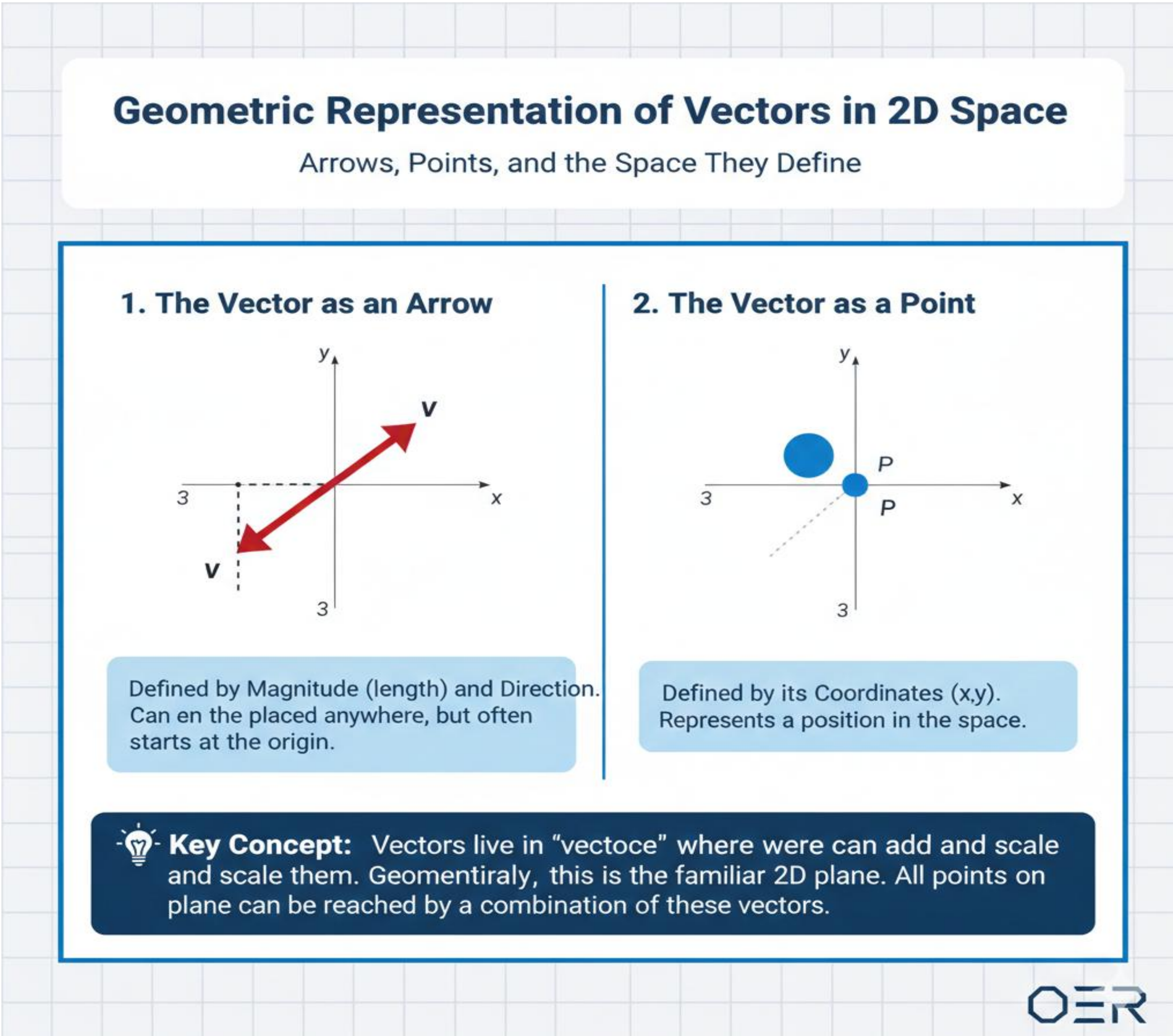
SLO 1.5 describe the structure of topological vector spaces and evaluate their role in functional analysis.

1.1 Definition And Axioms Of Vector Spaces

1.1.1 Basic definition of a vector space

A vector space is a set Vtogether with two operations: addition and scalar multiplication. Addition combines two vectors to produce another vector in the same set, while scalar multiplication scales a vector by a number from the underlying field (usually real or complex numbers). These operations must satisfy specific rules known as axioms.

Fill in the blank: A vector space is defined as a set equipped with __________ and scalar multiplication.



Geometric illustration of a vector space.

1.1.2 Axioms of vector spaces

The axioms of a vector space ensure consistency of operations. They include:

Closure: The sum of two vectors is also a vector in the space.

Associativity: Addition of vectors is associative.

Identity element: There exists a zero vector that acts as an additive identity.

Inverse element: Each vector has an additive inverse.

Distributivity: Scalar multiplication distributes over vector addition.

Compatibility: Scalar multiplication is compatible with field multiplication.

Fill in the blank: The existence of a zero vector ensures the presence of an additive __________ in a vector space.

Summary of vector space axioms

Closure under addition and scalar multiplication

Associativity of addition

Existence of zero vector and additive inverse

Distributivity of scalar multiplication

Compatibility with field multiplication

Examples of Vector Spaces

Space	Description
Euclidean Space ($\mathbb{R}^n$)	Standard arrows in 2D, 3D, or n-dimensions.
Polynomial Space (P_n)	The set of all polynomials of degree n or less.
Matrix Space ($M_{m,n}$)	All $m \times n$ matrices with real entries.
Function Space ($C[a, b]$)	Continuous functions on an interval; vital for differential equations.

1.1.3 Examples of vector spaces

Examples help to illustrate the abstract definition:

Rn: The set of all ordered n-tuples of real numbers with usual addition and scalar multiplication.

Polynomial spaces: The set of all polynomials of degree less than or equal to n.

Function spaces: The set of continuous functions on an interval, with pointwise addition and scalar multiplication.

Fill in the blank: The set of all polynomials of degree less than or equal to nforms a __________ space.

Caption: Table 1.1: Examples of vector spaces

Example	Elements	Operations
Rn	Ordered n-tuples	Usual addition and scalar multiplication
Polynomial space	Polynomials of degree ≤ n	Addition and scalar multiplication
Function space	Continuous functions	Pointwise addition and scalar multiplication

AI prompt: "Generate a table listing examples of vector spaces with elements and operations." Search string: "examples of vector spaces table OER"

Pilot questions 1.1

A vector space is defined as a set equipped with: A. Integration and differentiation B. Addition and scalar multiplication C. Limits and continuity D. Random operations

The existence of a zero vector ensures the presence of: A. A multiplicative identity B. An additive identity C. A scalar field D. A random element

The set of all polynomials of degree ≤ n forms a: A. Function space B. Polynomial space C. Hilbert space D. Banach space

Which of the following is an example of a vector space? A. Continuous functions on an interval B. Random numbers C. Prime numbers D. Natural numbers without operations

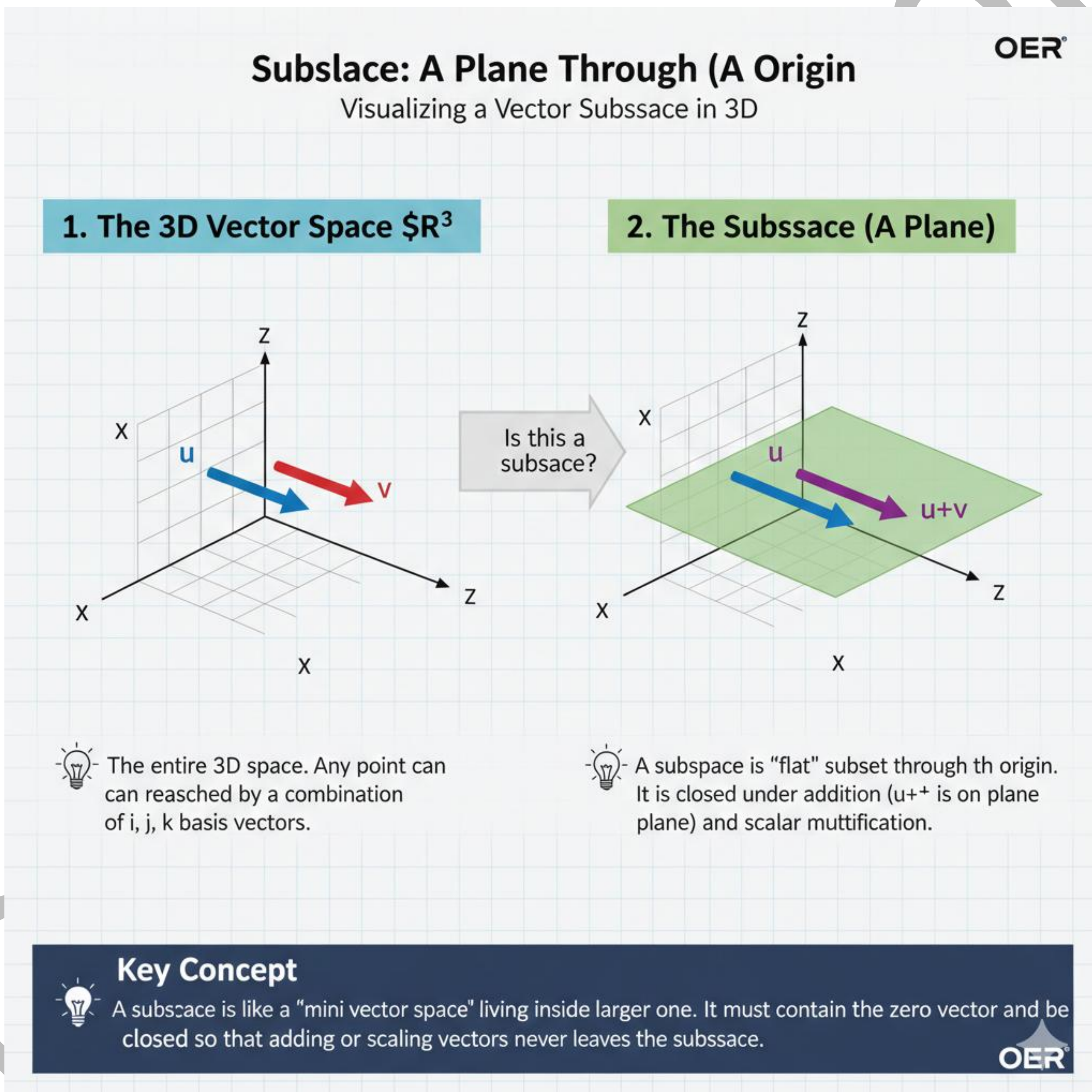
Closure under addition means: A. Adding two vectors produces another vector in the same space B. Adding two scalars produces a vector C. Adding two vectors produces a scalar D. Adding two scalars produces a scalar

1.2 Subspaces, Linear Independence, And Basis

1.2.1 Definition of subspaces

A subspace is a subset of a vector space that is itself a vector space under the same operations of addition and scalar multiplication. For example, in R3, the set of all vectors lying on a plane through the origin forms a subspace.

Fill in the blank: A subspace must contain the __________ vector to qualify as a vector space.



A plane through the origin as a subspace of R3.

1.2.2 Linear independence and dependence

A set of vectors is said to be linearly independent if no vector in the set can be expressed as a linear combination of the others. Conversely, if at least one vector can be expressed as a linear combination of the others, the set is linearly dependent.

Example: In R2, the vectors (1,0)and (0,1)are linearly independent, while (1,0), (0,1), and (1,1)are linearly dependent because the third vector can be written as a sum of the first two.

Fill in the blank: A set of vectors is linearly __________ if no vector can be expressed as a combination of the others.

Key points on linear independence

Independent: no vector is a linear combination of others

Dependent: at least one vector is a linear combination of others

Key Definitions

Term	Mathematical Condition	Conceptual Meaning
Linearly Independent	$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ only if all $c_i = 0$.	No vector in the set can be written as a combination of the others. They all point in "unique" directions.
Linearly Dependent	There exists at least one $c_i \neq 0$ such that the combination equals $\mathbf{0}$.	At least one vector is "redundant." It lies within the span of the other vectors.

1.2.3 Basis and dimension of a vector space

A basis of a vector space is a set of linearly independent vectors that span the entire space. The number of vectors in a basis is called the dimension of the space.

Example: In R3, the standard basis consists of the vectors (1,0,0), (0,1,0), and (0,0,1). These three vectors are linearly independent and span the whole space, so the dimension of R3is 3.

Fill in the blank: The number of vectors in a basis is called the __________ of the vector space.

[Insert table here] Caption: Table 1.2: Examples of bases and dimensions

Vector space	Basis	Dimension
R2	(1,0),(0,1)	2
R3	(1,0,0),(0,1,0),(0,0,1)	3
Polynomial space of degree ≤ 2	{1,x,x2}	3

Pilot questions 1.2

A subspace must contain which element? A. A scalar B. The zero vector C. A random vector D. A basis

A set of vectors is linearly independent if: A. No vector can be expressed as a linear combination of the others B. At least one vector can be expressed as a linear combination of the others C. All vectors are multiples of each other D. The set contains only one vector

The number of vectors in a basis is called the: A. Rank B. Dimension C. Order D. Identity

Which of the following is a basis for R2? A. (1,0),(0,1)B. (1,0),(1,1)C. (0,1),(0,0)D. (1,0),(2,0)

In R3, the standard basis consists of: A. Two vectors B. Three vectors C. Four vectors D. Infinite vectors

1.3 Normed Vector Spaces

1.3.1 Definition of a norm

A norm is a function xthat assigns a non-negative real number to each vector in a vector space, representing its length or magnitude. A normed vector space is a vector space equipped with such a norm.

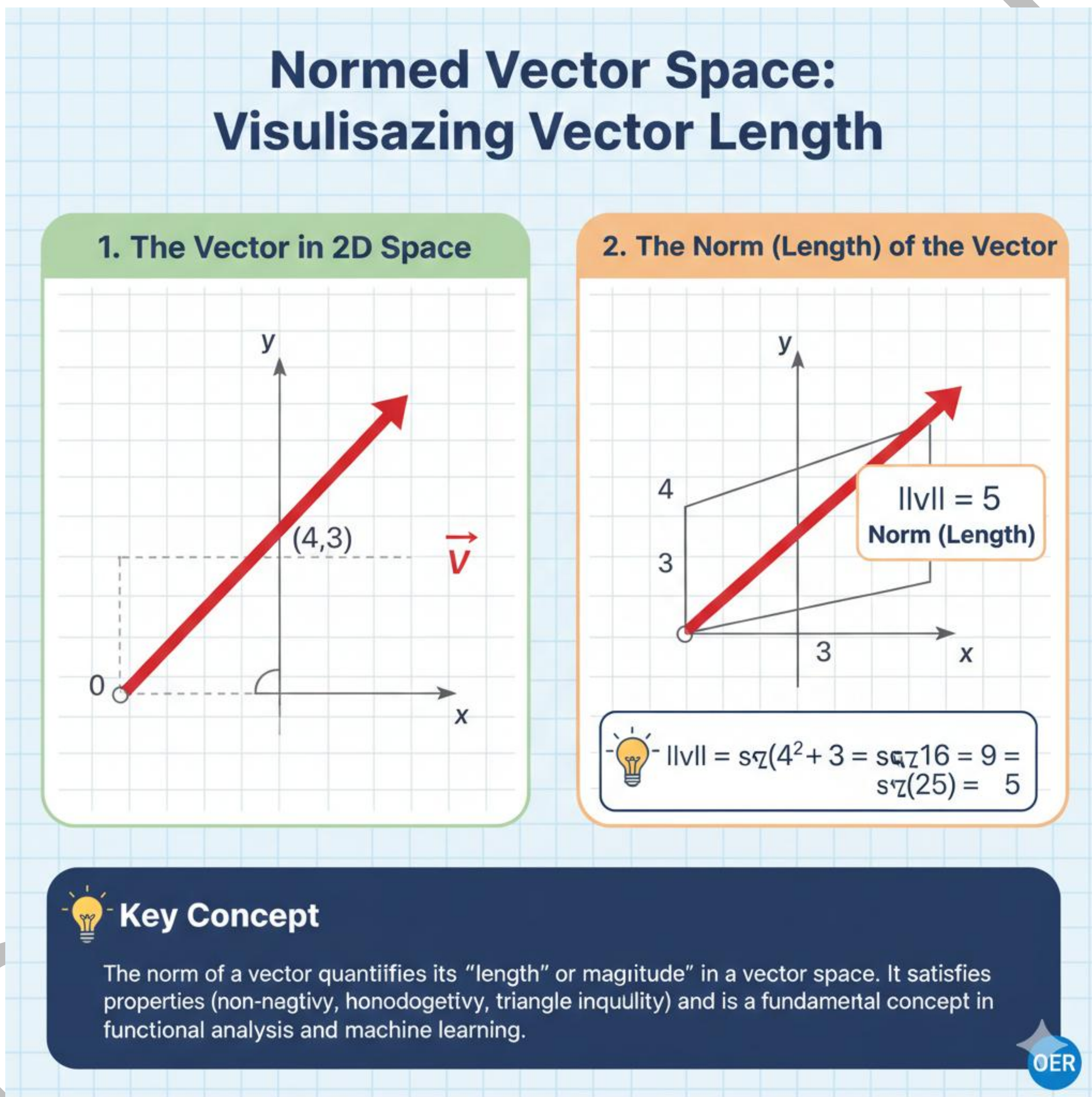
The norm must satisfy three properties:

Positivity: $x \geq 0$, and $x=0$if and only if xis the zero vector.

Homogeneity: αx=|||‖xfor any scalar .

Triangle inequality: x+yx+y.

Fill in the blank: A normed vector space is a vector space equipped with a __________ that measures the length of vectors.



Norm as the length of a vector.

1.3.2 Properties of norms

The properties of norms ensure consistency in measuring vector lengths:

Positivity guarantees that lengths are never negative.

Homogeneity ensures scaling behaves predictably.

Triangle inequality reflects the geometric idea that the direct path is shorter than or equal to any indirect path.

Fill in the blank: The triangle inequality states that x+yis less than or equal to x+y, reflecting the idea that the __________ path is the shortest.

Positivity: $x \geq 0$

Homogeneity: $\alpha x = |||\|x$

Triangle inequality: x+yx+yAI

Common Norms in Science & Engineering

Norm Name	Mathematical Definition	Physical/Practical Context
L_2 **Norm (Euclidean)**	$\sqrt{x_1^2 + x_2^2 + \dots}$	Standard "straight-line" distance.
L_1 **Norm (Manhattan)**	$	x_1
L_∞ **Norm (Max)**	$\max(	x_1
Function Norm ($L_2[a, b]$**)**	$\sqrt{\int_a^b	f(x)

1.3.3 Examples of normed spaces

Examples of normed vector spaces include:

Euclidean space Rn: Norm defined by $x=x12+x22+\cdots+xn2$.

Lpspaces: Norm defined by $fp=(\int|f(x)|pdx)1/p$.

Sup norm space C[a,b]: Norm defined by $f=\sup x\in[a,b]|f(x)|$.

Fill in the blank: In Euclidean space, the norm of a vector is given by the square root of the sum of the __________ of its components.

Normed Vector Spaces: Common Examples

In mathematical analysis and engineering, the "size" of an object depends entirely on which norm you choose. Below is a table of the most common normed vector spaces used to measure vectors, matrices, and functions.

Space	Elements	Norm Definition	Common Name / Use Case
$\mathbb{R}^n$	Vectors $(x_1, \ldots, x_n)$	$|x|2 = \sqrt{\sum{i=1}^n	x_i
$\mathbb{R}^n$	Vectors $(x_1, \ldots, x_n)$	$|x|1 = \sum{i=1}^n	x_i
$\mathbb{R}^n$	Vectors $(x_1, \ldots, x_n)$	$|x|_\infty = \max_i	x_i
$C[a, b]$	Continuous Functions	$|f|\infty = \sup{x \in [a,b]}	f(x)
$L^2[a, b]$	Square-integrable Functions	$|f|_2 = \left(\int_a^b	f(x)
$L^p[a, b]$	p-integrable Functions	$|f|_p = \left(\int_a^b	f(x)

Pilot questions 1.3

A normed vector space is a vector space equipped with: A. A topology B. A norm C. A derivative D. A boundary condition

Which property of norms ensures that lengths are never negative? A. Homogeneity B. Positivity C. Triangle inequality D. Closure

The triangle inequality reflects the idea that: A. The direct path is shortest B. The indirect path is shortest C. All paths are equal D. Vectors cannot be added

In Euclidean space, the norm of a vector is given by: A. The sum of its components B. The square root of the sum of squares of its components C. The maximum of its components D. The product of its components

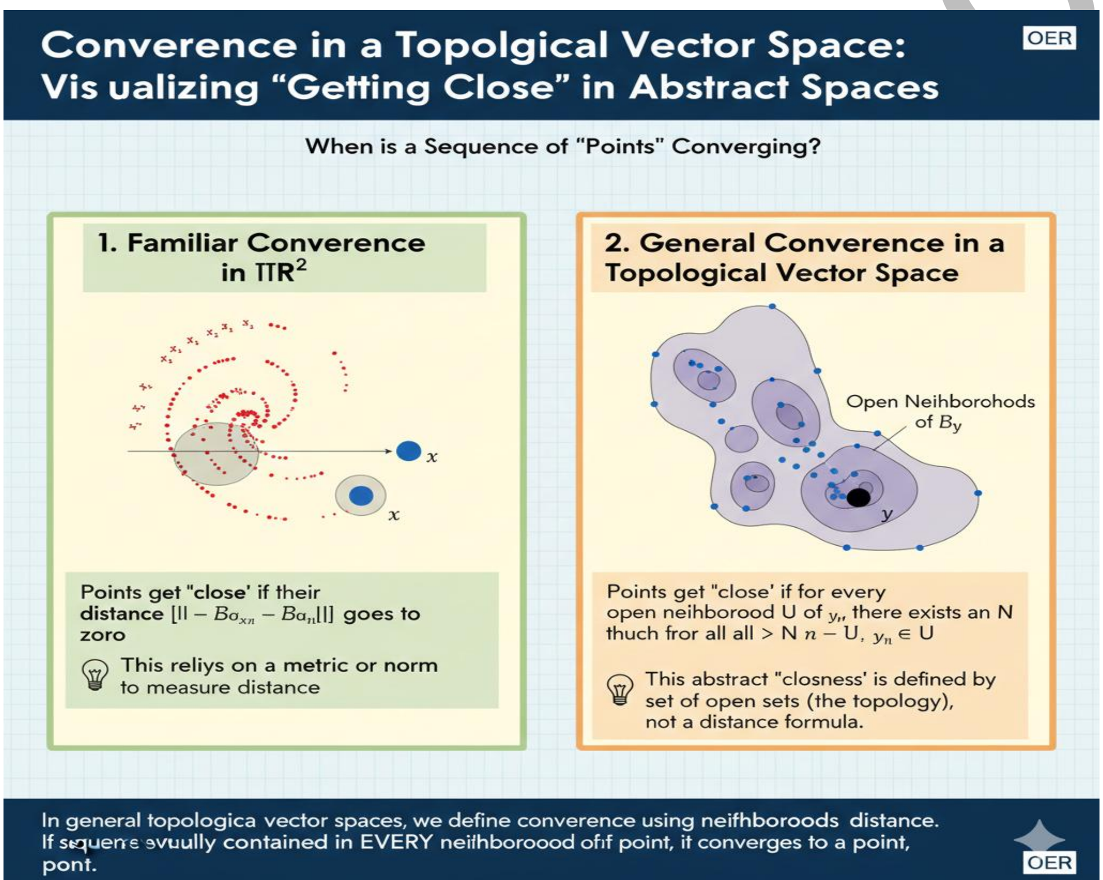
The sup norm of a continuous function is defined as: A. The integral of the function B. The maximum absolute value of the function over the interval C. The derivative of the function D. The square root of the sum of squares

1.4 Topological Vector Spaces

1.4.1 Definition of topology in vector spaces

A topological vector space is a vector space equipped with a topology, which is a structure that defines how sequences of vectors converge and how functions behave continuously. In such spaces, the operations of vector addition and scalar multiplication must be continuous with respect to the topology.

Fill in the blank: A topological vector space is a vector space equipped with a __________ that defines convergence and continuity.



Convergence in a topological vector space.

1.4.2 Continuity of addition and scalar multiplication

For a vector space to qualify as a topological vector space, both addition and scalar multiplication must be continuous functions. This means small changes in vectors or scalars lead to small changes in the results.

Example: In Rnwith the usual topology, addition and scalar multiplication are continuous, making it a topological vector space.

Fill in the blank: In a topological vector space, both addition and scalar multiplication must be __________ functions.

Conditions for a topological vector space

Addition is continuous

Scalar multiplication is continuous

Feature	Normed Vector Space	Topological Vector Space (General)
Distance	Measured by a specific formula (‖v).	Measured by "open sets" or neighborhoods.
Origin	Has a definite size/norm.	Only requires a "local base" of neighborhoods.
Strictness	Very strict (Every normed space is a TVS).	Much broader (Allows for spaces where no norm can be defined).

1.4.3 Examples and applications in functional analysis

Examples of topological vector spaces include:

Normed vector spaces: Every normed vector space is a topological vector space with the topology induced by the norm.

Hilbert spaces: These are complete inner product spaces and also topological vector spaces.

Banach spaces: Complete normed vector spaces that are also topological vector spaces.

Applications:

Studying convergence of sequences of functions.

Analysing continuity of linear operators.

Providing the framework for advanced structures in functional analysis.

Fill in the blank: Every normed vector space is automatically a __________ vector space with the topology induced by the norm.

[Examples of topological vector spaces

Space	Topology	Application
Normed vector space	Induced by norm	Convergence of sequences
Hilbert space	Induced by inner product	Fourier analysis
Banach space	Induced by norm	Operator theory

Pilot questions 1.4

A topological vector space is a vector space equipped with: A. A derivative B. A topology C. A boundary condition D. A random element

In a topological vector space, addition and scalar multiplication must be: A. Continuous functions B. Random operations C. Differentiable functions D. Non-linear functions

Every normed vector space is automatically a: A. Hilbert space B. Banach space C. Topological vector space D. Polynomial space

Which of the following is an example of a topological vector space? A. Banach space B. Hilbert space C. Normed vector space D. All of the above

Applications of topological vector spaces include: A. Studying convergence of sequences B. Analysing continuity of linear operators C. Providing framework for functional analysis D. All of the above

Series summary

In this Series, we explored the foundations of functional analysis through the study of vector spaces and their extensions. We began with the definition and axioms of vector spaces, supported by examples such as Rn, polynomial spaces, and function spaces. We then examined subspaces, linear independence, basis, and dimension, which provide the structural framework of vector spaces. Following that, we introduced normed vector spaces, highlighting the role of norms and their properties such as positivity, homogeneity, and the triangle inequality. Finally, we studied topological vector spaces, where continuity and convergence are incorporated, and we saw how these spaces underpin advanced structures like Hilbert and Banach spaces.

Glossary of terms

Vector space: A set with addition and scalar multiplication satisfying specific axioms.

Subspace: A subset of a vector space that is itself a vector space under the same operations.

Linear independence: A property of a set of vectors where no vector can be expressed as a linear combination of the others.

Basis: A set of linearly independent vectors that span a vector space.

Dimension: The number of vectors in a basis of a vector space.

Norm: A function that assigns a non-negative length to vectors, satisfying positivity, homogeneity, and triangle inequality.

Normed vector space: A vector space equipped with a norm.

Topological vector space: A vector space with a topology that makes addition and scalar multiplication continuous.

Concept check questions 1

(SLO 1.1) Define a vector space and state its axioms.

(SLO 1.2) Give one example of a subspace in R3.

(SLO 1.3) Explain the difference between linear independence and dependence.

(SLO 1.3) What is the dimension of R3and why?

(SLO 1.4) State the three properties of a norm.

(SLO 1.4) Give one example of a normed vector space.

(SLO 1.5) Describe the conditions required for a vector space to be a topological vector space.

(SLO 1.5) Explain why every normed vector space is automatically a topological vector space.

Answers to pilot questions 1

Pilot questions 1.1: 1B, 2A, 3B, 4A, 5A Pilot questions 1.2: 1B, 2A, 3B, 4A, 5B
Pilot questions 1.3: 1B, 2B, 3A, 4B, 5B Pilot questions 1.4: 1B, 2A, 3C, 4D, 5D

Series 2: Hilbert Spaces

Series outline

This Series contains four main Parts:

2.1 Definition and inner product structure

- 2.1.1 Definition of a Hilbert space
- 2.1.2 Inner product structure
- 2.1.3 Examples of Hilbert spaces

2.2 Orthogonality and orthonormal sets

- 2.2.1 Orthogonality in Hilbert spaces
- 2.2.2 Orthonormal sets
- 2.2.3 Examples of orthonormal sets

2.3 Completeness of Hilbert spaces

- 2.3.1 Concept of completeness
- 2.3.2 Importance of completeness
- 2.3.3 Examples of complete spaces

2.4 Applications of Hilbert spaces

- 2.4.1 Fourier analysis
- 2.4.2 Quantum mechanics
- 2.4.3 Operator theory

Introduction

In mathematics and physics, many problems require a framework where geometry and analysis meet. **Hilbert spaces** provide such a framework by extending the familiar ideas of Euclidean space into infinite dimensions. They are complete inner product spaces, meaning they combine algebraic, geometric, and analytic properties in a single structure. Hilbert spaces are central to functional analysis and play a vital role in quantum mechanics, Fourier analysis, and partial differential equations.

The aim of this Series is to introduce you to the concept of Hilbert spaces, explain their defining properties, and demonstrate their importance in both theoretical and applied contexts.

We shall commence this Series by defining Hilbert spaces and examining the role of inner products in their structure. Next, we will explore orthogonality and orthonormal sets, which allow us to generalise the idea of perpendicularity to functions. Subsequently, we will consider the completeness property, which ensures that Hilbert spaces are robust enough for analysis. Finally, we will conclude by discussing applications of Hilbert spaces in areas such as Fourier series, quantum mechanics, and operator theory.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 define a Hilbert space and explain the role of the inner product in its structure,

SLO 2.2 demonstrate the concept of orthogonality and construct orthonormal sets within Hilbert spaces,

SLO 2.3 analyse the completeness property of Hilbert spaces and explain its significance in functional analysis,

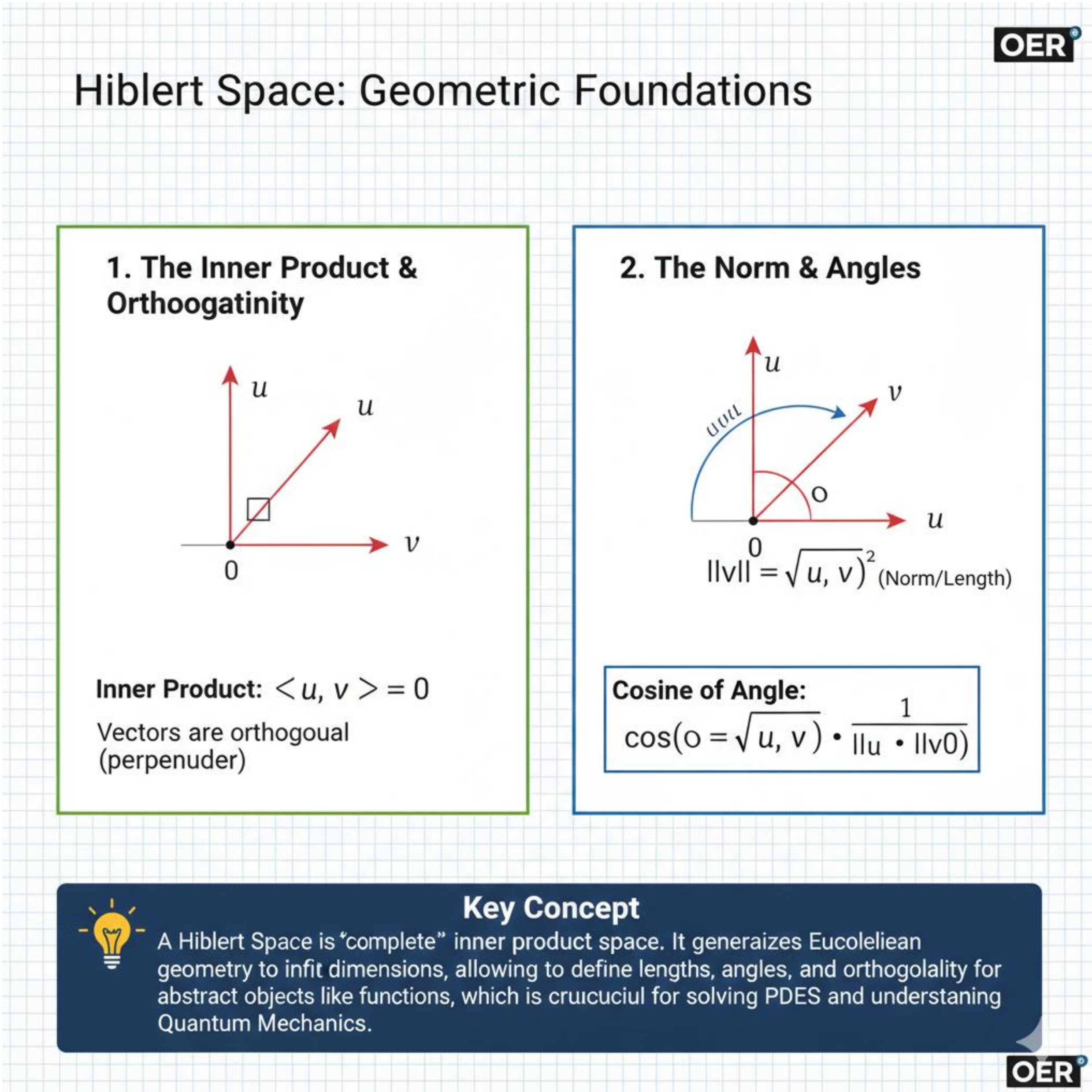
SLO 2.4 evaluate applications of Hilbert spaces in areas such as Fourier analysis, quantum mechanics, and operator theory.

2.1 Definition And Inner Product Structure

2.1.1 Definition of a Hilbert space

A **Hilbert space** is a complete inner product space. This means it is a vector space equipped with an **inner product**, which allows us to measure angles and lengths, and it is **complete**, meaning every Cauchy sequence in the space converges to a point within the space. Hilbert spaces generalise the familiar Euclidean spaces into infinite dimensions, making them essential in functional analysis.

Fill in the blank: A Hilbert space is defined as a complete __________ product space.



Geometric interpretation of a Hilbert space.

2.1.2 Inner product structure

An **inner product** is a function ⟨x,y⟩that takes two vectors and returns a scalar, satisfying the following properties:

Linearity: ⟨ax+by,z⟩=a⟨x,z⟩+b⟨y,z⟩.

Symmetry: ⟨x,y⟩=⟨y,x⟩$^{-}$.

Positivity: ⟨x,x⟩≥0, and equals zero only if xis the zero vector.

The inner product induces a **norm** defined by $x=\langle x,x\rangle$. This connects Hilbert spaces to normed vector spaces.

Fill in the blank: The norm in a Hilbert space is defined as the square root of the __________ of a vector with itself.

Properties of an inner product

Linearity

Symmetry

Positivity

Summary: Properties of an Inner Product

In a **Hilbert Space**, the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ is the engine that allows us to define "geometry." It generalizes the concept of a dot product to functions and infinite-dimensional spaces. For a mapping to be a valid inner product (on a complex vector space), it must satisfy these four fundamental properties.

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1. Conjugate Symmetry

- **The Rule:** $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$
- **The Logic:** If you swap the order of the vectors, you get the complex conjugate. For real-valued spaces, this simplifies to standard symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

2. Linearity in the First Argument

- **The Rule:** $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w}\rangle = a\langle \mathbf{u}, \mathbf{w}\rangle + b\langle \mathbf{v}, \mathbf{w}\rangle$
- **The Logic:** The inner product behaves linearly with respect to vector addition and scalar multiplication in the first slot.
- **Note:** In the second slot, scalars are pulled out as conjugates: $\langle \mathbf{u}, c\mathbf{v}\rangle = \overline{c}\langle \mathbf{u}, \mathbf{v}\rangle$.

3. Positive Definiteness

- **The Rule:** $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- **The Logic:** The "self-product" of a vector must always be a non-negative real number. This property is what allows us to define the **norm** (length) as $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$.

4. The Cauchy–Schwarz Inequality

- **The Rule:** $|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$
- **The Logic:** The absolute value of the inner product of two vectors is always less than or equal to the product of their lengths. This is the foundation for defining the **angle** between two functions or vectors.

Why this matters for Sturm–Liouville Theory

In S-L problems, we define the inner product of two functions $f(x)$ and $g(x)$ using a weight function $r(x)$:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}r(x)\,dx$$

- **Orthogonality:** If $\langle f, g\rangle = 0$, the functions are orthogonal.
- **Eigenfunction Expansion:** We use the inner product to "project" a complex function onto a basis of eigenfunctions, similar to finding the components of a 3D vector.

2.1.3 Examples of Hilbert spaces

Examples include:

Rn**with the dot product:** A finite-dimensional Hilbert space.

L2[a,b]**:** The space of square-integrable functions on an interval, with inner product ⟨f,g⟩=abf(x)g(x) dx.

Sequence space l2**:** The set of square-summable sequences, with inner product ⟨x,y⟩=n=1xnyn.

Fill in the blank: The space L2[a,b]consists of functions that are __________ integrable over the interval [a,b].

[Insert table here] Caption: Table 2.1: Examples of Hilbert spaces

Space	Inner product	Dimension
Rn	Dot product	Finite
L2[a,b]	abf(x)g(x) dx	Infinite
l2	n=1xnyn	Infinite

Pilot questions 2.1

A Hilbert space is defined as: A. A complete normed vector space B. A complete inner product space C. A topological vector space D. A Banach algebra

Which property of the inner product ensures ⟨x,y⟩=⟨y,x⟩‾? A. Linearity B. Symmetry C. Positivity D. Closure

The norm in a Hilbert space is defined as: A. ⟨x,y⟩B. ⟨x,x⟩C. ∑xnD. ∫f(x) dx

The space L2[a,b]consists of functions that are: A. Differentiable B. Square-integrable C. Continuous D. Polynomial

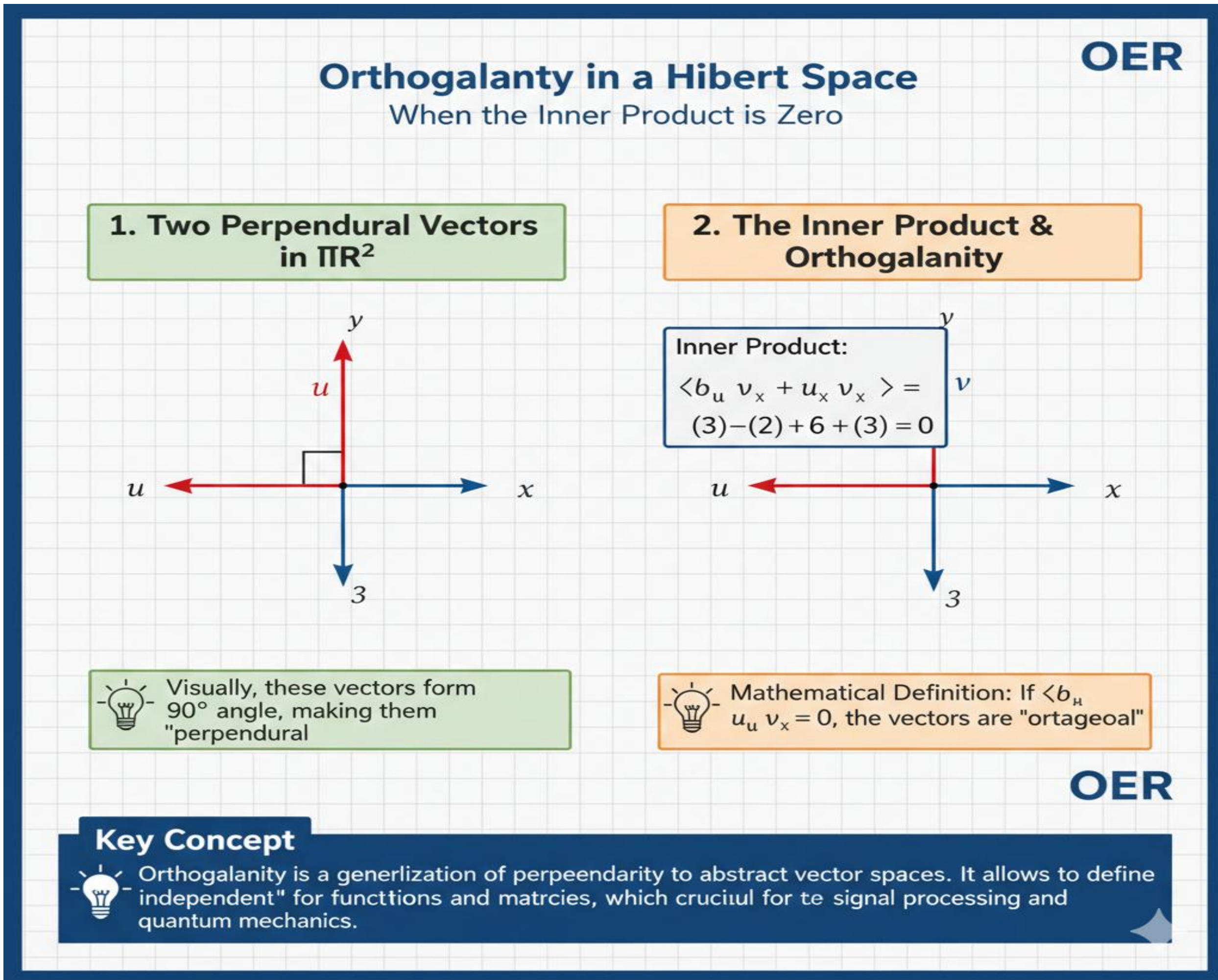
Which of the following is an example of a Hilbert space? A. Rnwith dot product B. L2[a,b]C. l2D. All of the above

2.2 Orthogonality And Orthonormal Sets

2.2.1 Orthogonality in Hilbert spaces

In a Hilbert space, two vectors xand yare said to be **orthogonal** if their inner product ⟨x,y⟩=0. This generalises the familiar idea of perpendicularity from Euclidean geometry to abstract spaces. Orthogonality is a powerful concept because it allows us to decompose vectors into independent components.

Fill in the blank: Two vectors are orthogonal if their __________ product equals zero.



Orthogonality of vectors in a Hilbert space.

2.2.2 Orthonormal sets

An **orthonormal set** is a collection of vectors that are both orthogonal and normalised to have unit length. Formally, a set {e1,e2,...,en}is orthonormal if:

$\langle ei,ej \rangle=0$for ij(orthogonality),

$\langle ei,ei \rangle=1$for all i(normalisation).

Orthonormal sets are important because they simplify computations. For example, in Fourier analysis, functions are expressed as sums of orthonormal basis functions.

Fill in the blank: An orthonormal set consists of vectors that are orthogonal and have __________ length.

Properties of orthonormal sets

Orthogonality: inner product of distinct vectors is zero

Normalisation: inner product of a vector with itself is one

1. The Kronecker Delta Condition

The defining characteristic of an orthonormal set is summarized by the inner product:

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}$$

Where δ_{ij} (the Kronecker Delta) means:

- **Orthogonal:** If $i \neq j$, the inner product is **0**.
- **Normalized:** If $i = j$, the inner product is **1** (meaning $\|\phi_i\| = 1$).

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2. Linear Independence

Every orthonormal set is automatically **linearly independent**. Because each vector points in a completely "new" direction that cannot be reached by any combination of the others, there is zero redundancy in the set.

3. Best Approximation (Projection)

If you want to approximate a complex function f using an orthonormal set, the "best" coefficients c_n are simply the inner products (projections):

$$c_n = \langle f, \phi_n \rangle$$

Unlike non-orthogonal sets, you don't need to solve a system of equations to find these coefficients; you just calculate each integral independently.

4. Bessel's Inequality

For any vector f in the Hilbert space and any orthonormal set, the sum of the squares of the components cannot exceed the total "energy" of the vector:

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2$$

5. Parseval's Identity (Completeness)

If the orthonormal set is **complete** (meaning it's a basis that spans the whole space), the inequality above becomes an equality:

$$\sum_{n=1}^{\infty} |c_n|^2 = \|f\|^2$$

- **Physical Meaning:** The total energy of a signal is equal to the sum of the energies of its individual harmonics.

Why this matters for Sturm–Liouville Problems

The eigenfunctions $y_n(x)$ of an S-L problem are always orthogonal, but they aren't always "normal" (length of 1). We turn them into an **orthonormal set** by dividing each function by its norm:

$$\phi_n(x) = \frac{y_n(x)}{\|y_n(x)\|}$$

Using these normalized versions makes the math for Fourier-type expansions much cleaner, as the denominator in your coefficient formula becomes 1.

2.2.3 Examples of orthonormal sets

Examples include:

Standard basis in Rn**:** Vectors (1,0,...,0),(0,1,0,...,0),...are orthonormal under the dot product.

Fourier basis in L2[-π,π]**:** Functions 12π,1cos(nx),1sin(nx)form an orthonormal set.

Sequence space l2**:** The set of sequences with a single 1 in one position and 0 elsewhere forms an orthonormal basis.

Fill in the blank: The standard basis vectors in Rnare orthonormal under the __________ product.

[Insert table here] Caption: Table 2.2: Examples of orthonormal sets

Space	**Orthonormal set**	**Inner product**
Rn	Standard basis vectors	Dot product
L2[-π,π]	Fourier basis functions	Integral inner product
l2	Sequences with one 1 and rest 0	Summation inner product

Pilot questions 2.2

Two vectors are orthogonal if: A. Their inner product equals zero B. Their sum equals zero C. Their difference equals zero D. Their product equals one

An orthonormal set consists of vectors that are: A. Orthogonal and of unit length B. Dependent and of unit length C. Orthogonal and of random length D. Independent and of infinite length

The standard basis vectors in Rnare orthonormal under: A. Integral inner product B. Dot product C. Summation inner product D. Random product

Which of the following is an example of an orthonormal set? A. Fourier basis functions in L2[-π,π]B. Standard basis in RnC. Sequences in l2with one 1 and rest 0 D. All of the above

Why are orthonormal sets important? A. They simplify computations in Hilbert spaces B. They complicate computations C. They prevent convergence D. They eliminate completeness

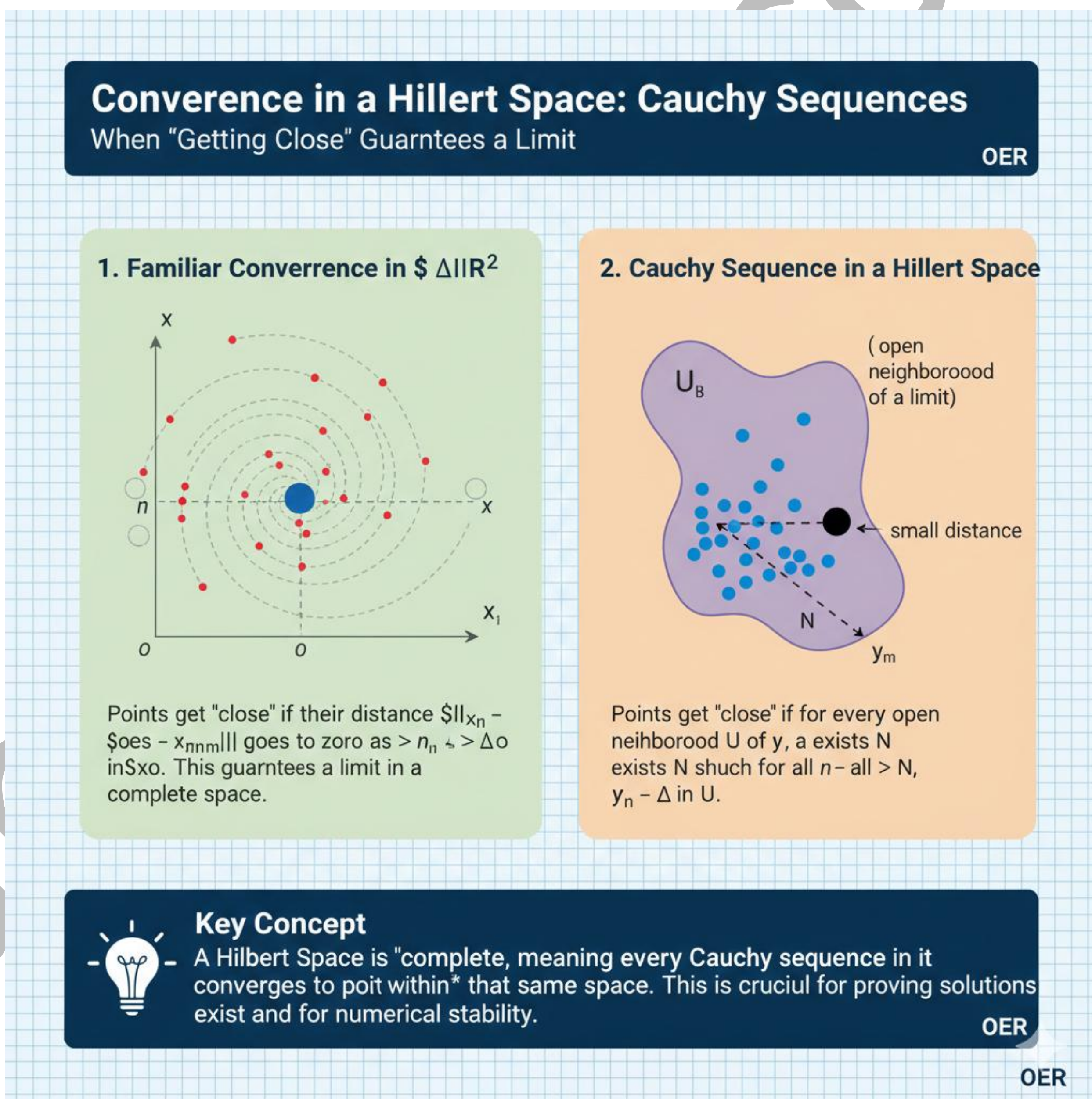
2.3 Completeness Of Hilbert Spaces

2.3.1 Concept of completeness

A Hilbert space is defined as a complete inner product space. **Completeness** means that every **Cauchy sequence** in the space converges to a limit that is also within the space. A **Cauchy sequence** is a sequence of vectors where the distance between successive terms becomes arbitrarily small as the sequence progresses.

This property ensures that Hilbert spaces are robust enough for analysis, since limits of sequences do not "escape" the space.

Fill in the blank: A Hilbert space is complete because every __________ sequence converges to a point in the space.



Convergence of a Cauchy sequence in a Hilbert space.

2.3.2 Importance of completeness

Completeness is crucial because it guarantees stability of the space under limits. Without completeness, sequences might converge to points outside the space, making analysis unreliable.

Example: The space of continuous functions on [0,1]with the sup norm is complete, hence a Banach space. Similarly, L2[a,b]is complete, making it a Hilbert space.

Fill in the blank: Completeness guarantees that limits of sequences remain __________ the space.

Importance of completeness

Ensures stability under limits

Guarantees convergence within the space

Provides reliability for analysis

Completeness vs. Incompleteness

Space	Complete?	Consequence
Rational Numbers ($\mathbb{Q}$)	**No**	A sequence of fractions can approach $\sqrt{2}$, which is not a fraction.
Real Numbers ($\mathbb{R}$)	**Yes**	Calculus works; limits of real numbers are real numbers.
Continuous Functions ($C[a,b]$)	**No** (under L^2 norm)	A sequence of continuous functions can converge to a discontinuous step function.
$L^2[a,b]$ **Space**	**Yes**	The fundamental space for Quantum Mechanics and Signal Processing.

2.3.3 Examples of complete spaces

Examples include:

Rn**:** Complete under the usual Euclidean norm.

L2[a,b]**:** Complete under the inner product norm.

l2**:** Complete under the summation norm.

Fill in the blank: The space l2consists of sequences that are __________ summable.

[Insert table here] Caption: Table 2.3: Examples of complete spaces

Space	Norm/Inner product	Completeness property
Rn	Euclidean norm	Complete
L2[a,b]	Integral inner product	Complete
l2	Summation inner product	Complete

Pilot questions 2.3

A Hilbert space is complete because: A. Every Cauchy sequence converges within the space B. Every sequence diverges C. Every sequence is bounded D. Every sequence is random

A Cauchy sequence is one where: A. Successive terms become arbitrarily close B. Successive terms diverge C. Successive terms remain constant D. Successive terms are random

Completeness guarantees that: A. Limits of sequences remain within the space B. Limits of sequences escape the space C. Limits of sequences are undefined D. Limits of sequences are random

Which of the following is a complete space? A. RnB. L2[a,b]C. l2D. All of the above

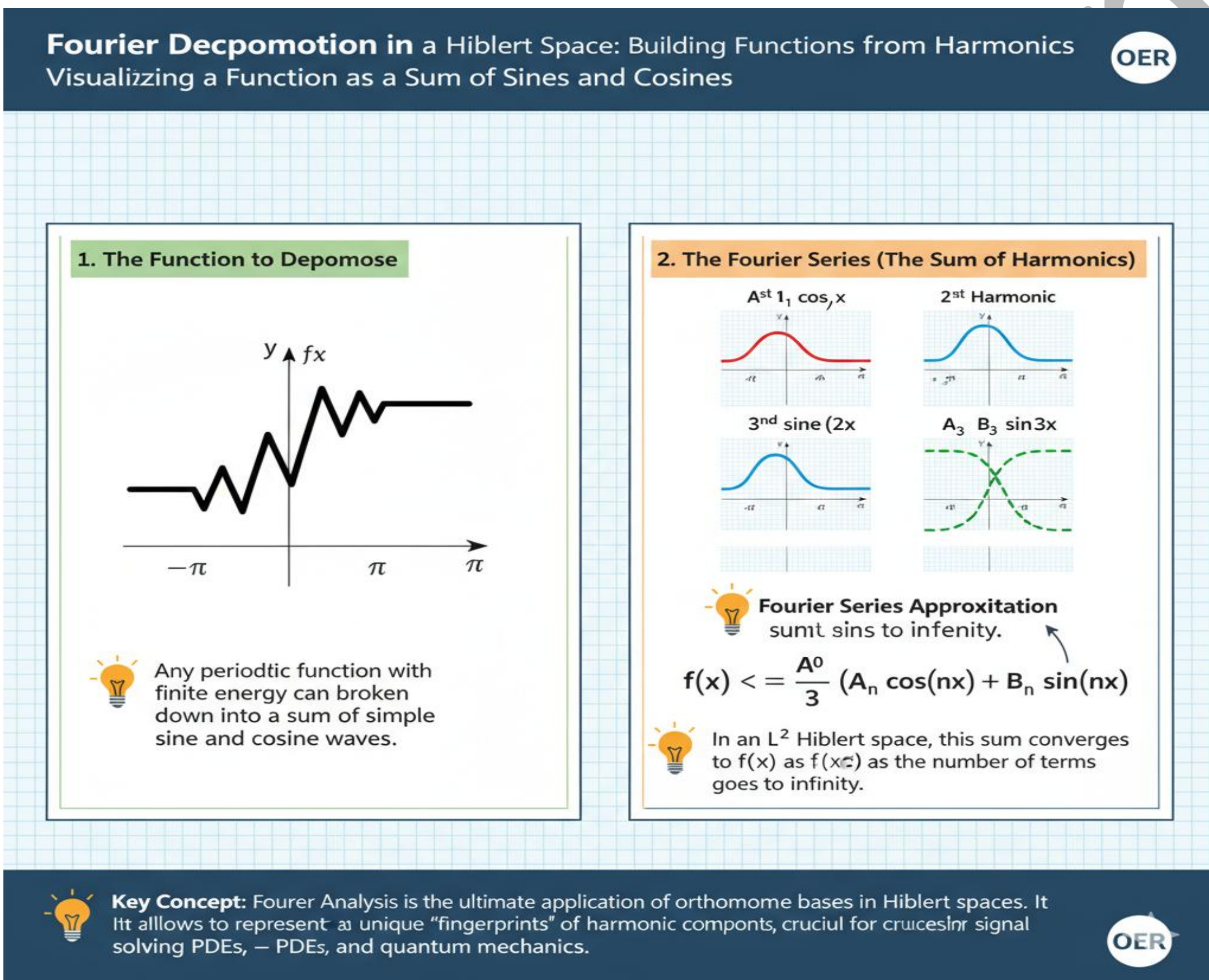
The space l2consists of sequences that are: A. Square-summable B. Continuous C. Differentiable D. Random

2.4 Applications Of Hilbert Spaces

2.4.1 Fourier analysis

One of the most important applications of Hilbert spaces is in **Fourier analysis**. In the Hilbert space L2[-π,π], functions can be expressed as infinite sums of orthonormal basis functions such as sines and cosines. This allows us to decompose complex signals into simpler components.

Fill in the blank: In Fourier analysis, functions are expressed as infinite sums of __________ basis functions.



Fourier decomposition in a Hilbert space. A

2.4.2 Quantum mechanics

Hilbert spaces provide the mathematical framework for **quantum mechanics**. The state of a quantum system is represented by a vector in a Hilbert space, and physical observables correspond to operators acting on these vectors. This structure allows predictions of probabilities and outcomes in quantum systems.

Fill in the blank: In quantum mechanics, the state of a system is represented by a __________ in a Hilbert space.

Role of Hilbert spaces in quantum mechanics

States represented by vectors

Observables represented by operators

Probabilities derived from inner products

Classical vs. Quantum Geometry

Feature	Classical Mechanics	Quantum Mechanics (Hilbert Space)
System State	Point in Phase Space (p, q)	Vector in Hilbert Space $
Variables	Real numbers (Position x, etc.)	Operators ($\hat{X}$, $\hat{P}$)
Measurement	Deterministic (exactly one value)	Probabilistic (eigenvalues of operators)
Geometry	Euclidean distance	Inner product (Overlap/Probability)

2.4.3 Operator theory

Hilbert spaces are also central to **operator theory**, where bounded linear operators are studied. These operators model transformations of vectors and functions, and their properties such as spectra and eigenvalues are analysed within Hilbert spaces.

Fill in the blank: In operator theory, bounded linear ___________ are studied within Hilbert spaces.

Applications of Hilbert spaces

Application	**Description**	**Example**
Fourier analysis	Decomposition into orthonormal basis functions	Signal processing
Quantum mechanics	States as vectors, observables as operators	Schrödinger equation
Operator theory	Study of bounded linear operators	Spectral analysis

Pilot questions 2.4

In Fourier analysis, functions are expressed as sums of: A. Random functions B. Orthonormal basis functions C. Polynomial functions D. Differentiable functions

In quantum mechanics, the state of a system is represented by: A. A scalar B. A vector in a Hilbert space C. A polynomial D. A random number

Observables in quantum mechanics correspond to: A. Scalars B. Operators acting on vectors C. Random variables D. Basis functions

In operator theory, Hilbert spaces are used to study: A. Bounded linear operators B. Random processes C. Continuous functions only D. Differentiable functions only

Which of the following is an application of Hilbert spaces? A. Fourier analysis B. Quantum mechanics C. Operator theory D. All of the above

Series summary

In this Series, we examined the structure and significance of Hilbert spaces. We began by defining Hilbert spaces as complete inner product spaces and explored how the inner product induces norms and geometric interpretations. We then studied orthogonality and orthonormal sets, which extend the idea of perpendicularity and provide powerful tools for decomposition and simplification. Following that, we considered the completeness property, which ensures that Cauchy sequences converge within the space, making Hilbert spaces reliable for analysis. Finally, we explored applications in Fourier analysis, quantum mechanics, and operator theory, highlighting the central role Hilbert spaces play in both pure and applied mathematics.

Glossary of terms

Hilbert space: A complete inner product space.

Inner product: A function that measures angles and lengths between vectors, satisfying linearity, symmetry, and positivity.

Norm: A function induced by the inner product that measures vector length.

Orthogonality: A property where two vectors have an inner product equal to zero.

Orthonormal set: A set of vectors that are orthogonal and of unit length.

Cauchy sequence: A sequence where successive terms become arbitrarily close as the sequence progresses.

Completeness: The property that every Cauchy sequence converges to a point within the space.

Operator theory: The study of linear operators acting on Hilbert spaces.

Concept check questions 2

(SLO 2.1) Define a Hilbert space and explain the role of the inner product.

(SLO 2.1) Give one example of a Hilbert space.

(SLO 2.2) State the conditions for a set to be orthonormal.

(SLO 2.2) Provide an example of an orthonormal set in Rn.

(SLO 2.3) Explain what it means for a Hilbert space to be complete.

(SLO 2.3) Give one example of a complete space.

(SLO 2.4) Describe the role of Hilbert spaces in Fourier analysis.

(SLO 2.4) Explain how Hilbert spaces are used in quantum mechanics.

Answers to pilot questions 2

Pilot questions 2.1: 1B, 2B, 3B, 4B, 5D **Pilot questions 2.2:** 1A, 2A, 3B, 4D, 5A **Pilot questions 2.3:** 1A, 2A, 3A, 4D, 5A **Pilot questions 2.4:** 1B, 2B, 3B, 4A, 5D

Series 3: Banach Spaces

Series outline

This Series contains four main Parts:

3.1 Definition and normed structure of Banach spaces

3.1.1 Definition of a Banach space
3.1.2 Normed structure
3.1.3 Examples of Banach spaces

3.2 Examples of Banach spaces

3.2.3 Function spaces
3.2.2 Sequence spaces
3.2.1 Finite-dimensional spaces

3.3 Linear operators on Banach spaces

3.3.3 Examples of linear operators
3.3.2 Boundedness and continuity
3.3.1 Definition of linear operators

3.4 Applications of Banach spaces

3.4.1 Differential equations
3.4.2 Optimisation problems
3.4.3 Functional analysis and operator theory

Introduction

In functional analysis, many problems require working with infinite-dimensional spaces where limits and convergence play a central role. While Hilbert spaces rely on inner products, **Banach spaces** generalise the idea of completeness to normed vector spaces. They provide a flexible framework for studying linear operators, differential equations, and approximation theory. Banach spaces are therefore indispensable in both pure mathematics and applied sciences.

The aim of this Series is to introduce you to Banach spaces, explain their defining properties, and highlight their importance in analysis and applications.

We shall commence this Series by defining Banach spaces and examining their normed structure. Next, we will explore examples of Banach spaces, including familiar function spaces. Subsequently, we will study linear operators on Banach spaces, focusing on boundedness and continuity. Finally, we will conclude with applications of Banach spaces in solving differential equations, optimisation problems, and functional analysis at large.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 3.1 define a Banach space and explain its normed structure,

SLO 3.2 illustrate examples of Banach spaces drawn from function spaces and sequence spaces,

SLO 3.3 analyse the behaviour of linear operators on Banach spaces with respect to boundedness and continuity,

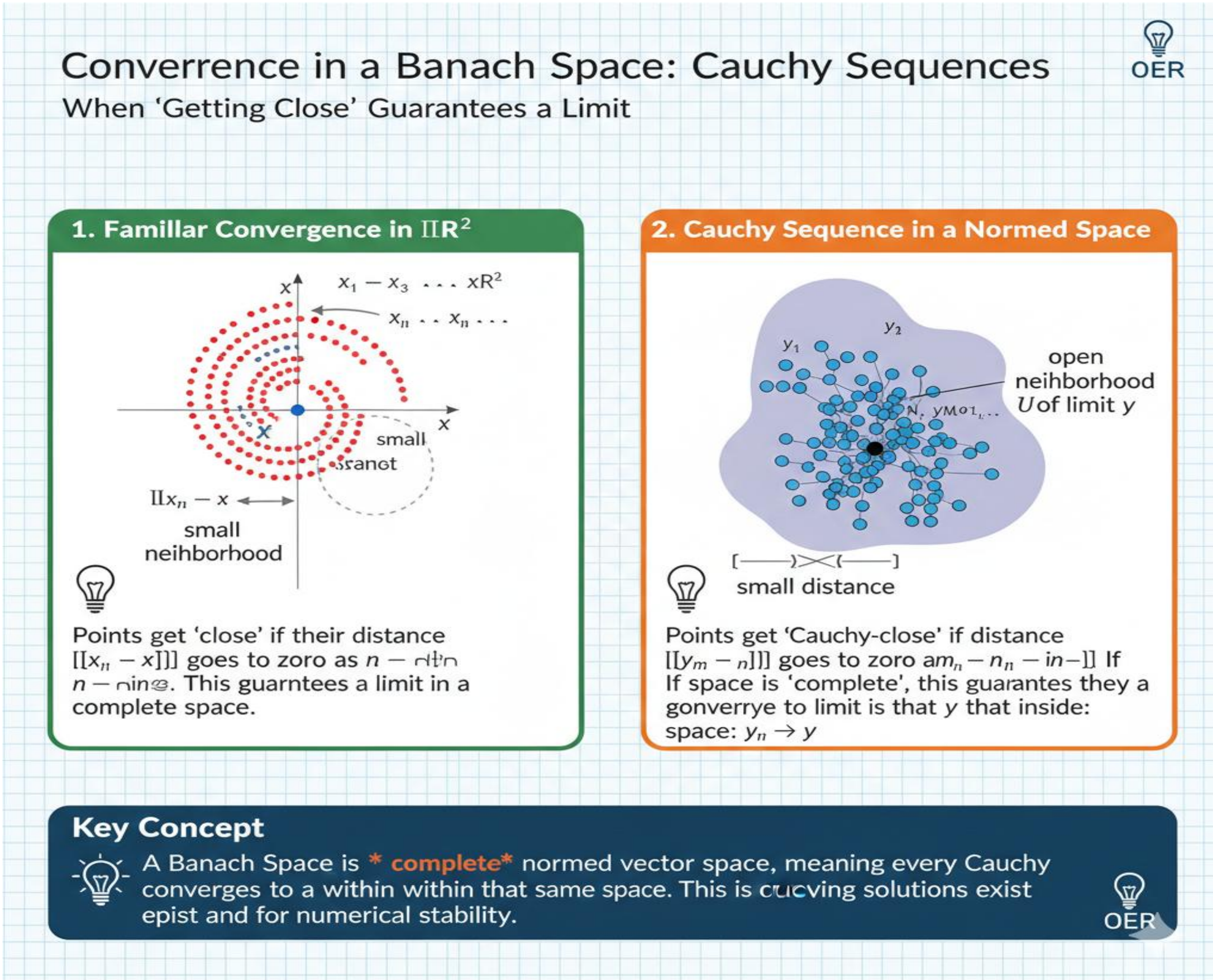
SLO 3.4 evaluate applications of Banach spaces in solving differential equations, optimisation problems, and functional analysis.

3.1 Definition And Normed Structure Of Banach Spaces

3.1.1 Definition of a Banach space

A **Banach space** is a **complete normed vector space**. This means it is a vector space equipped with a **norm** (a function that measures the length of vectors) and is **complete**, so every Cauchy sequence converges to a point within the space. Banach spaces generalise Hilbert spaces by focusing on norms rather than inner products, making them more flexible for analysis.

Fill in the blank: A Banach space is defined as a complete __________ vector space.



Completeness in a Banach space.

3.1.2 Normed structure

The **norm** in a Banach space must satisfy the following properties:

Positivity: x≥0, and x=0if and only if xis the zero vector.

Homogeneity: αx=|||‖xfor any scalar .

Triangle inequality: x+yx+y.

These properties ensure that the norm behaves consistently as a measure of vector length.

Fill in the blank: The triangle inequality states that the norm of a sum is less than or equal to the __________ of the norms.

Properties of norms in Banach spaces

Positivity

Homogeneity

Triangle inequality

3.1.3 Examples of Banach spaces

Examples include:

Rn**:** Complete under the Euclidean norm.

Lp[a,b]**spaces (for** p≥1**):** Complete under the Lpnorm.

lp**spaces:** Sequences whose p-th power is summable, complete under the lpnorm.

Fill in the blank: The space Lp[a,b]is complete under the __________ norm.

<table>
<tr><th>Space</th><th>Elements</th><th>Norm Definition</th><th>Completeness</th><th>Context / Application</th></tr>
<tr><td>$\mathbb{R}^n$ or $\mathbb{C}^n$</td><td>Finite vectors $(x_1, \ldots, x_n)$</td><td>$|x|_p = \left(\sum</td><td>x_i</td><td>^p \right)^{1/p}$</td></tr>
<tr><td>ℓ^p</td><td>Infinite sequences (x_n)</td><td>$|x|p = \left(\sum{n=1}^\infty</td><td>x_n</td><td>^p \right)^{1/p}$</td></tr>
<tr><td>ℓ^∞</td><td>Bounded sequences</td><td>$|x|_\infty = \sup_n</td><td>x_n</td><td>$</td></tr>
<tr><td>$C[a, b]$</td><td>Continuous functions</td><td>$|f|\infty = \max{x \in [a,b]}</td><td>f(x)</td><td>$</td></tr>
<tr><td>$L^p[a, b]$</td><td>Measurable functions</td><td>$|f|_p = \left(\int_a^b</td><td>f(x)</td><td>^p dx \right)^{1/p}$</td></tr>
</table>

Pilot questions 3.1

A Banach space is defined as: A. A complete inner product space B. A complete normed vector space C. A topological vector space D. A polynomial space

Which property of norms ensures that x=0only if xis the zero vector? A. Homogeneity B. Positivity C. Triangle inequality D. Closure

The triangle inequality states that: A. x+yx+yB. x+y=x+yC. x+yx+yD. x+y=0

The space Lp[a,b]is complete under: A. The sup norm B. The Lpnorm C. The dot product D. The random norm

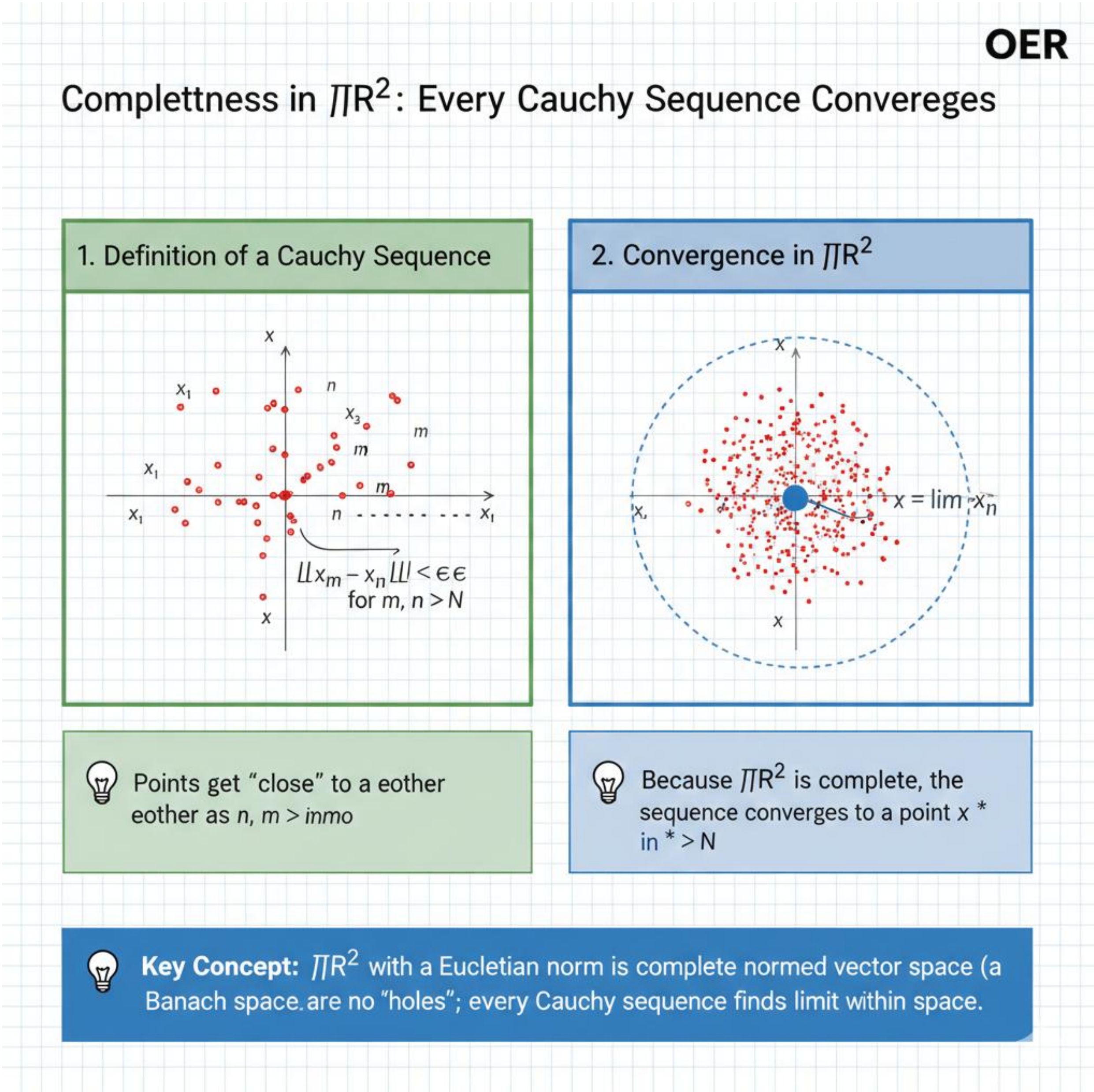
Which of the following is an example of a Banach space? A. RnB. Lp[a,b]C. lpD. All of the above

3.2 Examples Of Banach Spaces

3.2.1 Finite-dimensional spaces

Every finite-dimensional normed vector space is a Banach space because all norms in finite dimensions are equivalent, and completeness is guaranteed. For example, Rnwith the Euclidean norm is a Banach space.

Fill in the blank: Every finite-dimensional normed vector space is automatically __________ because completeness is guaranteed.



Finite-dimensional Banach space R2**.**

3.2.2 Sequence spaces

Sequence spaces provide important examples of Banach spaces:

l^pspaces ($p \geq 1$): Sequences (x_n)such that $\sum |x_n|^p <$. Equipped with the l^pnorm, these spaces are complete.

l: The space of bounded sequences, with norm defined as the supremum of absolute values.

Fill in the blank: The space lpconsists of sequences that are __________ summable.

3.2.3 Function spaces

Function spaces are central in analysis:

Lp[a,b]**spaces (**p≥1**):** Functions whose p-th power is integrable. These are Banach spaces under the Lpnorm.

C[a,b]**:** The space of continuous functions on [a,b], complete under the sup norm.

Fill in the blank: The space C[a,b]is complete under the __________ norm.

Space	Type of Functions	Norm Definition	Completeness	Application
$C[a,b]$	Continuous on $[a,b]$	$|f|\infty = \max{a \le x \le b}	f(x)	$
$L^1[a,b]$	Absolutely Integrable	$|f|_1 = \int_a^b	f(x)	, dx$
$L^2[a,b]$	Square-Integrable	$|f|_2 = \sqrt{\int_a^b	f(x)	^2 , dx}$
$L^p[a,b]$	p-Integrable	$|f|_p = \left(\int_a^b	f(x)	^p , dx \right)^{1/p}$
$L^\infty[a,b]$	Essentially Bounded	$|f|_\infty = \text{ess sup}	f(x)	$

Pilot questions 3.2

Every finite-dimensional normed vector space is: A. A Banach space B. An incomplete space C. A Hilbert space only D. A random space

The space lpconsists of sequences that are: A. Continuous B. Differentiable C. p-summable D. Random

The space lconsists of: A. Bounded sequences B. Continuous functions C. Differentiable functions D. Random sequences

The space Lp[a,b]is complete under: A. The sup norm B. The Lpnorm C. The dot product D. The random norm

The space C[a,b]is complete under: A. The sup norm B. The Lpnorm C. The dot product D. The random norm

3.3 Linear Operators On Banach Spaces

3.3.1 Definition of linear operators

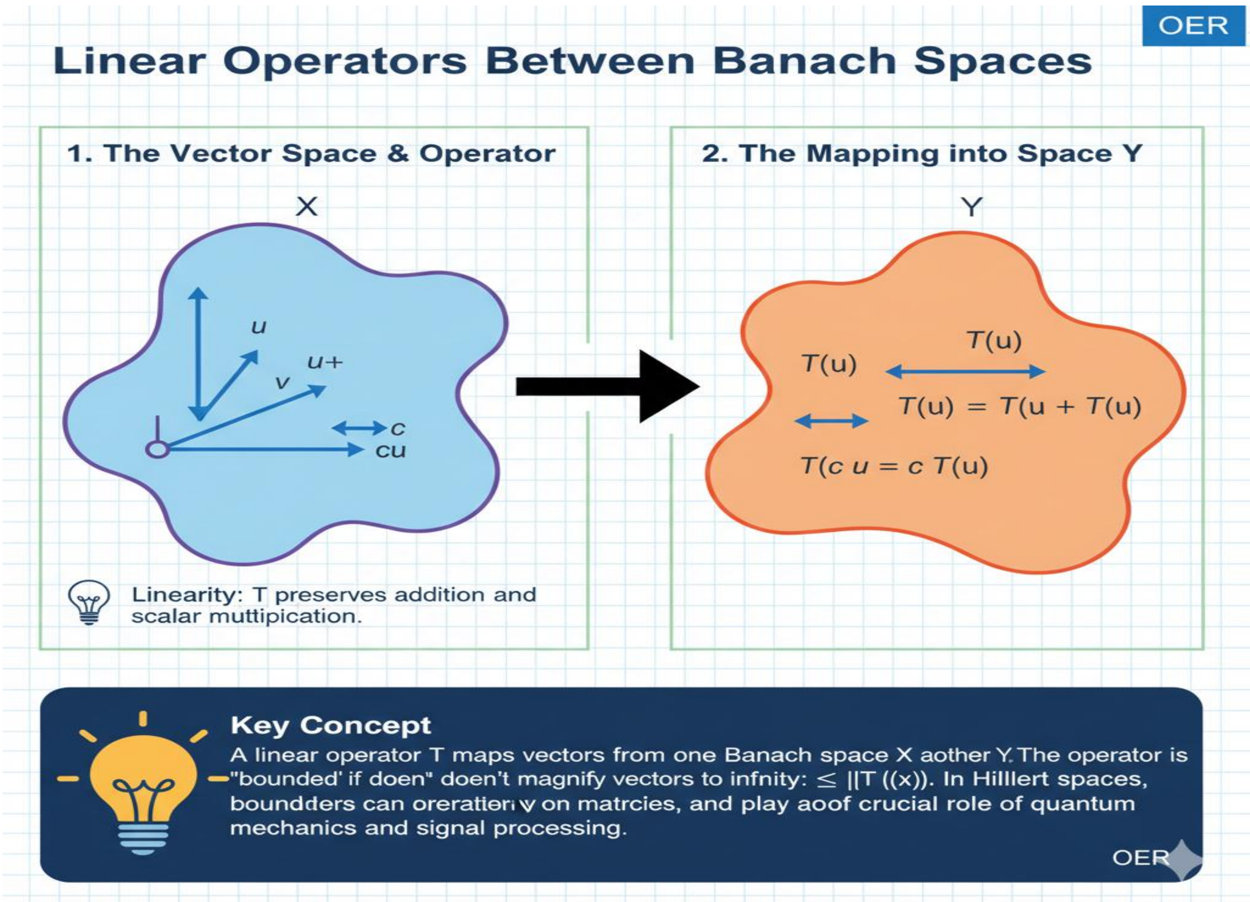
A **linear operator** is a mapping $T:X\rightarrow Y$between two vector spaces that satisfies:

Additivity: $T(x+y)=T(x)+T(y)$.

Homogeneity: $T(\alpha x)=\alpha T(x)$.

When Xand Yare Banach spaces, linear operators are studied with respect to their boundedness and continuity.

Fill in the blank: A linear operator satisfies both additivity and __________.



Linear operator between Banach spaces.

3.3.2 Boundedness and continuity

A linear operator Tis **bounded** if there exists a constant Csuch that $T(x) \leq Cx$for all x. Boundedness implies continuity, meaning small changes in input lead to small changes in output.

Fill in the blank: A linear operator is bounded if there exists a constant __________ such that $T(x) \leq Cx$.

Boundedness and continuity of linear operators

Boundedness: $\|T(x)\| \leq C\|x\|$

Continuity follows from boundedness

Comparison: Bounded vs. Unbounded

Feature	Bounded Operators	Unbounded Operators
Stability	Numerically stable; noise stays small.	Inherently unstable; noise is amplified.
Continuity	Always continuous.	Never continuous.
Domain	Usually the entire space X.	Usually a dense subspace (e.g., $C^1[a,b]$).
Example	Matrix multiplication, Integral operators.	Derivative operators ($\frac{d}{dx}$), Position/Momentum.

3.3.3 Examples of linear operators

Examples include:

- **Differentiation operator:** On C1[a,b], the operator T(f)=f'is linear but not bounded under the sup norm.
- **Integration operator:** On C[a,b], the operator T(f)=axf(t) dtis linear and bounded.
- **Shift operator:** On lp, the operator T((x1,x2,...))=(0,x1,x2,...)is linear and bounded.

Fill in the blank: The integration operator on C[a,b]is linear and __________.

[Insert table here] Caption: Table 3.3: Examples of linear operators on Banach spaces

Operator	Space	Bounded?
Differentiation	C1[a,b]	Not bounded
Integration	C[a,b]	Bounded
Shift	lp	Bounded

Pilot questions 3.3

- A linear operator satisfies: A. Additivity and homogeneity B. Continuity and boundedness C. Randomness and closure D. Differentiability and integration

A linear operator is bounded if: A. T(x)≤Cxfor some constant CB. T(x)=xalways C. T(x)xalways D. T(x)=0always

Boundedness of a linear operator implies: A. Continuity B. Randomness C. Divergence D. Non-linearity

The differentiation operator on C1[a,b]is: A. Linear and bounded B. Linear but not bounded C. Non-linear and bounded D. Non-linear and unbounded

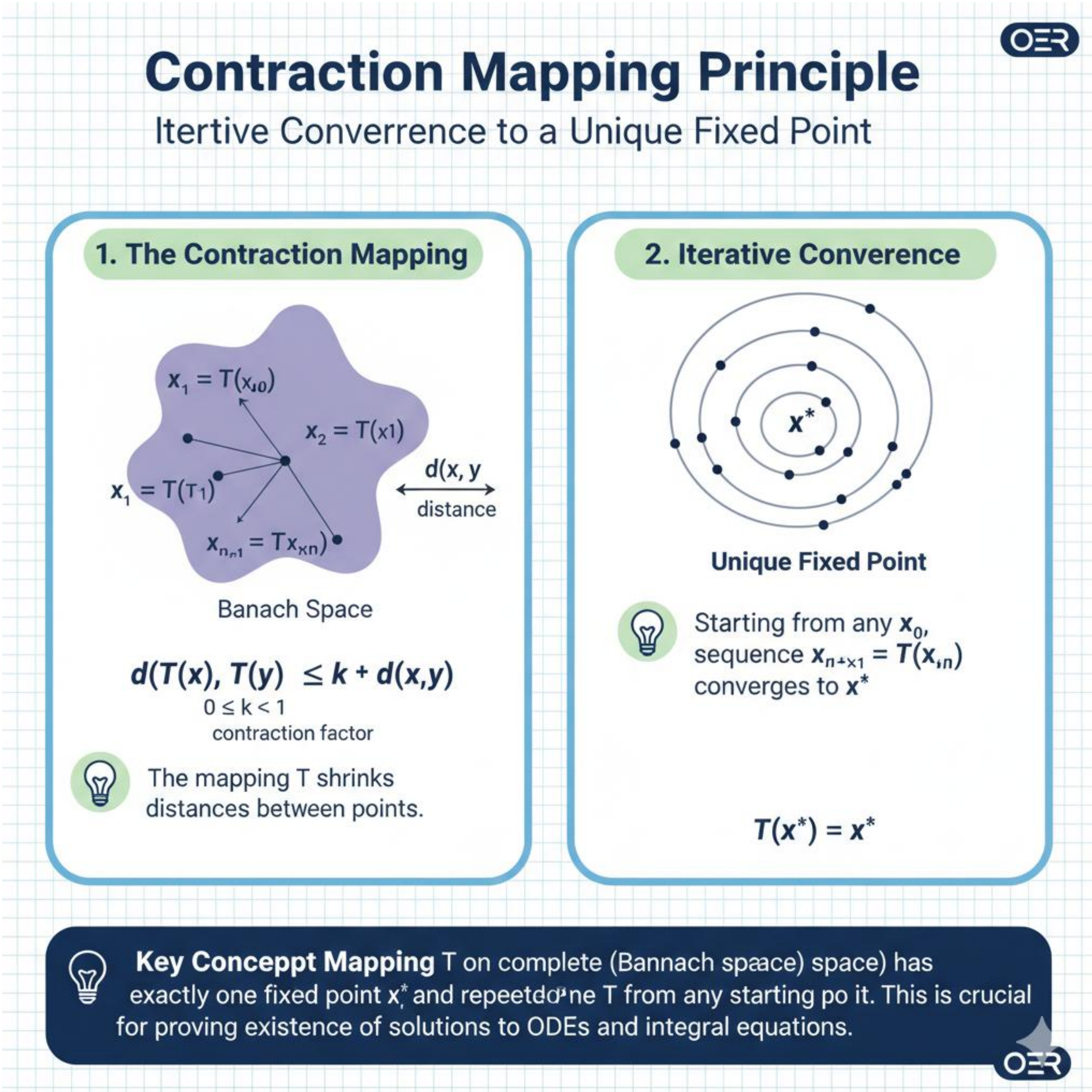
The integration operator on C[a,b]is: A. Linear and bounded B. Linear but not bounded C. Non-linear and bounded D. Non-linear and unbounded

3.4 Applications Of Banach Spaces

3.4.1 Differential equations

Banach spaces provide the framework for studying **differential equations**. Many existence and uniqueness theorems, such as the Banach fixed-point theorem, rely on the completeness of Banach spaces. This theorem guarantees solutions to certain equations by showing that iterative processes converge.

Fill in the blank: The Banach fixed-point theorem guarantees solutions by showing that iterative processes __________.



Banach fixed-point theorem applied to differential equations.

3.4.2 Optimisation problems

Banach spaces are widely used in **optimisation theory**, where functions are minimised or maximised subject to constraints. The completeness property ensures that optimal solutions exist within the space.

Fill in the blank: In optimisation problems, Banach spaces ensure that optimal solutions exist within the __________.

Role of Banach spaces in optimisation

Provide structure for minimisation and maximisation problems

Ensure existence of solutions

Support stability of optimisation methods

3.4.3 Functional analysis and operator theory

Banach spaces are central to **functional analysis**, where linear operators are studied in depth. They provide the setting for spectral theory, which analyses the behaviour of operators through their eigenvalues and spectra. This is essential in quantum mechanics, numerical analysis, and applied mathematics.

Fill in the blank: Spectral theory studies the behaviour of operators through their __________ and spectra.

[Insert table here] Caption: Table 3.4: Applications of Banach spaces

Application	Description	Example
Differential equations	Existence and uniqueness via fixed-point theorem	Ordinary differential equations
Optimisation	Ensures solutions exist within the space	Linear programming
Functional analysis	Study of operators and spectra	Quantum mechanics, numerical analysis

Pilot questions 3.4

The Banach fixed-point theorem guarantees solutions by showing that: A. Iterative processes converge B. Iterative processes diverge C. Iterative processes remain constant D. Iterative processes are random

In optimisation problems, Banach spaces ensure that: A. Optimal solutions exist within the space B. Solutions escape the space C. Solutions are undefined D. Solutions are random

Banach spaces are central to: A. Functional analysis B. Random processes C. Geometry only D. Algebra only

Spectral theory studies the behaviour of operators through their: A. Eigenvalues and spectra B. Random variables C. Derivatives D. Integrals

Which of the following is an application of Banach spaces? A. Differential equations B. Optimisation problems C. Functional analysis D. All of the above

Series summary

In this Series, we explored the structure and applications of Banach spaces. We began by defining Banach spaces as complete normed vector spaces, highlighting the role of norms and their properties. We then examined examples of Banach spaces, including finite-dimensional spaces, sequence spaces such as lp, and function spaces such as Lp[a,b]and C[a,b]. Following that, we studied linear operators on Banach spaces, focusing on boundedness and continuity, and illustrated with differentiation, integration, and shift operators. Finally, we considered applications of Banach spaces in differential equations, optimisation problems, and functional analysis, emphasising their importance in modern mathematics and applied sciences.

Glossary of terms

Banach space: A complete normed vector space.

Norm: A function that measures the length of vectors, satisfying positivity, homogeneity, and triangle inequality.

Cauchy sequence: A sequence where successive terms become arbitrarily close as the sequence progresses.

Bounded operator: A linear operator Tsuch that T(x)≤Cxfor some constant C.

Continuous operator: A linear operator where small changes in input lead to small changes in output.

Fixed-point theorem: A result guaranteeing the existence of solutions by showing iterative processes converge.

Spectral theory: The study of operators through their eigenvalues and spectra.

Concept check questions 3

(SLO 3.1) Define a Banach space.

(SLO 3.1) State the three properties of a norm.

(SLO 3.2) Give one example of a sequence space that is a Banach space.

(SLO 3.2) Explain why every finite-dimensional normed vector space is a Banach space.

(SLO 3.3) Define a bounded linear operator.

(SLO 3.3) Give one example of a bounded operator on a Banach space.

(SLO 3.4) State the Banach fixed-point theorem in simple terms.

(SLO 3.4) Explain one application of Banach spaces in optimisation problems.

Answers to pilot questions 3

Pilot questions 3.1: 1B, 2B, 3A, 4B, 5D **Pilot questions 3.2:** 1A, 2C, 3A, 4B, 5A
Pilot questions 3.3: 1A, 2A, 3A, 4B, 5A **Pilot questions 3.4:** 1A, 2A, 3A, 4A, 5D

Series 4: Linear Operators

Series outline

This Series contains four main Parts:

4.1 Definition and properties of linear operators

4.1.1 Definition of a linear operator
4.1.2 Properties of linear operators
4.1.3 Examples of linear operator

4.2 Bounded and unbounded operators

4.2.1 Bounded operators

4.2.2 Unbounded operators

4.2.3 Examples

4.3 Spectrum of linear operators

4.3.1 Definition of spectrum
4.3.2 Types of spectrum
4.3.3 Examples

4.4 Applications of linear operators

4.4.1 Differential equations

4.4.2 Quantum mechanics

4.4.3 Numerical analysis

Introduction

Linear operators are at the heart of functional analysis. They generalise the concept of functions by mapping elements from one vector space to another while preserving linearity. In Banach and Hilbert spaces, linear operators provide the framework for studying transformations, stability, and convergence.

This Series introduces the definition and properties of linear operators, distinguishing between bounded and unbounded cases. We then explore the spectrum of linear operators, which reveals deep insights into their behaviour, much like eigenvalues in finite-dimensional spaces. Finally, we examine applications of linear operators in differential equations, quantum mechanics, and numerical analysis.

By the end of this Series, you will understand how linear operators serve as the building blocks of functional analysis and why they are indispensable in both theory and applications.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 4.1 define linear operators and explain their fundamental properties,

SLO 4.2 distinguish between bounded and unbounded operators and analyse their behaviour in Banach and Hilbert spaces,

SLO 4.3 describe the spectrum of linear operators and explain its significance in functional analysis,

SLO 4.4 evaluate applications of linear operators in differential equations, quantum mechanics, and numerical analysis.

4.1 Definition And Properties Of Linear Operators

4.1.1 Definition of a linear operator

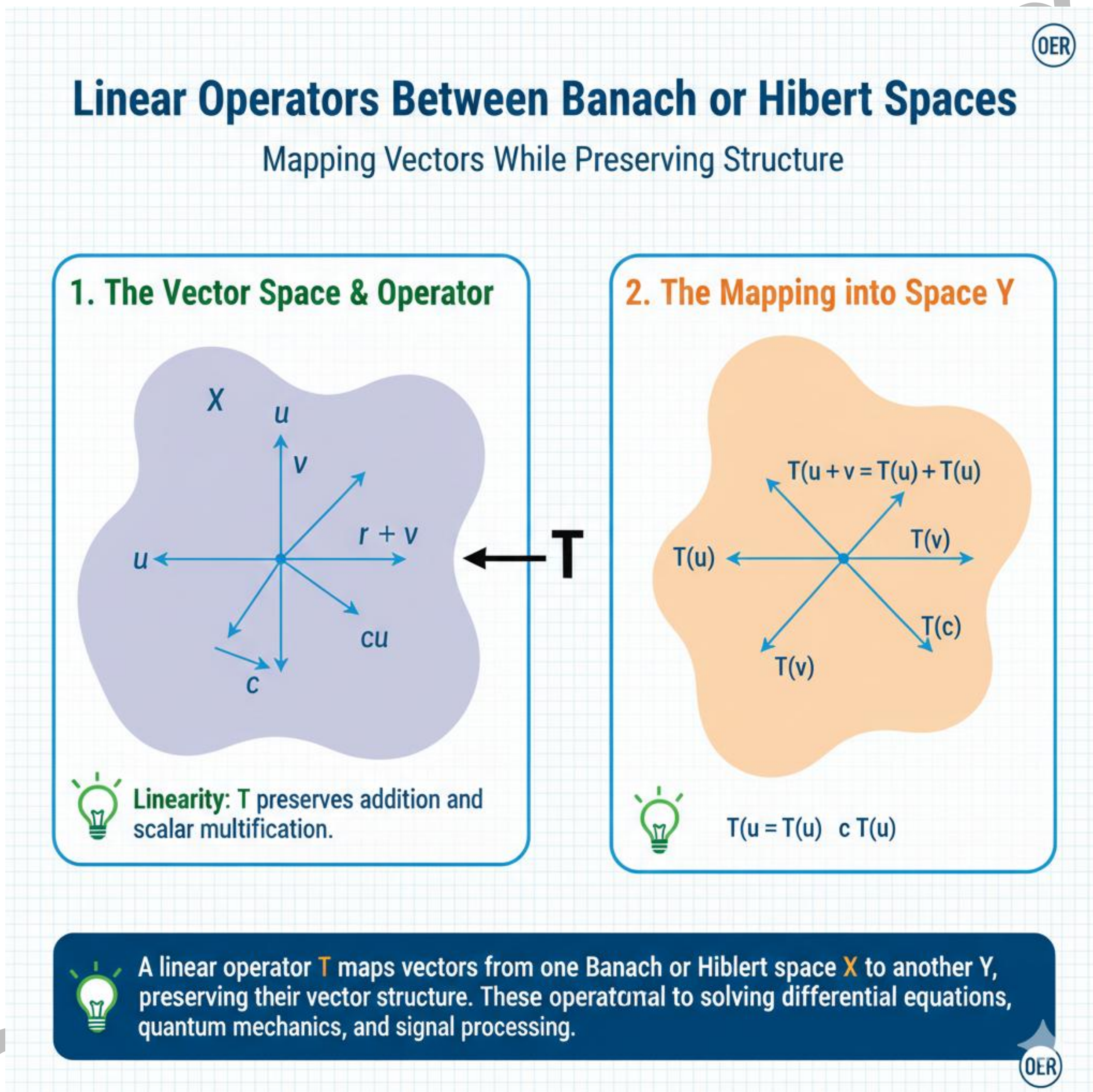
A linear operator is a mapping $T:X\to Y$between two vector spaces (often Banach or Hilbert spaces) that preserves linearity. Specifically:

Additivity: $T(x+y)=T(x)+T(y)$.

Homogeneity: $T(\alpha x)=\alpha T(x)$.

Linear operators generalise the concept of linear functions to infinite-dimensional spaces, making them central to functional analysis.

Fill in the blank: A linear operator preserves both additivity and __________.



Linear operator mapping vectors between spaces.

4.1.2 Properties of linear operators

Key properties include:

Boundedness: A linear operator Tis bounded if T(x)≤Cxfor some constant C.

Continuity: Boundedness implies continuity.

Kernel and range: The kernel (null space) of Tis the set of vectors mapped to zero, while the range is the set of all possible outputs.

Invertibility: An operator is invertible if there exists another operator T-1such that T-1(T(x))=x.

Fill in the blank: The kernel of a linear operator is the set of vectors mapped to __________.

[Insert box here] Caption: Box 4.1: Properties of linear operators

Boundedness

Continuity

Kernel and range

Invertibility

4.1.3 Examples of linear operators

Examples include:

Differentiation operator: T(f)=f'on C1[a,b].

Integration operator: T(f)(x)=axf(t) dton C[a,b].

Shift operator: T((x1,x2,...))=(0,x1,x2,...)on lp.

Fill in the blank: The differentiation operator on C1[a,b]is an example of a __________ operator.

[Insert table here] Caption: Table 4.1: Examples of linear operators

Operator	Space	Property
Differentiation	C1[a,b]	Linear, not bounded under sup norm
Integration	C[a,b]	Linear, bounded
Shift	lp	Linear, bounded

Pilot questions 4.1

A linear operator satisfies: A. Additivity and homogeneity B. Continuity and boundedness C. Randomness and closure D. Differentiability and integration

The kernel of a linear operator is: A. The set of vectors mapped to zero B. The set of vectors mapped to infinity C. The set of vectors mapped to one D. The set of vectors mapped to random values

Boundedness of a linear operator implies: A. Continuity B. Randomness C. Divergence D. Non-linearity

The differentiation operator on $C1[a,b]$is: A. Linear and bounded B. Linear but not bounded C. Non-linear and bounded D. Non-linear and unbounded

The integration operator on $C[a,b]$is: A. Linear and bounded B. Linear but not bounded C. Non-linear and bounded D. Non-linear and unbounded

4.2 Bounded And Unbounded Operators

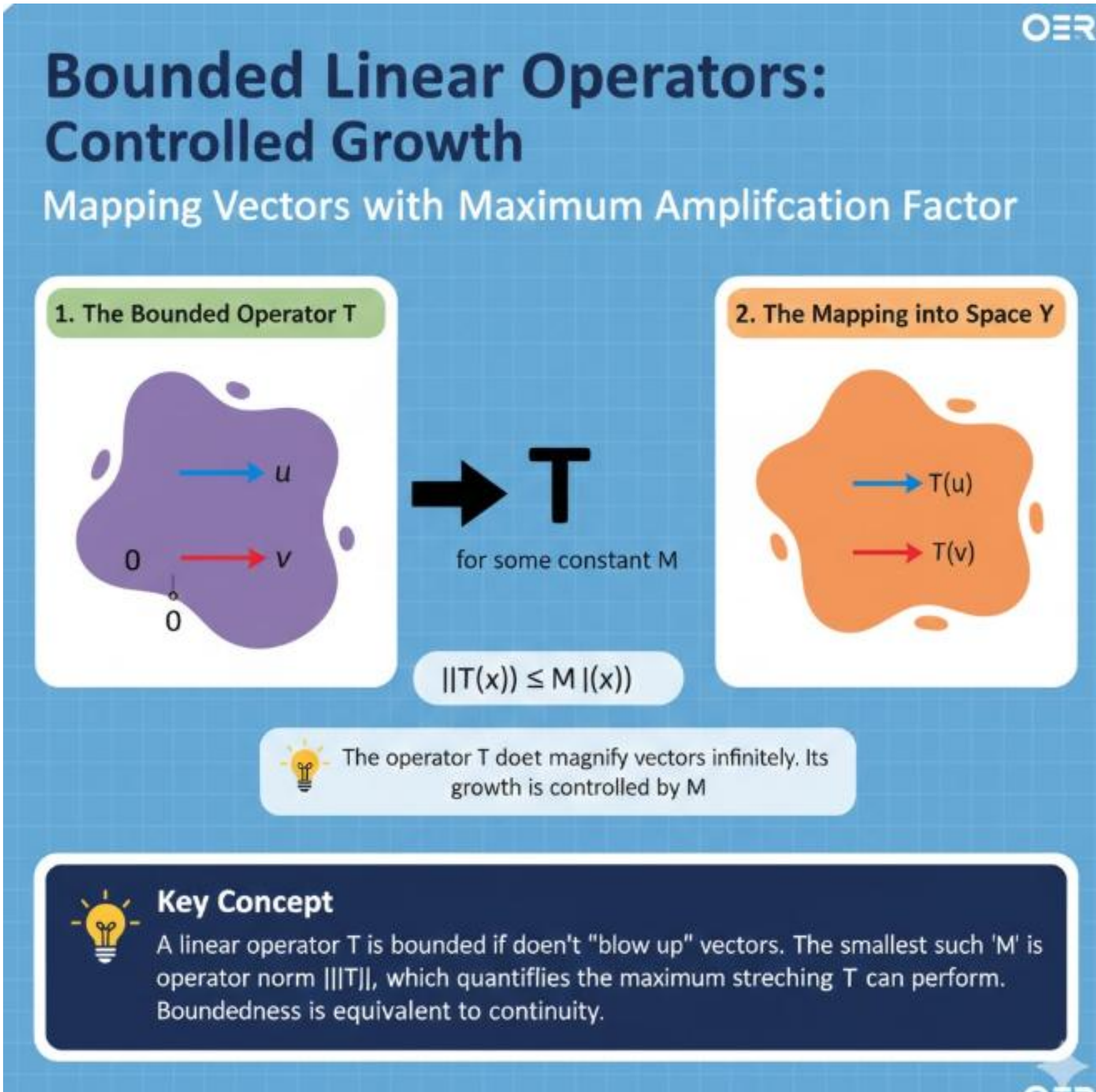
4.2.1 Bounded operators

A linear operator T:X→Ybetween Banach or Hilbert spaces is bounded if there exists a constant $C>0$such that:

$$T(x)\leq Cx \text{for all } x\in X$$

Bounded operators are continuous, meaning small changes in input lead to small changes in output.

Fill in the blank: A bounded operator satisfies $T(x)\leq Cx$for some constant __________.



Bounded operator ensures controlled growth of outputs.

4.2.2 Unbounded operators

An operator is unbounded if no such constant Cexists. Unbounded operators often arise in differential equations and quantum mechanics. For example, the differentiation operator on C1[a,b]under the sup norm is unbounded.

Fill in the blank: An operator is unbounded if no constant __________ can control the growth of outputs.

Bounded vs. unbounded operators

Bounded: growth controlled by a constant

Unbounded: growth not controlled by any constant

Feature	Bounded Operators	Unbounded Operators
Basic Condition	|T\mathbf{x}| \le M |\mathbf{ for some m.	No such M exists; $\|\|T\mathbf{x}$ can be "huge" for small $\mathbf{.
Continuity	**Always continuous.** Small input changes lead to small output changes.	**Not continuous.** Small high-frequency noise can cause large output spikes.
Domain	Usually the entire space X.	Usually defined only on a "dense" subset (e.g., differentiable functions).
Typical Example	**Integral Operators** (e.g., $\int K(x,t)f(t)dt$).	**Differential Operators** (e.g., $\frac{d}{dx}$ or ∇^2).
Numerical Stability	Very stable; errors do not amplify.	Inherently unstable; requires "regularization" to solve numerically.

4.2.3 Examples

Bounded operator: Integration operator on C[a,b].

Unbounded operator: Differentiation operator on C1[a,b].

Bounded operator: Shift operator on lp.

Fill in the blank: The differentiation operator on C1[a,b]is an example of an ___________ operator.

[Insert table here] Caption: Table 4.2: Examples of bounded and unbounded operators

Operator	Space	Bounded/Unbounded
Integration	C[a,b]	Bounded
Differentiation	C1[a,b]	Unbounded
Shift	lp	Bounded

Pilot questions 4.2

A bounded operator satisfies: A. T(x)≤Cxfor some constant CB. T(x)=xalways C. T(x)xalways D. T(x)=0always

An operator is unbounded if: A. No constant can control the growth of outputs B. Every constant controls the growth of outputs C. It is always continuous D. It is always bounded

Which of the following is a bounded operator? A. Differentiation on C1[a,b]B. Integration on C[a,b]C. Random operator D. None of the above

The differentiation operator on C1[a,b]is: A. Bounded B. Unbounded C. Random D. Continuous and bounded

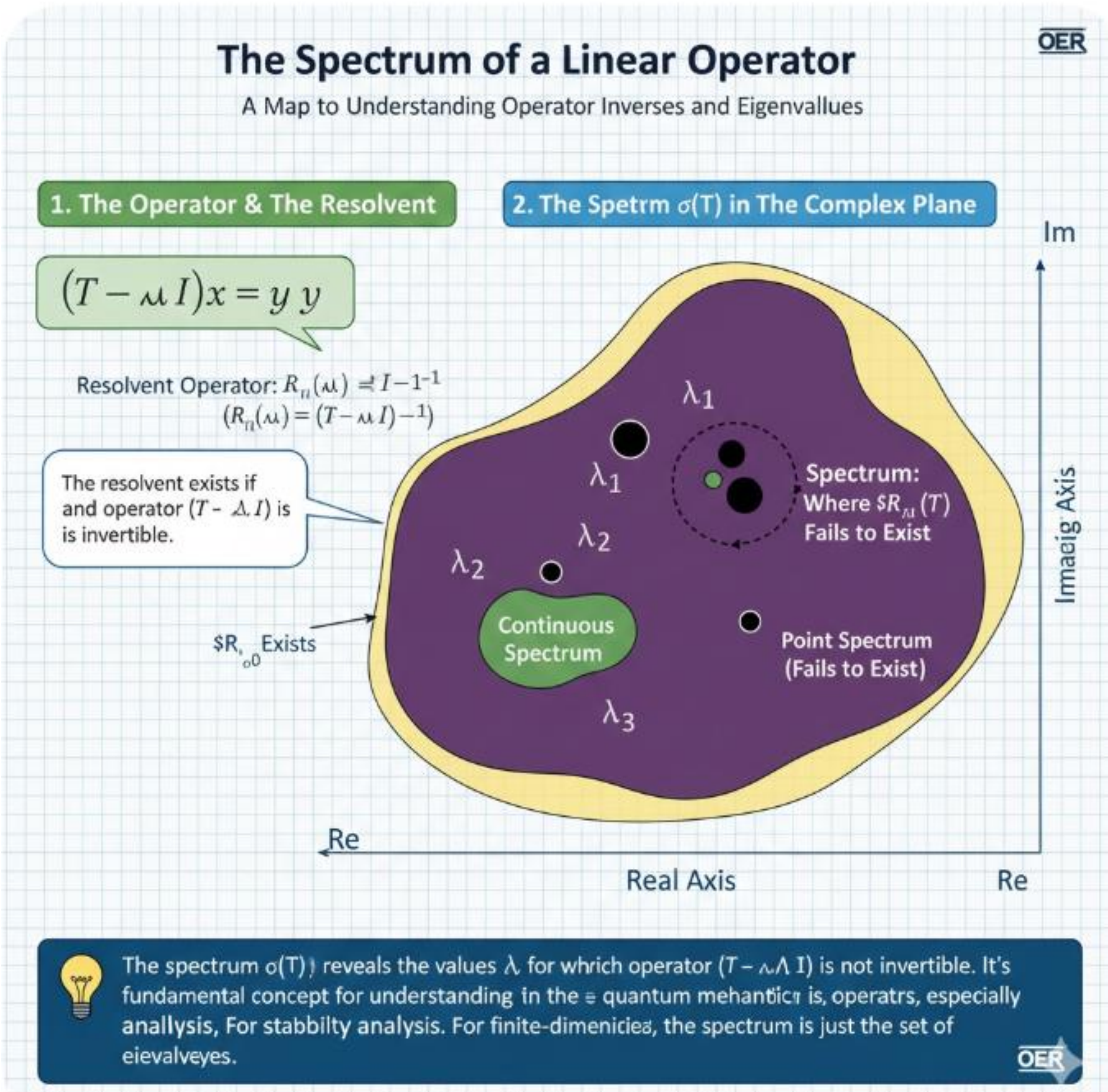
The shift operator on lpis: A. Bounded B. Unbounded C. Random D. Undefined

4.3 Spectrum Of Linear Operators

4.3.1 Definition of spectrum

For a bounded linear operator Ton a Banach space X, the spectrum of T, denoted σ(T), is the set of complex numbers such that T-λIis not invertible. This eneralizes the concept of eigenvalues from finite-dimensional spaces to infinite-dimensional settings.

Fill in the blank: The spectrum of an operator consists of all complex numbers such that T-λIis not __________.



Spectrum of a linear operator in the complex plane.

4.3.2 Types of spectrum

The spectrum can be divided into three types:

Point spectrum: Values of for which T-λIis not injective (eigenvalues).

Continuous spectrum: Values of for which T-λIis injective but not surjective, and the inverse is not bounded.

Residual spectrum: Values of for which T-λIis injective but has a non-dense range.

Fill in the blank: The point spectrum corresponds to the set of __________ of the operator.

Types of spectrum

Point spectrum: eigenvalues

Continuous spectrum: injective but not surjective

Residual spectrum: injective but non-dense range

Classification Table

Component	$(T - \lambda I)$ is Injective?	Range is Dense?	Inverse is Bounded?
Point (σ_p)	No (Eigenvalues exist)	No	No
Continuous (σ_c)	Yes	Yes	**No** (Unbounded)
Residual (σ_r)	Yes	**No**	No
Resolvent Set	Yes	Yes	**Yes** (Stable)

4.3.3 Examples

Finite-dimensional case: For a matrix A, the spectrum is the set of eigenvalues.

Infinite-dimensional case: For the shift operator on l2, the spectrum is the closed unit disk in the complex plane.

Differentiation operator: On L2[0,1], the spectrum is continuous and related to Fourier transforms.

Fill in the blank: The spectrum of the shift operator on l2is the closed __________ disk in the complex plane.

[Insert table here] Caption: Table 4.3: Examples of spectra of operators

Operator	Space	Spectrum
Matrix A	Rn	Eigenvalues

Shift operator	l2	Closed unit disk
Differentiation operator	L2[0,1]	Continuous spectrum

Pilot questions 4.3

The spectrum of an operator consists of values such that: A. T-λIis not invertible B. T-λIis always invertible C. T-λIis random D. T-λIis bounded

The point spectrum corresponds to: A. Eigenvalues of the operator B. Random values C. Continuous functions D. Non-dense ranges

The continuous spectrum occurs when: A. T-λIis injective but not surjective B. T-λIis surjective but not injective C. T-λIis random D. T-λIis bounded

The residual spectrum occurs when: A. T-λIis injective but has non-dense range B. T-λIis surjective but not injective C. T-λIis random D. T-λIis bounded

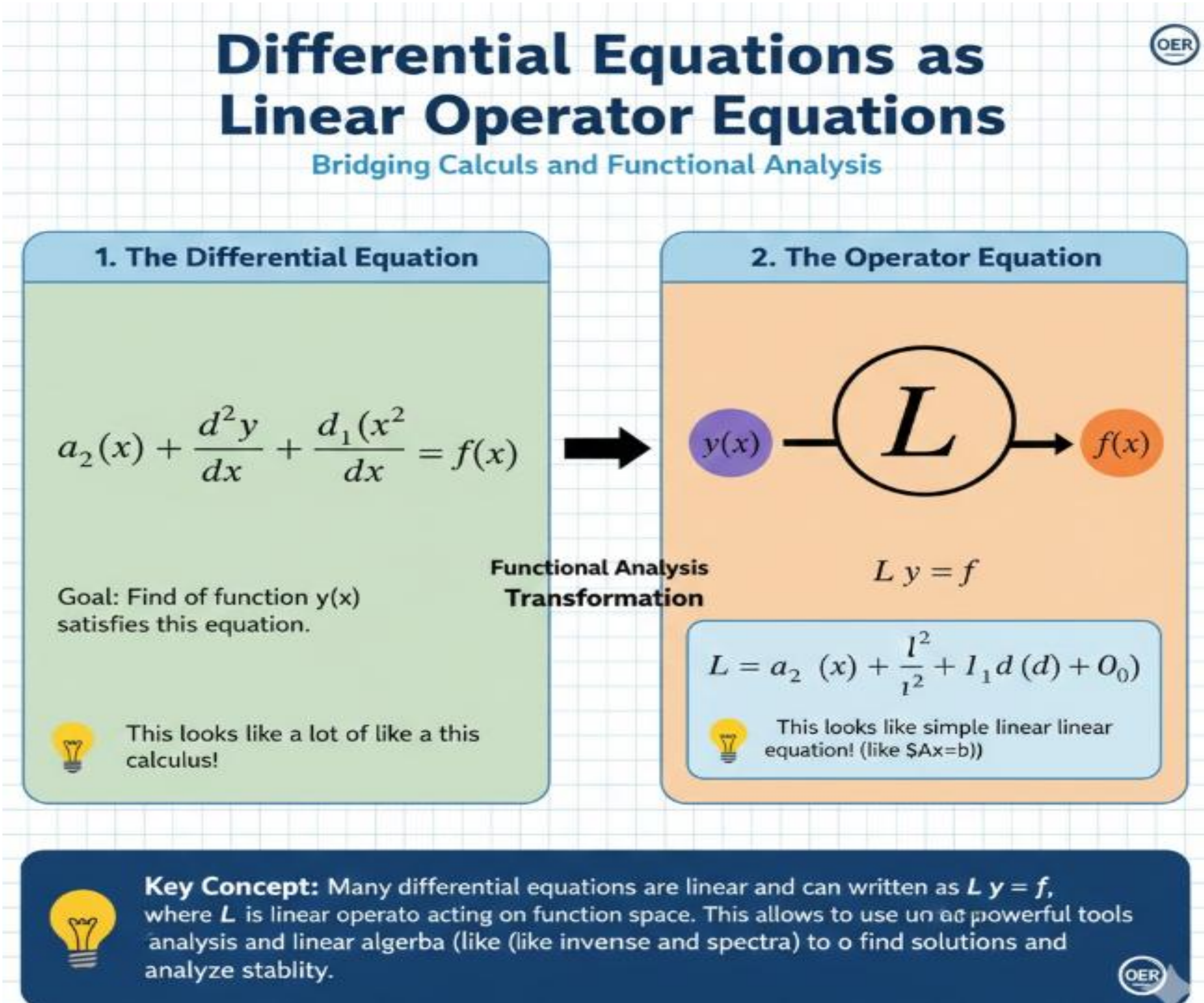
The spectrum of the shift operator on l2is: A. Closed unit disk B. Open unit disk C. Random set D. Continuous spectrum only

4.4 Applications Of Linear Operators

4.4.1 Differential equations

Linear operators provide the framework for solving differential equations. Many problems can be expressed in operator form, such as T(f)=f'for differentiation. Existence and uniqueness theorems often rely on properties of linear operators, especially boundedness and spectral analysis.

Fill in the blank: Differential equations can often be expressed in __________ form using linear operators.



Differential equations expressed in operator form.

4.4.2 Quantum mechanics

In quantum mechanics, physical observables such as momentum and energy are represented by linear operators acting on Hilbert spaces. The spectral properties of these operators correspond to measurable quantities like eigenvalues, which represent possible outcomes of experiments.

Fill in the blank: In quantum mechanics, physical observables are represented by __________ operators acting on Hilbert spaces.

[Insert box here] Caption: Box 4.4: Role of linear operators in quantum mechanics

Observables represented by operators

Spectral properties correspond to measurable outcomes

Operators act on Hilbert spaces

Summary: The Role of Linear Operators in Quantum Mechanics

In the mathematical formulation of quantum mechanics, **linear operators** act as the bridge between abstract state vectors and the physical world we can measure. Every measurable property of a system is tied to an operator acting on a Hilbert space.

---

1. Observables as Self-Adjoint Operators

In classical mechanics, properties like momentum or energy are simple numbers. In quantum mechanics, these are **Hermitian (self-adjoint) operators**:

- **Property:** A self-adjoint operator A satisfies $A = A^\dagger$.
- **Significance:** This ensures that all calculated measurement results (eigenvalues) are **real numbers**, which is a requirement for physical reality.

2. Eigenvalues and Measurement Outcomes

When you measure a physical quantity (an "observable"), the only possible outcomes are the **eigenvalues** (λ) of its corresponding operator $\hat{A}$.

- **The Equation:** $\hat{A}|\psi\rangle = \lambda|\psi\rangle$
- **State Collapse:** Immediately after a measurement yields λ, the system's state "collapses" into the corresponding eigenvector $|\psi\rangle$.

3. Evolution via the Hamiltonian Operator

The most important operator in any quantum system is the **Hamiltonian** ($\hat{H}$), representing the total energy.

- **Dynamics:** It dictates how the system changes over time through the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

- **Stationary States:** Finding the eigenvectors of the Hamiltonian tells us the stable energy levels of an atom or molecule.

4. Commutation and Uncertainty

Linear operators do not always commute (meaning $\hat{A}\hat{B}$ is not always equal to $\hat{B}\hat{A}$).

- **Commutator:** $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$
- **Heisenberg Uncertainty Principle:** If two operators do not commute (like Position $\hat{X}$ and Momentum $\hat{P}$), it is mathematically impossible to know both properties simultaneously with perfect precision.

Mapping Observables to Operators

Physical Quantity	Operator Symbol	Form (in Position Space)
Position	$\hat{x}$	$x \cdot \psi(x)$
Momentum	$\hat{p}$	$-i\hbar\frac{\partial}{\partial x}$
Kinetic Energy	$\hat{T}$	$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
Total Energy	$\hat{H}$	$\hat{T} + V(x)$

4.4.3 Numerical analysis

Linear operators are used in numerical analysis to approximate solutions of equations. Iterative methods, such as power iteration or Krylov subspace methods, rely on operator theory to compute eigenvalues and approximate solutions efficiently.

Fill in the blank: Iterative methods in numerical analysis often rely on __________ theory to compute eigenvalues.

[Insert table here] Caption: Table 4.4: Applications of linear operators

Application	Description	Example
Differential equations	Expressed in operator form	Existence and uniqueness theorems
Quantum mechanics	Observables as operators	Momentum, energy operators
Numerical analysis	Iterative methods using operator theory	Power iteration, Krylov methods

***g*Pilot questions 4.4**

Differential equations can often be expressed in: A. Operator form B. Random form C. Polynomial form D. Continuous form

In quantum mechanics, physical observables are represented by: A. Scalars B. Linear operators C. Random variables D. Polynomials

Spectral properties of operators in quantum mechanics correspond to: A. Measurable outcomes B. Random values C. Continuous functions D. Non-dense ranges

Iterative methods in numerical analysis often rely on: A. Operator theory B. Random processes C. Geometry only D. Algebra only

Which of the following is an application of linear operators? A. Differential equations B. Quantum mechanics C. Numerical analysis D. All of the above

Series summary

In this Series, we studied linear operators, which are fundamental in functional analysis. We began by defining linear operators and examining their properties such as boundedness, continuity, kernel, range, and invertibility. We then distinguished between bounded and unbounded operators, highlighting their roles in Banach and Hilbert spaces. Next, we explored the spectrum of linear operators, generalising eigenvalues to infinite-dimensional spaces and categorising the spectrum into point, continuous, and residual types. Finally, we examined applications of linear operators in differential equations, quantum mechanics, and numerical analysis, showing their central role in both theoretical and applied mathematics.

Glossary of terms

Linear operator: A mapping between vector spaces that preserves additivity and homogeneity.

Bounded operator: An operator Tsuch that T(x)≤Cxfor some constant C.

Unbounded operator: An operator for which no constant can control the growth of outputs.

Kernel: The set of vectors mapped to zero by an operator.

Range: The set of all possible outputs of an operator.

Spectrum: The set of complex numbers such that T-λIis not invertible.

Point spectrum: The set of eigenvalues of an operator.

Continuous spectrum: Values where T-λIis injective but not surjective, and inverse is not bounded.

Residual spectrum: Values where T-λIis injective but has non-dense range.

Concept check questions 4

(SLO 4.1) Define a linear operator and state its two main properties.

(SLO 4.1) What is the kernel of a linear operator?

(SLO 4.2) Distinguish between bounded and unbounded operators.

(SLO 4.2) Give one example of a bounded operator and one example of an unbounded operator.

(SLO 4.3) Define the spectrum of a linear operator.

(SLO 4.3) Name the three types of spectrum.

(SLO 4.4) Explain how linear operators are used in quantum mechanics.

(SLO 4.4) Give one application of linear operators in numerical analysis.

Answers to pilot questions 4

Pilot questions 4.1: 1A, 2A, 3A, 4B, 5A Pilot questions 4.2: 1A, 2A, 3B, 4B, 5A
Pilot questions 4.3: 1A, 2A, 3A, 4A, 5A Pilot questions 4.4: 1A, 2B, 3A, 4A, 5D

Series 5: Spectral Theory

Series outline

This Series contains four main Parts:

5.1 Definition and classification of spectrum

5.1.2 Classification of spectrum
5.1.2 Classification of spectrum
5.1.3 Examples

5.2 Spectral properties of bounded operators

5.2.1 Spectral radius
5.2.2 Spectral radius formula
5.2.3 Properties of spectrum for bounded operators

5.3 Spectral theorem in Hilbert spaces

5.3.1 Statement of the spectral theorem

5.3.2 Spectral decomposition

5.3.3 Applications of the spectral theorem

5.4 Applications of spectral theory

5.4.1 Quantum mechanics
5.4.2 Differential equations
5.4.3 Numerical analysis

Introduction

Spectral theory extends the concept of eigenvalues and eigenvectors from finite-dimensional linear algebra to infinite-dimensional spaces such as Banach and Hilbert spaces. It provides a powerful framework for analysing linear operators by studying their spectra, which reveal deep insights into stability, convergence, and physical interpretations.

In this Series, we begin by defining the spectrum of an operator and classifying it into point, continuous, and residual spectra. We then examine spectral properties of bounded operators, focusing on their behaviour in Banach spaces. Next, we study the spectral theorem in Hilbert spaces, which generalises diagonalisation of matrices to infinite dimensions. Finally, we explore applications of spectral theory in quantum mechanics, differential equations, and numerical analysis.

By the end of this Series, you will understand how spectral theory connects functional analysis with practical applications in science and engineering.

Series Learning Outcomes

Upon completing this Series, you should be able to:

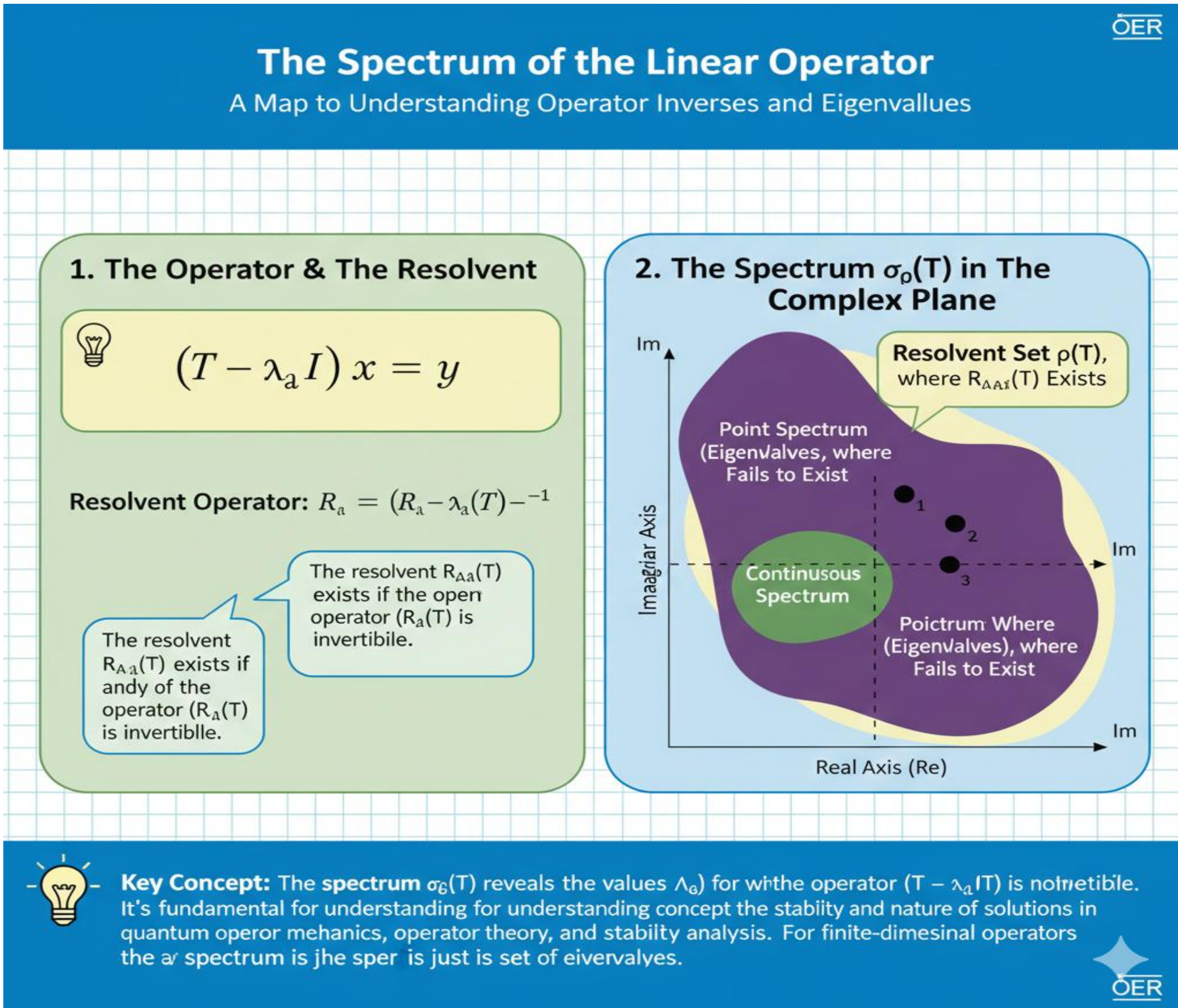
- **SLO 5.1** define the spectrum of a linear operator and classify it into point, continuous, and residual spectra,
- **SLO 5.2** analyse spectral properties of bounded operators in Banach spaces,
- **SLO 5.3** explain the spectral theorem in Hilbert spaces and its significance,
- **SLO 5.4** evaluate applications of spectral theory in quantum mechanics, differential equations, and numerical analysis.

5.1 Definition And Classification Of Spectrum

5.1.1 Definition of spectrum

For a bounded linear operator Ton a Banach space X, the **spectrum** of T, denoted $\sigma(T)$, is the set of complex numbers such that $T-\lambda I$is not invertible. This extends the idea of eigenvalues from finite-dimensional matrices to infinite-dimensional operators.

Fill in the blank: The spectrum of an operator consists of all complex numbers such that $T-\lambda I$is not __________.



Spectrum of a linear operator in the complex plane.

5.1.2 Classification of spectrum

The spectrum can be classified into three types:

- **Point spectrum (p):** Values of for which T-λIis not injective (eigenvalues).
- **Continuous spectrum (c):** Values of for which T-λIis injective but not surjective, and the inverse is not bounded.
- **Residual spectrum (r):** Values of for which T-λIis injective but has a non-dense range.

Fill in the blank: The point spectrum corresponds to the set of __________ of the operator.

[Insert box here] Caption: Box 5.1: Classification of spectrum

Point spectrum: eigenvalues

Continuous spectrum: injective but not surjective

Residual spectrum: injective but non-dense range

Spectrum Classification Table

Component	Injective?	Range Dense?	Inverse Bounded?	Physical/Intuitive Analogy
Point (σ_p)	**No**	No	No	Discrete energy levels (Bound states)
Continuous (σ_c)	Yes	**Yes**	**No**	Free particle energy (Scattering states)
Residual (σ_r)	Yes	**No**	No	Mathematical "mismatch" (Rare in Physics)
Resolvent Set (ρ)	Yes	Yes	**Yes**	Stable, well-posed system

5.1.3 Examples

Matrix case: For a finite-dimensional matrix A, the spectrum is the set of eigenvalues.

Shift operator on l2**:** Spectrum is the closed unit disk in the complex plane.

Differentiation operator on L2[0,1]**:** Spectrum is continuous, related to Fourier analysis.

Fill in the blank: The spectrum of the shift operator on l2is the closed __________ disk in the complex plane.

[Insert table here] Caption: Table 5.1: Examples of spectra of operators

Operator	**Space**	**Spectrum**
Matrix A	Rn	Eigenvalues
Shift operator	l2	Closed unit disk
Differentiation operator	L2[0,1]	Continuous spectrum

Pilot questions 5.1

The spectrum of an operator consists of values such that: A. T-λIis not invertible B. T-λIis always invertible C. T-λIis random D. T-λIis bounded

The point spectrum corresponds to: A. Eigenvalues of the operator B. Random values C. Continuous functions D. Non-dense ranges

The continuous spectrum occurs when: A. T-λIis injective but not surjective B. T-λIis surjective but not injective C. T-λIis random D. T-λIis bounded

The residual spectrum occurs when: A. T-λIis injective but has non-dense range B. T-λIis surjective but not injective C. T-λIis random D. T-λIis bounded

The spectrum of the shift operator on l2is: A. Closed unit disk B. Open unit disk C. Random set D. Continuous spectrum only

5.2 Spectral Properties Of Bounded Operators

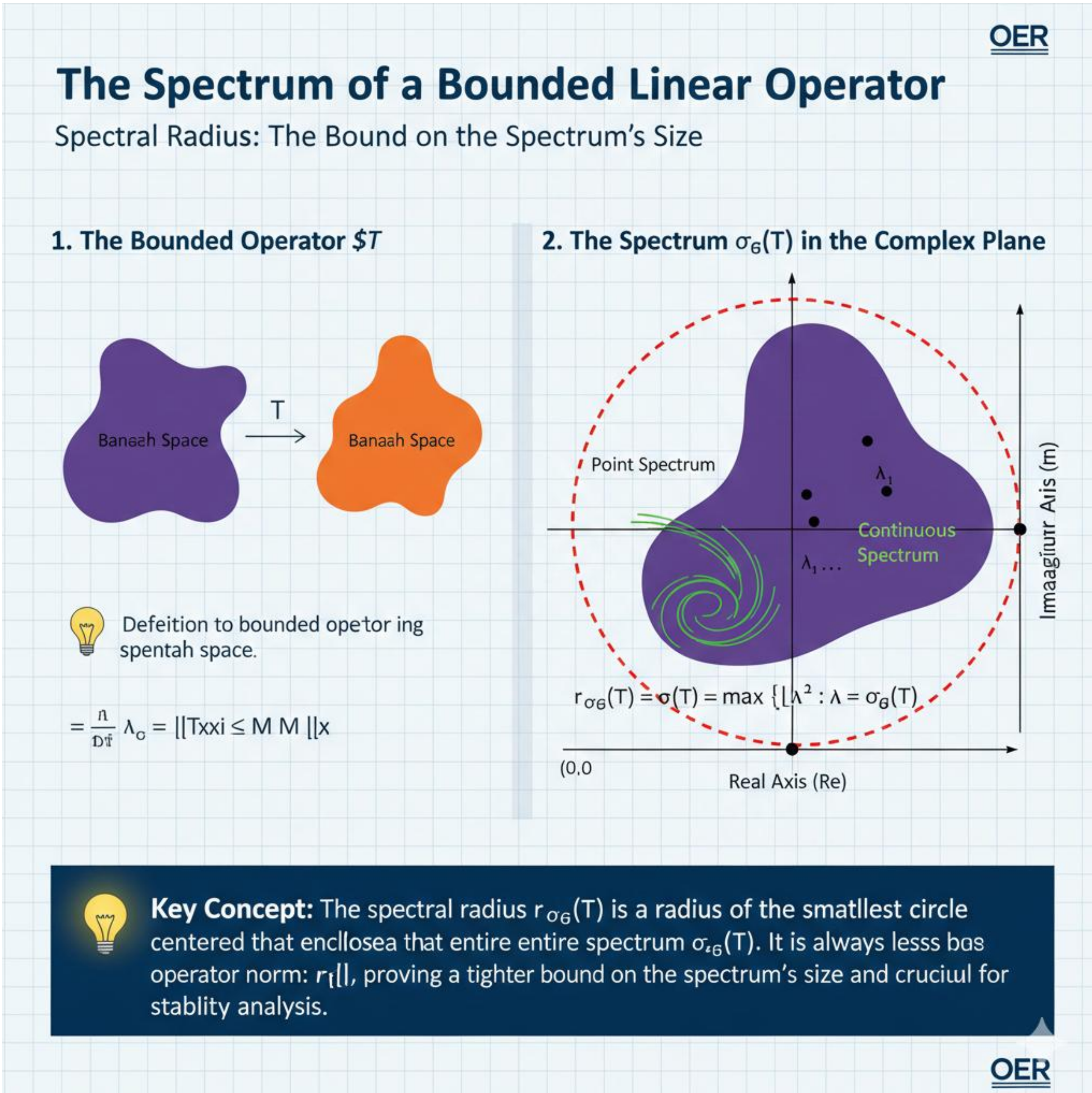
5.2.1 Spectral radius

For a bounded operator Ton a Banach space, the **spectral radius** r(T)is defined as:

$$r(T)=\sup\{||:\lambda\in\sigma(T)\}$$

It measures the “size” of the spectrum in the complex plane.

Fill in the blank: The spectral radius of an operator is the supremum of the __________ of elements in its spectrum.



Spectral radius of a bounded operator.

5.2.2 Spectral radius formula

The spectral radius can be computed using the **Gelfand formula**:

$$r(T)=\lim n\to Tn1/n$$

This connects the spectral radius to the operator norm.

Fill in the blank: The Gelfand formula states that $r(T)=\lim n\to Tn1/n$.

[Insert box here] Caption: Box 5.2: Spectral radius formula

$r(T)=\sup\{|\,|:\lambda\in\sigma(T)\}$

$r(T)=\lim n\to T n 1/n$

Comparison: Norm vs. Spectral Radius

Feature	Operator Norm $\|T$	Spectral Radius $r(T)$
Meaning	Maximum growth in a single step.	Long-term asymptotic growth rate.
Calculation	$\sup_{\|x\|=1} \|$	\lim_{n \to \infty} |T^n|^{1/
Topological View	Bounds the "box" the operator fits in.	Bounds the "spectrum" in $\mathbb{C}$.
Sensitivity	Changes if you change the norm on X.	Does **not** change (it is an intrinsic property).

5.2.3 Properties of spectrum for bounded operators

The spectrum $\sigma(T)$is always a non-empty compact subset of the complex plane.

The spectral radius is bounded above by the operator norm: $r(T)\leq T$.

For normal operators in Hilbert spaces, the spectral radius equals the operator norm.

Property	Description	Mathematical Formulation
Non-emptiness	For a complex Banach space, the spectrum is never an empty set.	$\sigma(T) \neq \emptyset$
Compactness	The spectrum is a **closed** and **bounded** subset of the complex plane.	$\sigma(T)$ is compact in $\mathbb{C}$
Boundedness	The magnitude of any element in the spectrum is bounded by the operator norm.	$\lambda \in \sigma(T) \implies
Spectral Radius	The radius of the smallest disk centered at the origin that contains $\sigma(T)$.	$r(T) = \sup {
Gelfand Formula	The spectral radius can be computed via the limit of the norms of the operator's powers.	r(T) = \lim_{n \to \infty} |T^n|^{1/
Spectral Mapping	The spectrum of a polynomial (or analytic function) of T is the image of the spectrum.	$\sigma(f(T)) = f(\sigma(T))$
Boundary Property	The boundary of the spectrum is always contained in the approximate point spectrum.	$\partial\sigma(T) \subseteq \sigma_{ap}(T)$

Fill in the blank: For normal operators in Hilbert spaces, the spectral radius equals the __________ norm.

Pilot questions 5.2

The spectral radius of an operator is: A. The supremum of the absolute values of elements in its spectrum B. The infimum of the absolute values of elements in its spectrum C. Always equal to zero D. Random

The Gelfand formula states that: A. r(T)=lim n→Tn1/nB. r(T)=Talways C. r(T)=0always D. r(T)=

The spectrum of a bounded operator is always: A. Non-empty and compact B. Empty and unbounded C. Random D. Undefined

The spectral radius is bounded above by: A. The operator norm B. Zero C. Infinity D. Random values

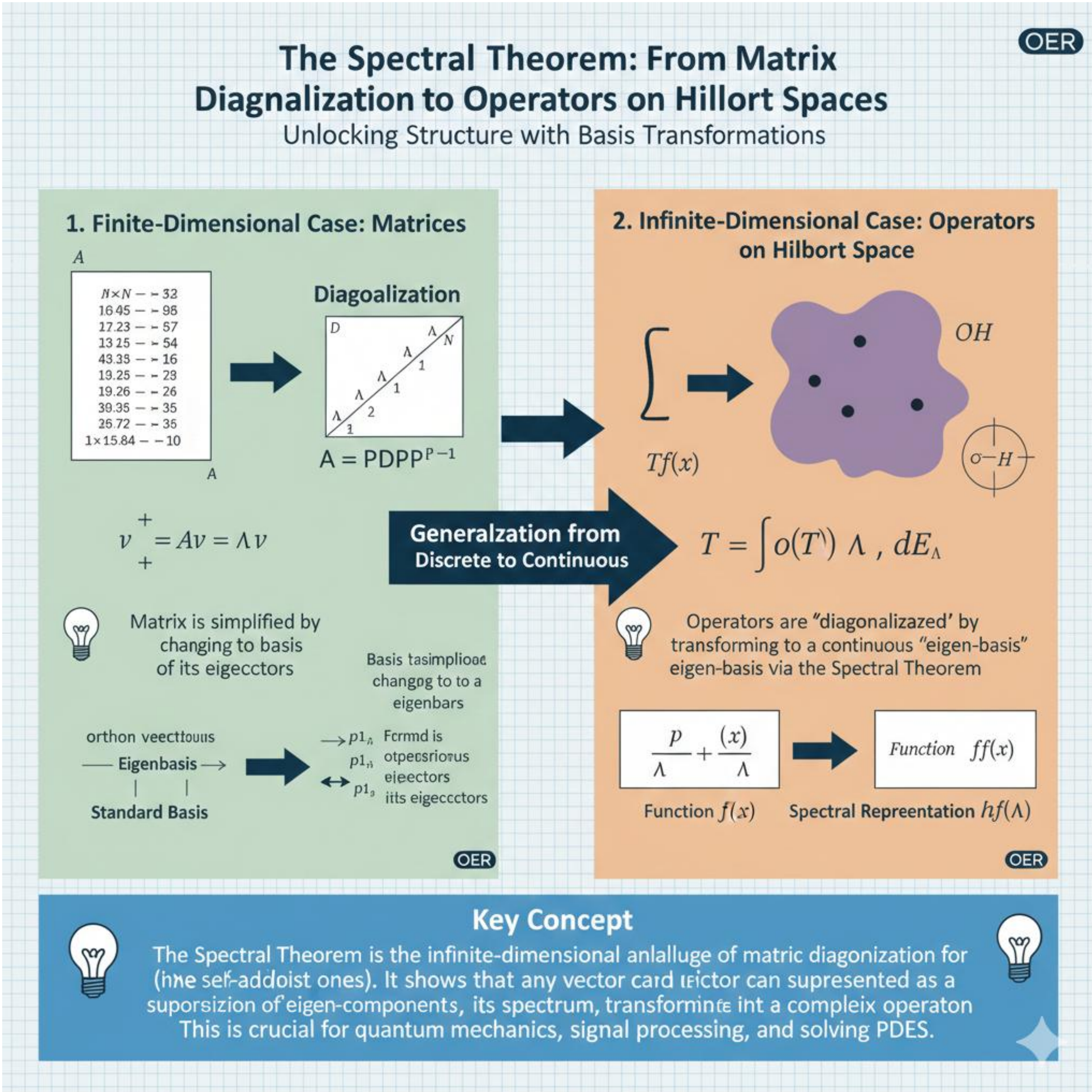
For normal operators in Hilbert spaces, the spectral radius equals: A. The operator norm B. Zero C. Infinity D. Random values

5.3 Spectral Theorem In Hilbert Spaces

5.3.1 Statement of the spectral theorem

The **spectral theorem** is a cornerstone of functional analysis. It states that for a bounded self-adjoint operator Ton a Hilbert space H, there exists a representation of Tin terms of its spectral measure. This generalises the diagonalisation of matrices to infinite-dimensional Hilbert spaces.

Fill in the blank: The spectral theorem generalises the __________ of matrices to infinite-dimensional Hilbert spaces.



Spectral theorem generalises diagonalisation to Hilbert spaces.

5.3.2 Spectral decomposition

The spectral theorem allows us to write a self-adjoint operator Tas:

$$T=\sigma(T)\ dE(\lambda)$$

where $E(\lambda)$is a projection-valued measure. This decomposition expresses Tin terms of its spectrum, similar to how a matrix is expressed in terms of its eigenvalues and eigenvectors.

Fill in the blank: The spectral decomposition expresses an operator as an integral with respect to its __________ measure.

[Insert box here] Caption: Box 5.3: Spectral decomposition

T=σ(T) dE(λ)

E(λ)is a projection-valued measure

5.3.3 Applications of the spectral theorem

Quantum mechanics: Observables are represented by self-adjoint operators, and the spectral theorem provides the mathematical foundation for measurement.

Fourier analysis: The spectral theorem underlies the decomposition of functions into orthogonal components.

Differential equations: It helps solve equations involving self-adjoint operators by decomposing them into spectral components.

Fill in the blank: In quantum mechanics, the spectral theorem provides the mathematical foundation for __________.

[Insert table here] Caption: Table 5.3: Applications of the spectral theorem

Field	**Application**	**Example**
Quantum mechanics	Foundation for measurement	Position and momentum operators
Fourier analysis	Decomposition into orthogonal components	Fourier transform
Differential equations	Solution via spectral decomposition	Sturm–Liouville problems

Pilot questions 5.3

The spectral theorem generalises: A. Diagonalisation of matrices B. Randomisation of matrices C. Integration of matrices D. Differentiation of matrices

The spectral decomposition expresses an operator as: A. An integral with respect to its projection-valued measure B. A sum of random variables C. A derivative of functions D. A polynomial expansion

In quantum mechanics, the spectral theorem provides the foundation for: A. Measurement B. Randomness C. Differentiation D. Integration

The spectral theorem applies to: A. Self-adjoint operators on Hilbert spaces B. Random operators on Banach spaces C. Non-linear operators only D. Continuous functions only

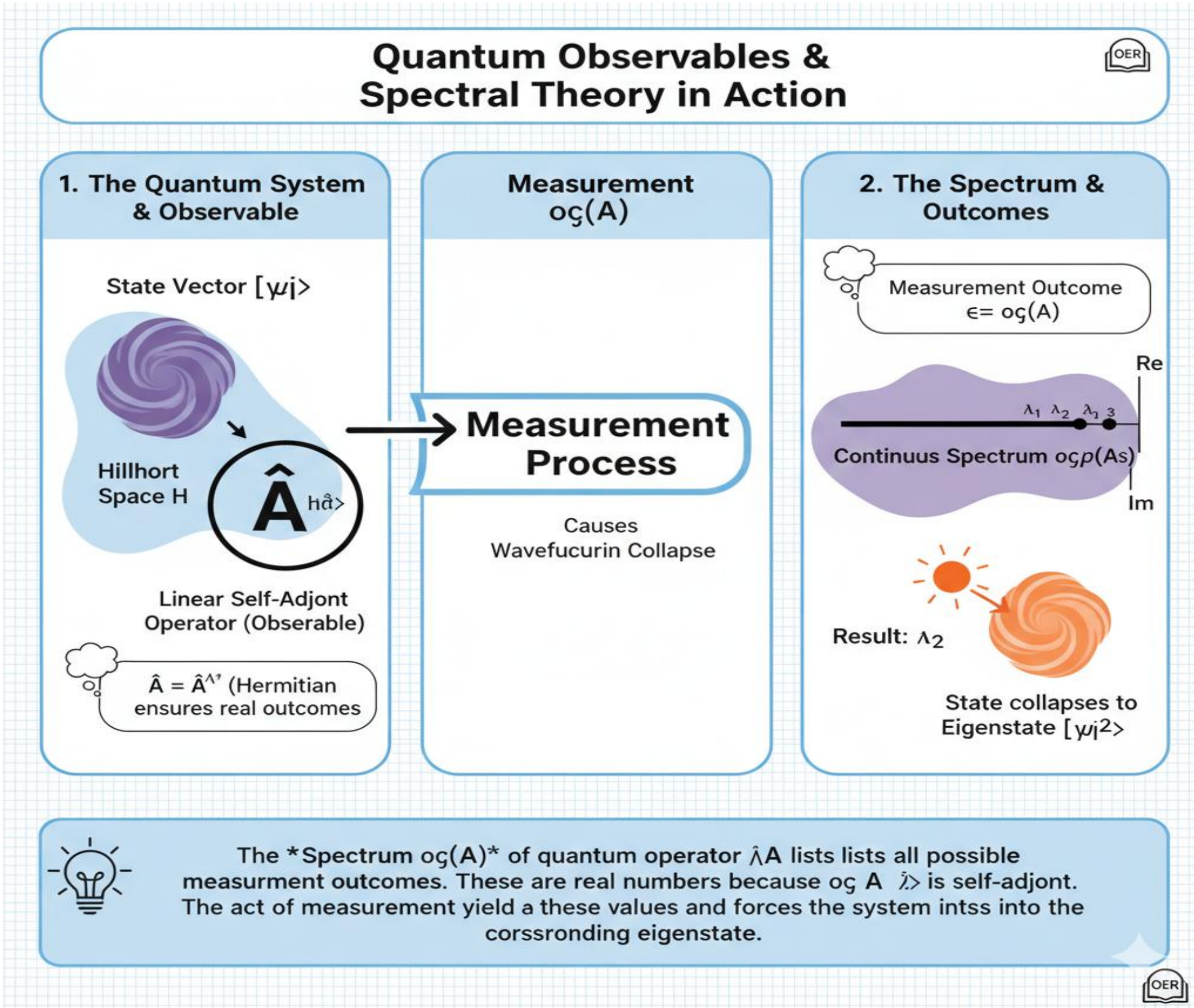
Which of the following is an application of the spectral theorem? A. Quantum mechanics B. Fourier analysis C. Differential equations D. All of the above

5.4 Applications Of Spectral Theory

5.4.1 Quantum mechanics

Spectral theory is fundamental in **quantum mechanics**. Physical observables such as energy, momentum, and position are represented by self-adjoint operators on Hilbert spaces. The spectral theorem ensures that these operators have real spectra, corresponding to measurable values in experiments.

Fill in the blank: In quantum mechanics, physical observables are represented by __________ operators on Hilbert spaces.



Spectral theory applied to quantum mechanics.

5.4.2 Differential equations

Spectral theory provides tools for solving **differential equations**, especially those involving self-adjoint operators. Sturm–Liouville problems, for example, can be solved using spectral decomposition, where solutions are expressed in terms of eigenfunctions and eigenvalues.

Fill in the blank: Sturm–Liouville problems can be solved using spectral __________.

[Insert box here] Caption: Box 5.4: Role of spectral theory in differential equations

Provides eigenfunction expansions

Solves Sturm–Liouville problems

Connects operator theory with PDEs

Key Applications

Problem Type	Role of Spectral Theory	Resulting Mathematical Tool
Vibration Analysis	Finding natural frequencies of structures.	Fourier Series / Modal Analysis
Quantum Mechanics	Determining energy levels of an atom.	Schrödinger Operator Spectrum
Signal Processing	Decomposing signals into frequencies.	Fourier and Wavelet Transforms
Heat Transfer	Predicting how quickly a material cools.	Eigenvalue decay ($e^{-\lambda t}$)

5.4.3 Numerical analysis

In **numerical analysis**, spectral theory underpins iterative methods for computing eigenvalues and eigenvectors. Algorithms such as the power method and Lanczos method rely on spectral properties to approximate solutions efficiently.

Fill in the blank: The power method in numerical analysis relies on spectral __________ to approximate eigenvalues.

[Insert table here] Caption: Table 5.4: Applications of spectral theory

Field	**Application**	**Example**
Quantum mechanics	Observables as operators	Energy, momentum operators
Differential equations	Eigenfunction expansions	Sturm–Liouville problems
Numerical analysis	Iterative eigenvalue methods	Power method, Lanczos method

Pilot questions 5.4

In quantum mechanics, physical observables are represented by: A. Self-adjoint operators B. Random variables C. Polynomials D. Scalars

Sturm–Liouville problems can be solved using: A. Spectral decomposition B. Randomisation C. Differentiation only D. Integration only

Spectral theory provides eigenfunction expansions for solving: A. Differential equations B. Random processes C. Algebraic identities D. Continuous functions only

The power method in numerical analysis relies on: A. Spectral properties B. Random values C. Continuous functions D. Non-dense ranges

Which of the following is an application of spectral theory? A. Quantum mechanics B. Differential equations C. Numerical analysis D. All of the above

Series summary

In this Series, we explored the foundations and applications of spectral theory. We began by defining the spectrum of a linear operator and classifying it into point, continuous, and residual spectra. We then studied spectral properties of bounded operators, including the spectral radius and its computation via the Gelfand formula. Next, we examined the spectral theorem in Hilbert spaces, which generalises matrix diagonalisation to infinite dimensions through spectral decomposition. Finally, we considered applications of spectral theory in quantum mechanics, differential equations, and numerical analysis, highlighting its central role in both theoretical and applied mathematics.

Glossary of terms

Spectrum ($\sigma(T)$): The set of complex numbers such that $T-\lambda I$is not invertible.

Point spectrum: The set of eigenvalues of an operator.

Continuous spectrum: Values where $T-\lambda I$is injective but not surjective, and inverse is not bounded.

Residual spectrum: Values where $T-\lambda I$is injective but has non-dense range.

Spectral radius ($r(T)$): The supremum of the absolute values of elements in the spectrum.

Gelfand formula: $r(T)=\lim_{n\to\infty}\|T^n\|^{1/n}$.

Spectral theorem: A result stating that self-adjoint operators on Hilbert spaces can be represented via spectral decomposition.

Spectral decomposition: Expression of an operator as an integral with respect to a projection-valued measure.

Concept check questions 5

(SLO 5.1) Define the spectrum of a linear operator.

(SLO 5.1) Name the three types of spectrum.

(SLO 5.2) State the formula for the spectral radius.

(SLO 5.2) What inequality relates the spectral radius and the operator norm?

(SLO 5.3) State the spectral theorem for self-adjoint operators in Hilbert spaces.

(SLO 5.3) What is spectral decomposition?

(SLO 5.4) Explain one application of spectral theory in quantum mechanics.

(SLO 5.4) Give one application of spectral theory in numerical analysis.

Answers to pilot questions 5

Pilot questions 5.1: 1A, 2A, 3A, 4A, 5A **Pilot questions 5.2:** 1A, 2A, 3A, 4A, 5A
Pilot questions 5.3: 1A, 2A, 3A, 4A, 5D **Pilot questions 5.4:** 1A, 2A, 3A, 4A, 5D