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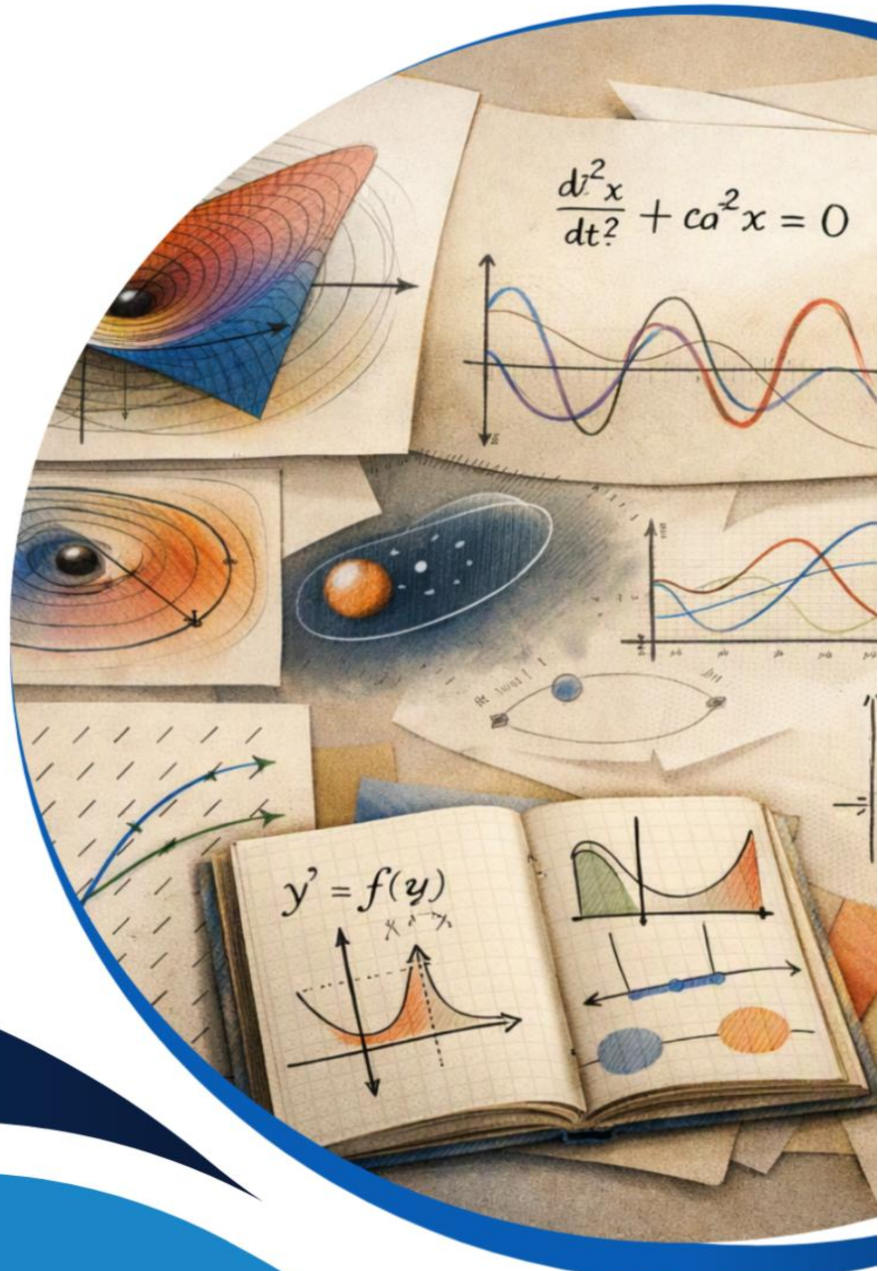
MTH401

THEORY OF ORDINARY DIFFERENTIAL EQUATIONS



Series 1: Existence, Uniqueness, and Dependence of Solutions in MTH 401: Theory of Ordinary Differential Equations

1.1 Existence Theorems

Definition and significance

Conditions for existence of solutions

1.2 Uniqueness Theorems

Lipschitz condition

Examples of uniqueness in initial value problems

1.3 Dependence of Solutions on Initial Data and Parameters

Continuous dependence on initial conditions

Sensitivity to parameter changes

1.4 Applications and Illustrative Examples

Worked examples of existence and uniqueness

Practical implications in modelling

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 1.1 define the existence theorems for ordinary differential equations and explain their significance,

SLO 1.2 state and apply uniqueness theorems, with emphasis on the Lipschitz condition,

SLO 1.3 analyse how solutions depend on initial data and parameters, demonstrating continuous dependence, and

SLO 1.4 evaluate illustrative examples that show the practical implications of existence and uniqueness results.

Introduction

In mathematics and applied sciences, ordinary differential equations often arise when modelling real-world phenomena such as population growth, chemical reactions, or mechanical vibrations. However, solving these equations is not only about finding a formula, it is also about knowing whether a solution exists at all and whether that solution is unique. Without such guarantees, mathematical models may become unreliable or ambiguous.

The aim of this Series is to equip you with the ability to define and explain existence and uniqueness theorems, and to analyse how solutions depend on initial data and parameters. By the end of this Series, you should be able to apply these concepts to ordinary differential equations and appreciate their importance in ensuring well-posed problems.

We shall commence this Series by exploring the existence theorems that provide conditions under which solutions to ordinary differential equations are guaranteed. Next, we will move on to uniqueness theorems, focusing on the role of the Lipschitz condition. Following that, we will examine how solutions depend on initial data and parameters, highlighting the stability and sensitivity of models. Finally, we will wrap up with applications and illustrative examples that demonstrate how these theorems are used in practice.

1.1 Existence Theorems

Ordinary differential equations often arise in modelling physical, biological, and economic systems. Before attempting to solve such equations, it is important to know whether a solution actually exists. The existence theorem provides conditions under which a solution to an initial value problem is guaranteed.

1.1.1 Definition and significance

The existence theorem states that if a function $f(x,y)$ is continuous in a region containing the point $(x0,y0)$, then there exists at least one solution to the initial value problem

$$y'(x)=f(x,y), y(x0)=y0$$

in some interval around $x0$.

This result is significant because it assures us that the mathematical model is meaningful. Without existence, the model would be unreliable, as no solution could be found.

Fill in the blank: The existence theorem requires that the function f(x,y)is __________ in a region containing the initial point.

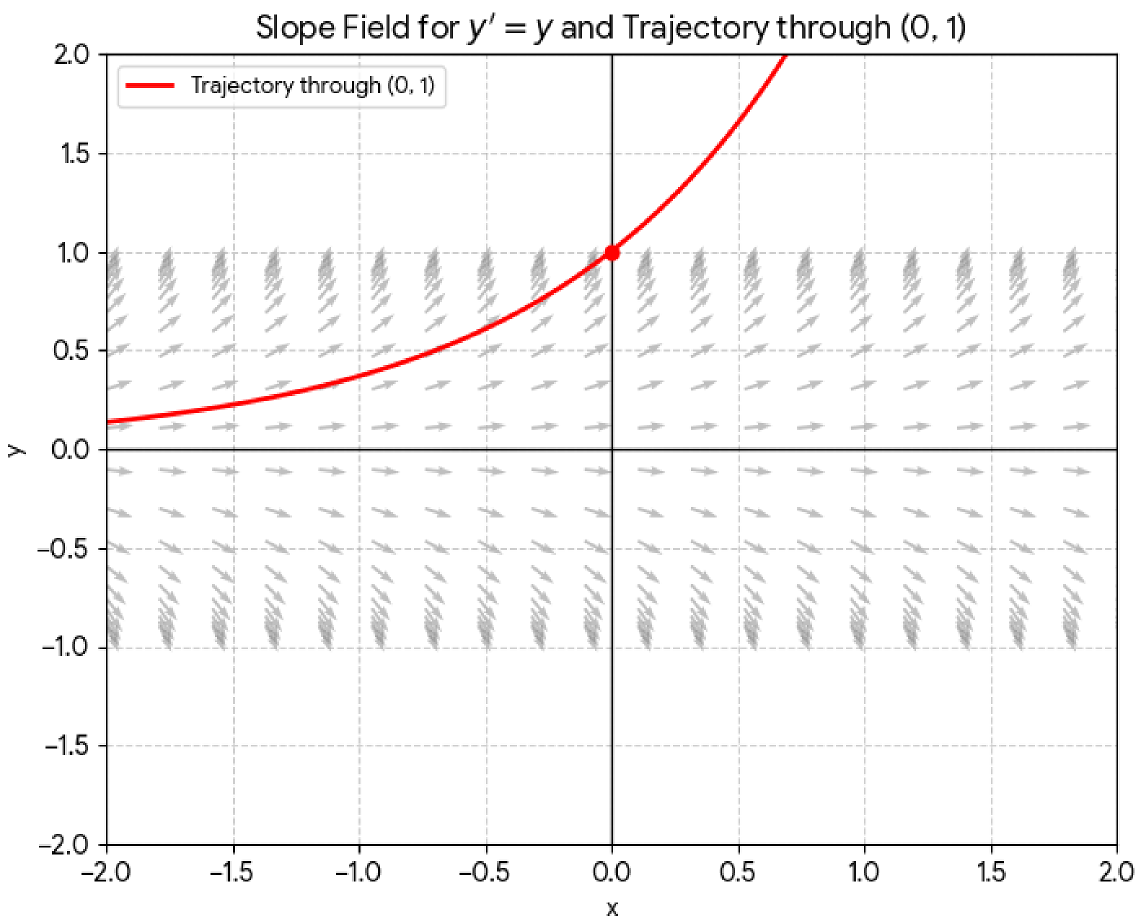


Illustration of existence of a solution through an initial point.

1.1.2 Conditions for existence of solutions

The key condition for existence is the continuity of the function f(x,y). If fis continuous in a rectangle around the initial point, then a solution exists.

For example, consider the equation

$$y'=x+y, y(0)=1$$

Here, f(x,y)=x+yis continuous everywhere, so a solution exists.

Fill in the blank: A solution to an initial value problem exists if the function f(x,y)is __________ in a neighbourhood of the initial point.

Key conditions for existence of solutions

Existence and Uniqueness of Solutions to ODEs

For a first-order ordinary differential equation (ODE) given by $\frac{dy}{dx} = f(x, y)$ with an initial condition $y(x_0) = y_0$, we need to know if a solution even exists and if it is the *only* one. This is determined by the **Existence and Uniqueness Theorem** (often called the Picard-Lindelöf Theorem).

The Necessary Conditions

To ensure a solution exists and is unique within a region R surrounding the initial point (x_0, y_0), the following conditions must be met:

Condition	Requirement	What it Guarantees
Existence	$f(x, y)$ must be **continuous** in the region R.	There is at least one function $y(x)$ that satisfies the ODE through the point.
Uniqueness	The partial derivative $\frac{\partial f}{\partial y}$ must also be **continuous** in R.	There is **exactly one** solution curve. Curves cannot branch or cross at that point.

f(x,y)must be continuous in a region around (x0,y0).

The initial value problem must be properly specified.

The solution exists in some interval around x_0. AI prompt: “Create a summary box listing conditions for existence of solutions to ODEs.” Search string: “existence theorem conditions ODE summary OER”

Pilot Questions 1.1

The existence theorem requires that the function $f(x,y)$ is: A. Differentiable B. Continuous C. Monotone D. Lipschitz

A solution to an initial value problem exists if $f(x,y)$ is continuous in a: A. Line segment B. Neighbourhood of the initial point C. Entire domain D. Single point

Which of the following equations satisfies the condition for existence? A. $y'=1x$, $y(0)=1$ B. $y'=x+y$, $y(0)=1$ C. $y'=\tan(y)$, $y(0)=\pi/2$ D. $y'=x$, $y(0)=-1$

The existence theorem guarantees: A. That a solution exists and is unique B. That at least one solution exists C. That solutions are bounded D. That solutions are periodic

The condition of continuity in the existence theorem applies to: A. The initial condition only B. The function $f(x,y)$ C. The derivative of $f(x,y)$ D. The solution itself

1.2 Uniqueness Theorems

While the existence theorem assures us that a solution to an ordinary differential equation exists, it does not guarantee that the solution is unique. The uniqueness theorem provides conditions under which the solution is the only one that passes through the given initial point. This is essential in ensuring that mathematical models are well-posed and reliable.

1.2.1 The Lipschitz condition

The uniqueness theorem states that if the function $f(x,y)$ is continuous in a region containing (x_0,y_0) and satisfies the Lipschitz condition in y, then the solution to the initial value problem is unique.

The Lipschitz condition means that there exists a constant $L>0$ such that for all y_1,y_2 in the region,

$$|f(x,y_1)-f(x,y_2)|\leq L|y_1-y_2|$$

This condition restricts how rapidly f can change with respect to y.

Fill in the blank: The uniqueness theorem requires that the function f(x,y)satisfies the __________ condition in y.

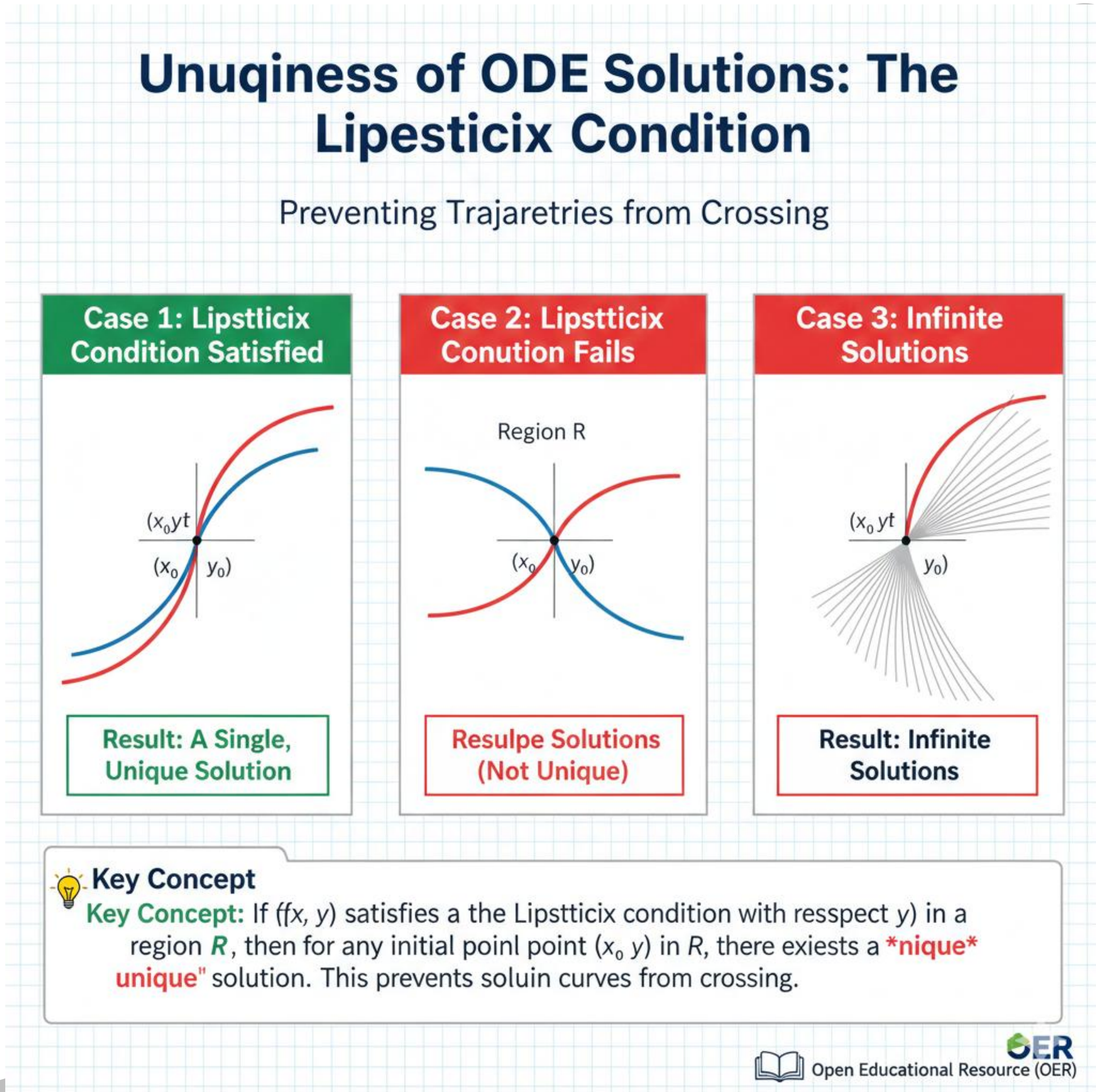


Illustration of uniqueness of solution under the Lipschitz condition.

1.2.2 Examples of uniqueness in initial value problems

Consider the equation

$$y'=y,y(0)=1$$

Here, $f(x,y)=y$. The function is continuous and satisfies the Lipschitz condition with $L=1$. Therefore, the solution is unique:

$$y(x)=ex$$

Now consider

$$y'=|y|,y(0)=0$$

Here, $f(y)=|y|$is continuous but does not satisfy the Lipschitz condition near $y=0$. As a result, multiple solutions exist, such as $y(x)=0$and $y(x)=x24$.

Fill in the blank: The equation $y'=|y|,y(0)=0$admits __________ solutions because the Lipschitz condition fails.

Key points about uniqueness of solutions

Continuity alone does not guarantee uniqueness.

Lipschitz condition in yensures uniqueness.

Failure of Lipschitz condition may lead to multiple solutions.

Uniqueness of ODE Solutions: The Lipschitz Condition

While the **Existence Theorem** tells us that a path through a point (x_0, y_0) exists, the **Uniqueness Theorem** ensures that there is *only one* such path. The primary condition used to guarantee this is the **Lipschitz Condition.**

Core Requirements for Uniqueness

For the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$:

Condition	Mathematical Definition	Physical Interpretation
Continuity of f	$f(x, y)$ must be continuous in a region R.	Ensures at least one solution (Existence).
Lipschitz Continuity	$	f(x, y_1) - f(x, y_2)
Sufficient Condition	$\frac{\partial f}{\partial y}$ is continuous in the region R.	A continuous partial derivative implies the Lipschitz condition is met.

Pilot Questions 1.2

The uniqueness theorem requires that the function f(x,y)satisfies the: A. Continuity condition B. Lipschitz condition C. Differentiability condition D. Monotonicity condition

The Lipschitz condition restricts how rapidly f(x,y)can change with respect to: A. xB. yC. both xand yD. neither xnor y

Which of the following equations has a unique solution? A. y'=y,y(0)=1B. y'=|y|,y(0)=0C. y'=tan(y),y(0)=π/2D. y'=1x,y(0)=0

The Lipschitz condition requires the existence of a constant: A. MB. LC. KD. C

If the Lipschitz condition fails, the initial value problem may have: A. No solution B. Exactly one solution C. Multiple solutions D. A periodic solution

1.3 Dependence Of Solutions On Initial Data And Parameters

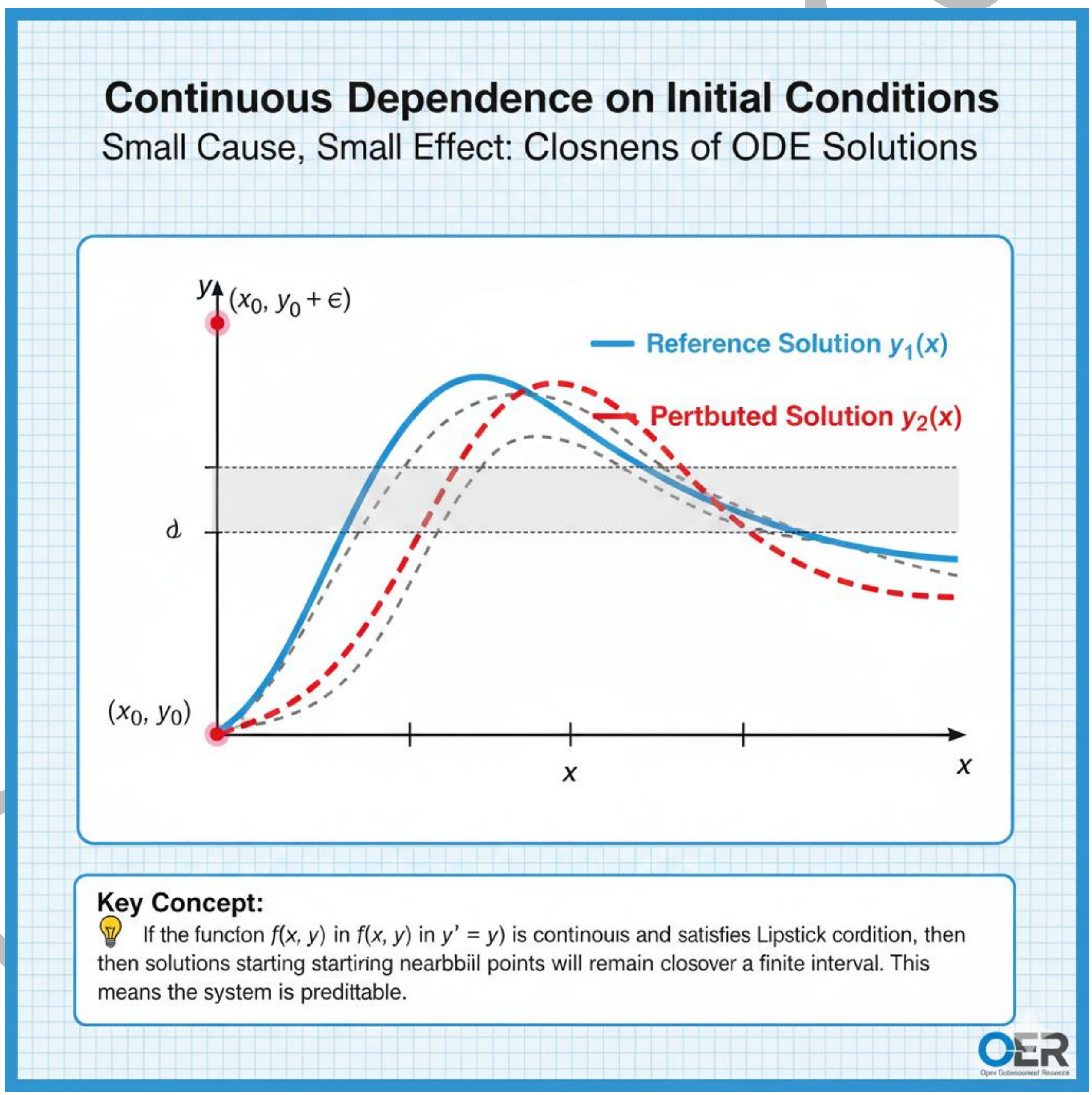
When solving ordinary differential equations, it is not enough to know that solutions exist and are unique. We must also understand how these solutions behave when the initial conditions or parameters of the system change. This property is known as continuous dependence, and it ensures that small changes in input lead to small changes in output.

1.3.1 Continuous dependence on initial conditions

The theorem of continuous dependence states that if f(x,y)is continuous and satisfies the Lipschitz condition in y, then the solution of the initial value problem depends continuously on the initial data.

This means that if we slightly change the initial condition y(x0)=y0, the resulting solution curve will remain close to the original solution curve.

Fill in the blank: Continuous dependence ensures that small changes in the initial condition produce __________ changes in the solution.



Continuous dependence of solutions on initial conditions.

1.3.2 Sensitivity to parameter changes

In many models, parameters such as growth rates, decay constants, or external forces influence the behaviour of solutions. Continuous dependence also applies to parameters: small changes in parameters lead to small changes in solutions.

For example, consider the logistic equation:

$$y'=ry(1-y), y(0)=y_0$$

Here, the parameter r controls the growth rate. If r changes slightly, the solution curve changes slightly as well, but the overall qualitative behaviour remains similar.

Fill in the blank: In the logistic equation, the parameter that controls the growth rate is denoted by __________.

Key points about dependence of solutions

Solutions depend continuously on initial conditions.

Solutions depend continuously on parameters.

Small changes in inputs produce small changes in outputs.

Continuous Dependence in ODEs

The concept of **continuous dependence** addresses a vital practical question: if our measurements of a system's starting state or its physical constants are slightly off, will our prediction be "close enough," or will it be completely wrong?

Core Principles of Stability

For an ODE $\frac{dy}{dt} = f(t, y, \alpha)$ with initial condition $y(t_0) = y_0$:

Type of Dependence	Description	Practical Implication
On Initial Conditions	How the solution $y(t)$ changes if the starting value y_0 is shifted by a small amount ϵ.	Small errors in initial sensor readings lead to only small errors in future state predictions.
On Parameters	How the solution changes if a constant α (like gravity or friction) is slightly varied.	Engineering designs are "robust"; a 1% change in material stiffness won't cause a 100% change in behavior.

Pilot Questions 1.3

Continuous dependence ensures that small changes in initial conditions produce: A. Large changes in solutions B. Small changes in solutions C. No changes in solutions D. Oscillatory changes in solutions

The theorem of continuous dependence requires that f(x,y)satisfies the: A. Monotonicity condition B. Lipschitz condition C. Periodicity condition D. Differentiability condition

In the logistic equation y'=ry(1-y), the parameter rcontrols: A. Initial condition B. Growth rate C. Stability D. Oscillation

Continuous dependence applies to: A. Initial conditions only B. Parameters only C. Both initial conditions and parameters D. Neither initial conditions nor parameters

If continuous dependence fails, small changes in inputs may lead to: A. Predictable solutions B. Unstable or unpredictable solutions C. Unique solutions D. No solutions

1.4 Applications And Illustrative Examples

Theorems of existence, uniqueness, and dependence are not abstract results alone, they have practical implications in modelling real-world systems. By applying these theorems, we can ensure that the mathematical models we construct are reliable, predictable, and meaningful.

1.4.1 Worked examples of existence and uniqueness

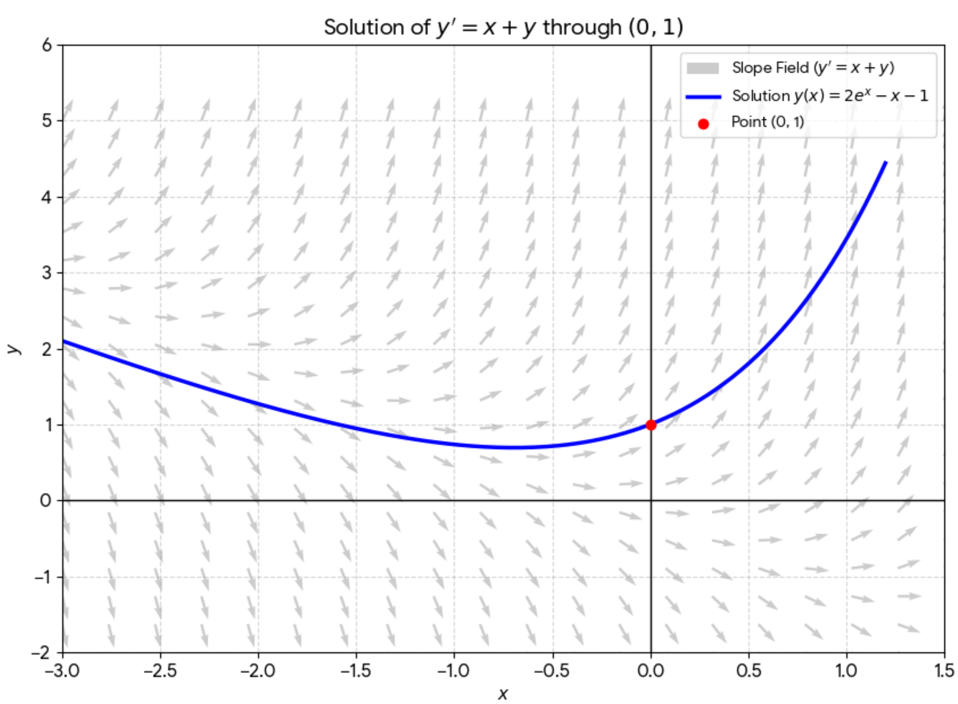
Consider the initial value problem:

$$y'=x+y,y(0)=1$$

Here, $f(x,y)=x+y$is continuous everywhere and satisfies the Lipschitz condition in y. Therefore, by the existence and uniqueness theorems, there is a unique solution. Solving gives:

$$y(x)=2ex-x-1$$

Fill in the blank: The function $f(x,y)=x+y$satisfies the __________ condition in y, ensuring uniqueness.



Unique solution curve for the initial value problem.

1.4.2 Practical implications in modelling

In population dynamics, consider the logistic equation:

$$y'=ry(1-y),y(0)=y0$$

The existence theorem guarantees that a solution exists for all y0. The uniqueness theorem ensures that the population trajectory is uniquely determined by the initial population size and growth rate. Continuous dependence shows that small changes in y0or rlead to small changes in the solution, making the model stable and predictable.

Fill in the blank: In the logistic equation, the uniqueness theorem ensures that the population trajectory is uniquely determined by the initial __________ size and growth rate.

[Insert Box here] Caption: Box 1.4: Applications of existence and uniqueness theorems

Guarantee reliability of population models.

Ensure predictability in chemical reaction rates.

Provide stability in mechanical vibration models.

Applications: Existence & Uniqueness in Mathematical Modelling

In mathematical modelling, the **Existence and Uniqueness Theorem** is not just a theoretical formality; it is the "Safety Check" that ensures a model is scientifically valid and computationally solvable. If a model lacks these properties, it is considered **ill-posed.**

Critical Roles in Applied Science

Field	Modeling Application	Why the Theorem is Essential
Physics	**Classical Mechanics**	Guarantees **Determinism**. For a given set of forces and initial positions ($F = ma$), there must be exactly one future trajectory for a planet or projectile.
Epidemiology	**SIR Models**	Ensures that for any initial infected population, the model predicts a single, definitive outcome for the spread of a disease.
Control Theory	**Autonomous Systems**	Engineers must know that a drone's flight controller will produce a unique response to a sensor input. Non-uniqueness would cause the system to "glitch" or crash.
Climate Science	**Weather Prediction**	While systems can be chaotic, the underlying ODEs must have unique solutions over short terms to allow for any degree of reliable forecasting.

Pilot Questions 1.4

For the equation y'=x+y,y(0)=1, the solution is: A. y(x)=exB. y(x)=2ex-x-1C. y(x)=x2+1D. y(x)=sin(x)

The logistic equation models: A. Population growth B. Chemical reactions C. Mechanical vibrations D. Heat conduction

Existence and uniqueness theorems ensure that mathematical models are: A. Ambiguous B. Reliable C. Unstable D. Undefined

Continuous dependence in modelling ensures: A. Large changes in outputs for small changes in inputs B. Small changes in outputs for small changes in inputs C. No changes in outputs D. Oscillatory changes in outputs

The uniqueness theorem ensures that the population trajectory in the logistic equation is determined by: A. Growth rate only B. Initial size only C. Both initial size and growth rate D. Neither

Series Summary

In this Series, we examined the foundational theorems that guarantee the reliability of ordinary differential equations. We began with the existence theorem, which assures us that a solution exists under continuity conditions. We then studied the uniqueness theorem, emphasising the importance of the Lipschitz condition in ensuring that only one solution passes through a given initial point. Following that, we explored the dependence of solutions on initial data and parameters, highlighting the principle of continuous dependence and its role in stability. Finally, we applied these theorems to practical examples, demonstrating their importance in modelling population dynamics, chemical reactions, and mechanical systems. Together, these results form the backbone of ODE theory, ensuring that mathematical models are both meaningful and predictable.

Glossary of Terms

Existence theorem: A result that guarantees at least one solution to an initial value problem under continuity conditions.

Uniqueness theorem: A result that ensures the solution to an initial value problem is the only one passing through the initial point, provided the Lipschitz condition holds.

Lipschitz condition: A restriction on how rapidly a function can change with respect to a variable, ensuring uniqueness of solutions.

Continuous dependence: The property that small changes in initial conditions or parameters lead to small changes in solutions.

Initial value problem (IVP): An ordinary differential equation together with specified values at a starting point.

Concept Check Questions 1

Upon completing this Series, you should be able to:

Linked to SLO 1.1

Define the existence theorem for ordinary differential equations and explain its significance.

Give an example of an ODE that satisfies the conditions for existence.

Linked to SLO 1.2

State the uniqueness theorem and identify the condition required for uniqueness.

Provide an example of an ODE that fails the Lipschitz condition and explain the consequence.

Linked to SLO 1.3

Analyse how solutions depend on initial conditions using the concept of continuous dependence.

Explain how parameter changes affect solutions in the logistic equation.

Linked to SLO 1.4

Evaluate the practical implications of existence and uniqueness theorems in population modelling.

Apply the theorems to demonstrate why the logistic equation provides a stable model.

Answers to Pilot Questions 1

Pilot Questions 1.1 (Existence Theorems)

B. Continuous

B. Neighbourhood of the initial point

B. $y'=x+y, y(0)=1$

B. That at least one solution exists

B. The function $f(x,y)$

Pilot Questions 1.2 (Uniqueness Theorems)

B. Lipschitz condition

B. y

A. $y'=y, y(0)=1$

B. L

C. Multiple solutions

Pilot Questions 1.3 (Dependence of Solutions)

B. Small changes in solutions

B. Lipschitz condition

B. Growth rate

C. Both initial conditions and parameters

B. Unstable or unpredictable solutions

Pilot Questions 1.4 (Applications and Illustrative Examples)

B. y(x)=2ex-x-1

A. Population growth

B. Reliable

B. Small changes in outputs for small changes in inputs

C. Both initial size and growth rate

Series 2: Properties of Solutions and Sturm–Sonin–Polya Theorems

2.1 Properties of Solutions

Continuity and differentiability of solutions

Behaviour under varying conditions

2.2 Sturm Comparison Theorem

Statement of the theorem

Applications in oscillation theory

2.3 Sonin–Polya Theorem

Statement of the theorem

Implications for bounds and oscillatory behaviour

2.4 Illustrative Examples and Applications

Worked examples of Sturm and Sonin–Polya theorems

Practical modelling contexts

Introduction

Ordinary differential equations often describe dynamic systems whose solutions exhibit important properties such as continuity, differentiability, and oscillatory behaviour. Understanding these properties allows us to predict how solutions behave under different conditions and to compare the behaviour of different systems.

The aim of this Series is to explain the properties of solutions of ordinary differential equations and to introduce two important results: the Sturm comparison theorem and the Sonin–Polya theorem. These results provide powerful tools for analysing oscillations and bounding solutions, which are essential in both theoretical and applied contexts.

We shall commence this Series by examining the general properties of solutions, focusing on continuity and differentiability. Next, we will study the Sturm comparison theorem, which allows us to compare the zeros of solutions of different equations. Following that, we will explore the Sonin–Polya theorem, which provides bounds and insights into oscillatory behaviour. Finally, we will conclude with illustrative examples and applications that demonstrate how these theorems are used in practice.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 2.1 describe the fundamental properties of solutions of ordinary differential equations, including continuity and differentiability,

SLO 2.2 state the Sturm comparison theorem and explain its application in analysing oscillatory behaviour,

SLO 2.3 define the Sonin–Polya theorem and analyse its implications for bounding and oscillatory properties of solutions, and

SLO 2.4 apply Sturm and Sonin–Polya theorems to illustrative examples and evaluate their practical significance in modelling contexts.

2.1 Properties Of Solutions

Solutions of ordinary differential equations possess important properties that determine their behaviour and usefulness in modelling. These properties include continuity, differentiability, and predictable behaviour under varying conditions. Understanding these features helps us to trust the solutions and apply them confidently in practical contexts.

2.1.1 Continuity and differentiability of solutions

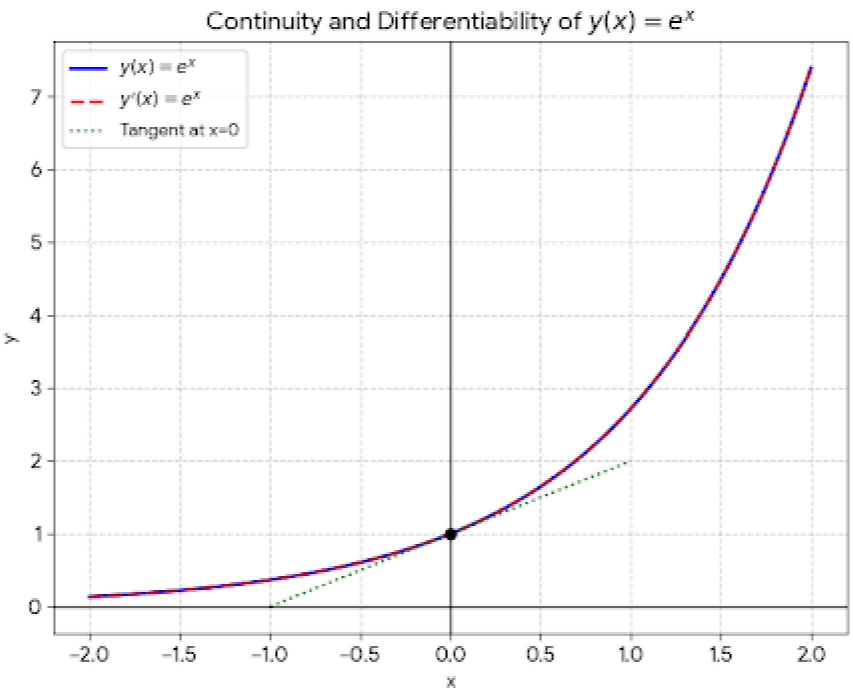
A solution to an ordinary differential equation is typically a function that is both continuous and differentiable.

Continuous function: A function with no breaks or jumps in its graph.

Differentiable function: A function that has a well-defined derivative at every point in its domain.

For example, the solution to $y'=y, y(0)=1$is $y(x)=ex$. This function is continuous and differentiable everywhere.

Fill in the blank: A solution to an ordinary differential equation is expected to be both __________ and differentiable.



Example of a continuous and differentiable solution.

Solutions also exhibit predictable behaviour when initial conditions or parameters change. This is linked to the principle of continuous dependence studied earlier.

Small changes in initial conditions produce small changes in the solution curve.

Small changes in parameters lead to small changes in the trajectory of the solution.

For example, in the logistic equation $y'=ry(1-y)$, changing the parameter rslightly alters the growth rate but does not fundamentally change the qualitative behaviour of the solution.

Fill in the blank: In the logistic equation, the parameter that influences the growth rate of the solution is __________.

Key properties of solutions

Solutions are continuous and differentiable.

Solutions depend continuously on initial conditions.

Solutions depend continuously on parameters.

Key Properties of ODE Solutions

When we solve an ordinary differential equation (ODE), we aren't just looking for a random formula; we are looking for a function that satisfies specific mathematical "rigor" tests. These properties ensure that the solution is physically meaningful and mathematically sound.

Core Mathematical Properties

Property	Requirement	Significance
Continuity	The solution $y(x)$ must be continuous over the interval of existence.	Ensures there are no instantaneous "teleportations" or gaps in the system's state.
Differentiability	$y(x)$ must be differentiable such that y' satisfies the ODE.	By definition, an ODE involves a rate of change. If y' doesn't exist, the equation itself becomes meaningless.
Existence	A solution must exist for the given initial condition (x_0, y_0).	Confirms that the mathematical model actually describes a possible physical reality.
Uniqueness	Only one solution curve passes through a specific initial point.	Guarantees **determinism**: given the same starting conditions, the system will always behave the same way.

Pilot Questions 2.1

A solution to an ordinary differential equation is expected to be: A. Continuous and differentiable B. Discontinuous and differentiable C. Continuous but not differentiable D. Neither continuous nor differentiable

The solution y(x)=exis: A. Continuous only B. Differentiable only C. Both continuous and differentiable D. Neither continuous nor differentiable

In the logistic equation y'=ry(1-y), the parameter rcontrols: A. Initial condition B. Growth rate C. Stability D. Oscillation

Continuous dependence ensures that small changes in initial conditions produce: A. Large changes in solutions B. Small changes in solutions C. No changes in solutions D. Oscillatory changes in solutions

Which of the following is a key property of solutions of ODEs? A. They are always periodic B. They are continuous and differentiable C. They are always bounded D. They are always linear

2.2 Sturm Comparison Theorem

The Sturm comparison theorem is a powerful result in the theory of ordinary differential equations. It allows us to compare the zeros of solutions of two second-order linear differential equations. This theorem is particularly useful in oscillation theory, where the spacing of zeros indicates the oscillatory nature of solutions.

2.2.1 Statement of the theorem

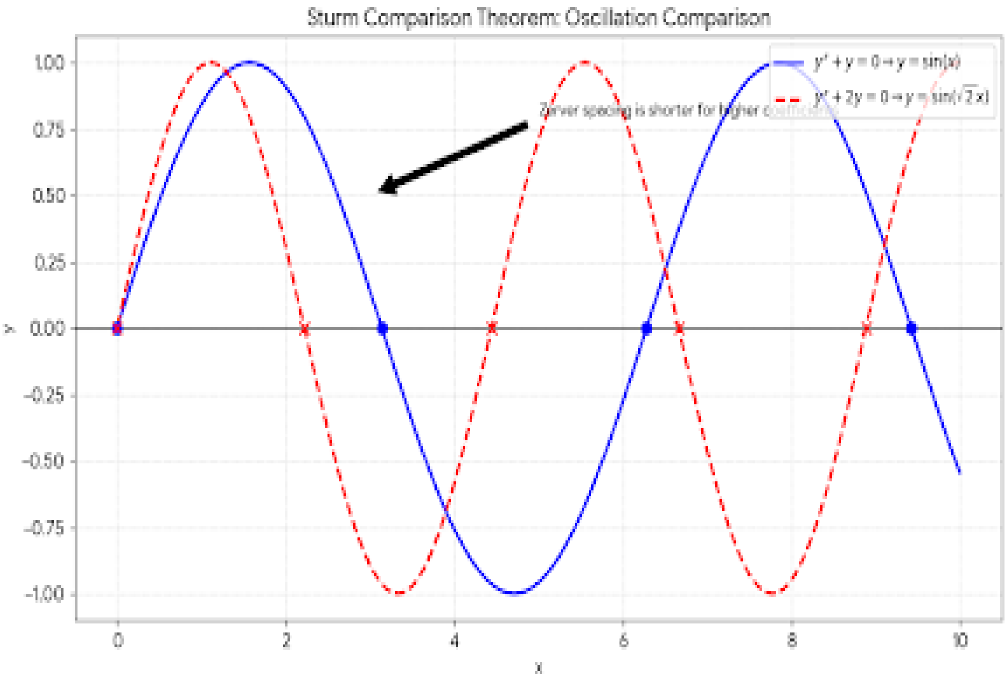
The Sturm comparison theorem states that if we have two second-order linear differential equations of the form:

$$y''+p(x)y=0,z''+q(x)z=0$$

with continuous functions p(x)and q(x), and if p(x)≤q(x)for all xin an interval, then between any two consecutive zeros of z(x), there exists at least one zero of y(x).

This result provides a way to compare the oscillatory behaviour of two systems.

Fill in the blank: The Sturm comparison theorem compares the __________ of solutions of two second-order linear differential equations.



2.2.2 Applications in oscillation theory

Oscillation theory studies the behaviour of solutions that cross the axis repeatedly. The Sturm comparison theorem helps us determine how frequently solutions oscillate by comparing them with solutions of simpler equations.

For example, the equation y''+y=0has solutions sin(x)and cos(x), which oscillate with period 2π. The equation y''+2y=0has solutions that oscillate more rapidly, with period 2. By applying the Sturm comparison theorem, we can conclude that the zeros of the faster oscillating solution are interlaced with those of the slower one.

Fill in the blank: The Sturm comparison theorem is particularly useful in __________ theory, where solutions cross the axis repeatedly.

[Insert Box here] Caption: Box 2.2: Key points about Sturm comparison theorem

- ***Compares zeros of solutions of two second-order linear ODEs.***
- ***Provides insights into oscillatory behaviour.***
- ***Useful in physics and engineering applications involving vibrations and waves.***

Key Conditions and Results

Requirement	Condition	The Result (The "Comparison")
Coefficient Comparison	$q_2(x) \geq q_1(x)$ for all x.	The solution to Equation 2 oscillates **faster** than Equation 1.
The Zero Property	Let x_1 and x_2 be consecutive zeros of $y(x)$.	There is at least one zero of $v(x)$ in the interval (x_1, x_2).
Strict Comparison	If $q_2(x) > q_1(x)$ at any point.	The distance between the zeros of $v(x)$ is strictly smaller than the distance between the zeros of $y(x)$.

Summary: The Sturm Comparison Theorem

The **Sturm Comparison Theorem** provides a way to describe the qualitative behavior (specifically the spacing of roots) of second-order linear ODEs without solving them. It is the mathematical foundation for understanding why "stiffer" or "tighter" systems vibrate at higher frequencies.

The Comparison Framework

Given two second-order homogeneous linear equations on an interval $[a, b]$:

1. **Equation 1:** $y'' + q_1(x)y = 0$
2. **Equation 2:** $v'' + q_2(x)v = 0$

The Sturm comparison theorem compares: A. Derivatives of solutions B. Zeros of solutions C. Integrals of solutions D. Limits of solutions

If p(x)≤q(x), then between any two consecutive zeros of z(x), there exists: A. No zero of y(x)B. At least one zero of y(x)C. Exactly two zeros of y(x)D. A maximum of y(x)

The equation y''+y=0has solutions that oscillate with period: A. B. 2πC. 2D. /2

The Sturm comparison theorem is particularly useful in: A. Stability theory B. Oscillation theory C. Existence theory D. Reduction theory

The solutions of $y''+2y=0$oscillate: A. More slowly than those of $y''+y=0$B. At the same rate as those of $y''+y=0$C. More rapidly than those of $y''+y=0$D. Without oscillation

2.3 Sonin–Polya Theorem

The Sonin–Polya theorem is another important result in the study of ordinary differential equations. It provides bounds and insights into the oscillatory behaviour of solutions, complementing the Sturm comparison theorem. This theorem is particularly useful in understanding how solutions behave when they oscillate and how their amplitudes are controlled.

2.3.1 Statement of the theorem

The Sonin–Polya theorem states that for certain second-order linear differential equations, the amplitude of oscillations of solutions can be bounded, and the spacing of zeros can be estimated. This result helps us to understand not only that solutions oscillate, but also how their oscillations are distributed and limited.

Fill in the blank: The Sonin–Polya theorem provides __________ on the oscillatory behaviour of solutions of certain ODEs.

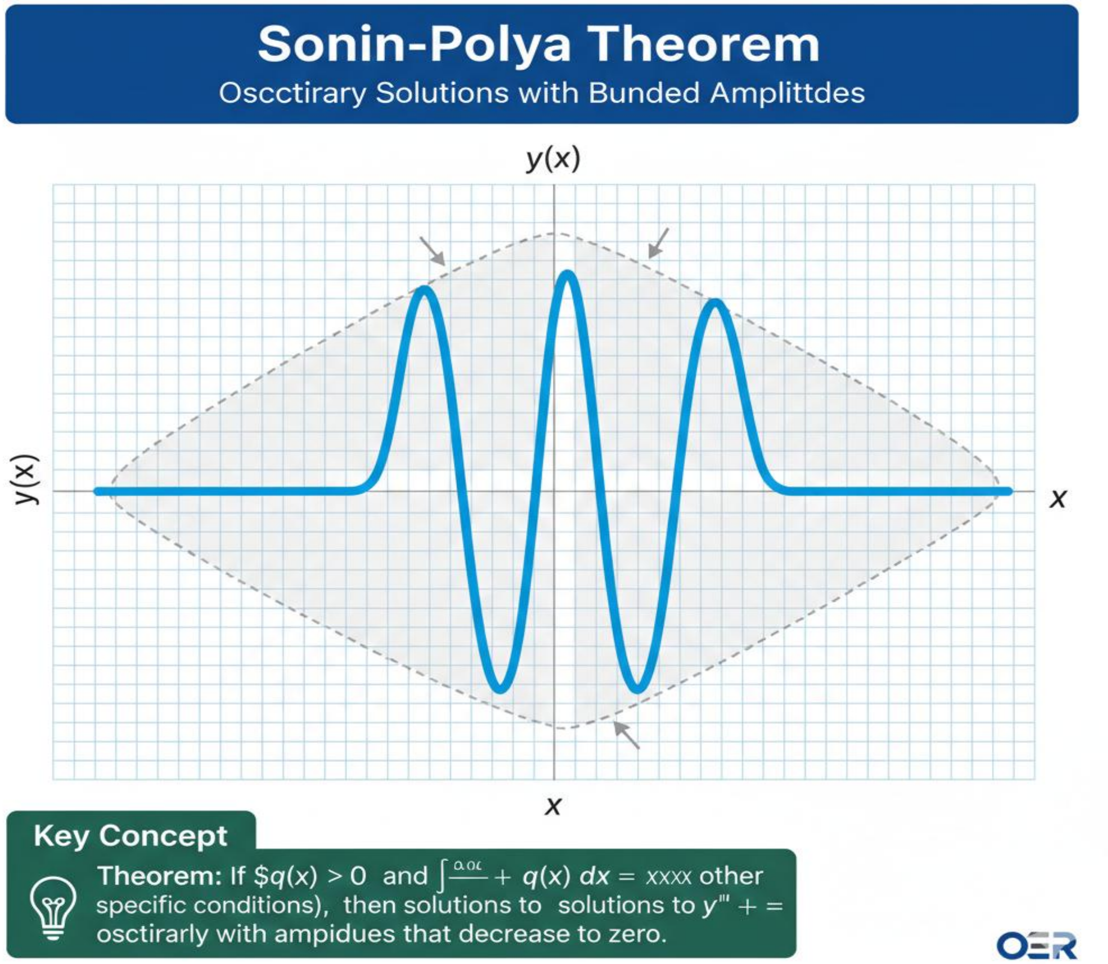


Illustration of bounded oscillations described by the Sonin–Polya theorem. :

2.3.2 Implications for bounds and oscillatory behaviour

The implications of the Sonin–Polya theorem are significant:

- It shows that oscillatory solutions do not grow without limit, their amplitudes are bounded.
- It provides estimates for the spacing of zeros, which is important in oscillation theory.
- It helps in predicting the qualitative behaviour of solutions in physical systems such as vibrations and wave motion.

For example, in mechanical systems, the theorem ensures that oscillations remain within predictable bounds, making the system stable and controllable.

Fill in the blank: The Sonin–Polya theorem ensures that oscillatory solutions remain within predictable __________.

Key points about Sonin–Polya theorem

Summary: The Sonin–Polya Theorem

The **Sonin–Polya Theorem** provides a powerful tool for analyzing the **amplitudes** of oscillating solutions to second-order differential equations. While the Sturm Comparison Theorem focuses on the frequency (zeros), Sonin–Polya tells us whether the "peaks" and "valleys" of a wave are growing, shrinking, or staying constant.

The Core Theorem

Consider the differential equation in the form:

$$(p(x)y')' + q(x)y = 0$$

If $p(x)$ and $q(x)$ are positive and have continuous derivatives, the theorem describes the behavior of the relative extrema (local maxima and minima) of the solution $|y(x)|$.

Condition on $p(x)q(x)$	Effect on Amplitude	Visual Behavior
$(p(x)q(x))'$ is **positive**	Amplitudes are **decreasing**.	The wave "dies out" or dampens over time.
$(p(x)q(x))'$ is **negative**	Amplitudes are **increasing**.	The oscillations grow larger (potentially unstable).
$(p(x)q(x))$ is **constant**	Amplitudes are **constant**.	A steady, uniform sine-like wave (e.g., simple harmonic motion).

Provides bounds on oscillatory solutions.

Estimates spacing of zeros.

Useful in physical systems involving vibrations and waves.

Pilot Questions 2.3

The Sonin–Polya theorem provides: A. Bounds on oscillatory behaviour B. Exact solutions to ODEs C. Stability conditions for non-linear systems D. Reduction to integral equations

The theorem ensures that oscillatory solutions: A. Grow without limit B. Remain bounded C. Are always periodic D. Do not exist

The spacing of zeros in oscillatory solutions can be estimated using: A. Existence theorem B. Uniqueness theorem C. Sturm comparison theorem D. Sonin–Polya theorem

The Sonin–Polya theorem is particularly useful in analysing: A. Population growth models B. Vibrations and wave motion C. Chemical reaction rates D. Stability of equilibrium points

The theorem complements the: A. Existence theorem B. Uniqueness theorem C. Sturm comparison theorem D. Reduction theorem

2.4 Illustrative Examples And Applications

The Sturm comparison theorem and the Sonin–Polya theorem are not only theoretical results, they also have practical applications in analysing oscillatory behaviour in physical and mathematical systems. By working through examples, we can see how these theorems provide insights into the spacing of zeros, boundedness of oscillations, and stability of systems.

2.4.1 Worked examples of Sturm comparison theorem

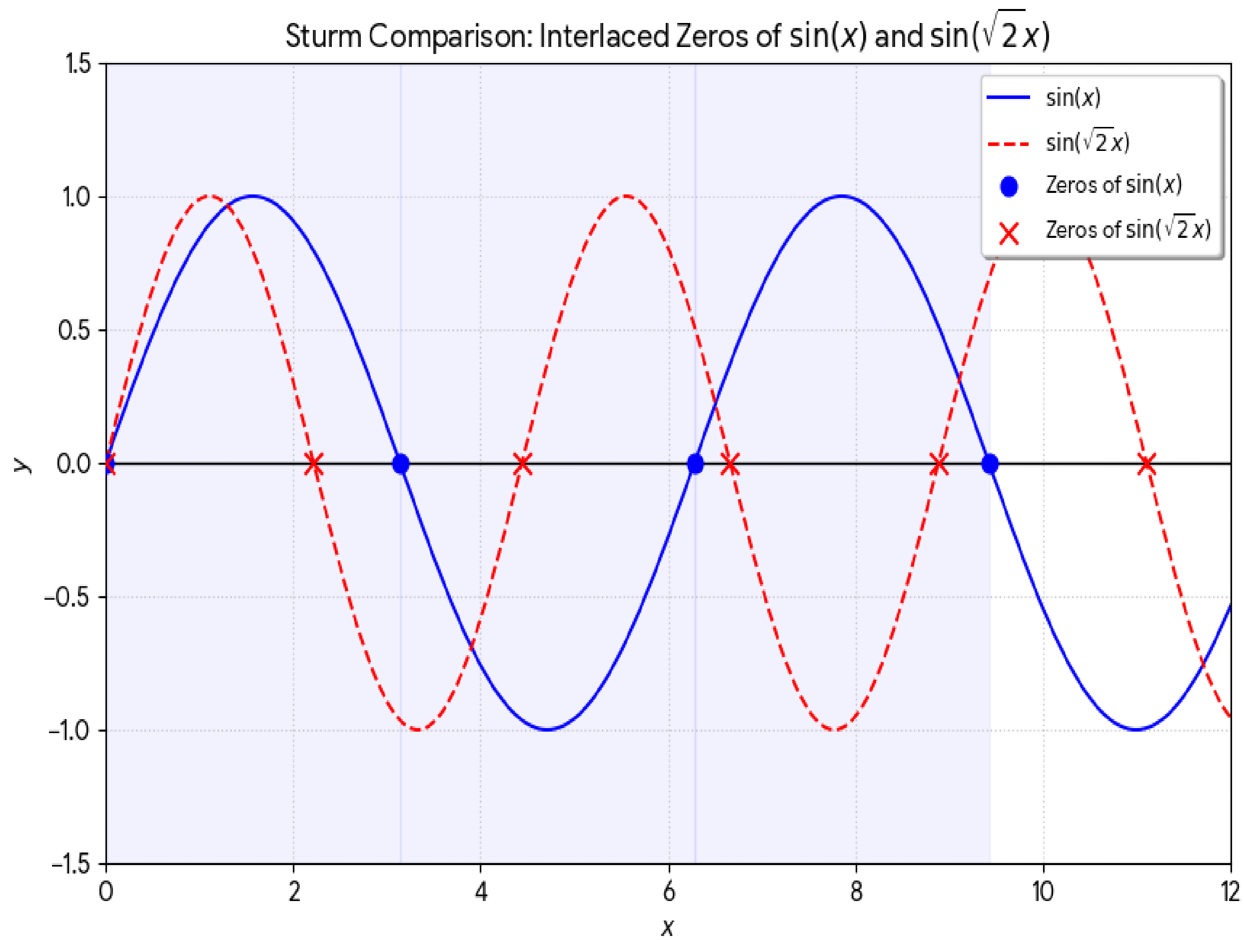
Consider the equations:

$$y''+y=0,z''+2y=0$$

The solution to the first equation is $y(x)=\sin(x)$, which has zeros at integer multiples of . The solution to the second equation is $z(x)=\sin(2x)$, which has zeros at integer multiples of $/2$.

By the Sturm comparison theorem, between any two consecutive zeros of $z(x)$, there exists at least one zero of $y(x)$. This illustrates how the theorem interlaces the zeros of solutions with different oscillatory rates.

Fill in the blank: The Sturm comparison theorem shows that the zeros of solutions with different oscillatory rates are __________.



Interlacing of zeros in Sturm comparison theorem.

2.4.2 Worked examples of Sonin–Polya theorem

Consider an oscillatory solution of a second-order equation where amplitude is bounded. For instance, solutions of $y''+y=0$remain bounded between -1and 1. The Sonin–Polya theorem confirms that oscillations remain within predictable bounds and that the spacing of zeros can be estimated.

This is important in mechanical systems such as vibrating strings, where oscillations must remain bounded to avoid instability.

Fill in the blank: The Sonin–Polya theorem ensures that oscillatory solutions remain within predictable __________.

Applications of Sturm and Sonin–Polya theorems

Applications of Sturm & Sonin–Polya Theorems

While these theorems may seem like abstract calculus, they are the backbone of **qualitative analysis** in physics and engineering—allowing us to predict the behavior of complex systems without needing an exact formula.

Theorem	***Key Focus***	***Primary Application***
Sturm Comparison	***Frequency & Zeros***	***Estimating natural frequencies, counting "nodes" (zeros) in vibration, and bounding the roots of special functions (like Bessel or Legendre).***
Sonin–Polya	***Amplitude & Peaks***	***Determining if oscillations will grow (unstable), decay (stable), or remain constant in varying environments.***

Sturm theorem: compares zeros of solutions, useful in oscillation theory.

Sonin–Polya theorem: provides bounds and estimates spacing of zeros.

Applications: vibrations, wave motion, stability analysis.

Pilot Questions 2.4

The Sturm comparison theorem shows that the zeros of solutions with different oscillatory rates are: A. Identical B. Interlaced C. Independent D. Non-existent

The solution y(x)=sin(x)has zeros at: A. Integer multiples of B. Integer multiples of /2C. Integer multiples of 2D. Non-integer values only

The Sonin–Polya theorem ensures that oscillatory solutions remain: A. Unbounded B. Within predictable bounds C. Periodic with infinite amplitude D. Non-oscillatory

Which theorem is particularly useful in estimating the spacing of zeros? A. Existence theorem B. Uniqueness theorem C. Sturm comparison theorem D. Sonin–Polya theorem

Applications of Sturm and Sonin–Polya theorems include: A. Population growth models B. Vibrations and wave motion C. Chemical reaction rates D. Heat conduction

Series Summary

In this Series, we explored the properties of solutions of ordinary differential equations and two important theorems that govern their oscillatory behaviour. We began by examining the fundamental properties of solutions, emphasising continuity, differentiability, and predictable behaviour under varying conditions. We then studied the Sturm comparison theorem, which allows us to compare the zeros of solutions of two second-order linear equations and provides insights into oscillation theory. Following that, we introduced the Sonin–Polya theorem, which establishes bounds on oscillatory solutions and estimates the spacing of zeros. Finally, we applied both theorems to illustrative examples, showing their practical significance in contexts such as vibrations, wave motion, and stability analysis.

Glossary of Terms

Continuity: A property of a function where its graph has no breaks or jumps.

Differentiability: A property of a function that ensures it has a well-defined derivative at every point in its domain.

Oscillation theory: The study of solutions of differential equations that cross the axis repeatedly, often in periodic fashion.

Sturm comparison theorem: A theorem that compares the zeros of solutions of two second-order linear differential equations.

Sonin–Polya theorem: A theorem that provides bounds on oscillatory solutions and estimates the spacing of their zeros.

Zero of a solution: A point where the solution function crosses the horizontal axis.

Concept Check Questions 2

Linked to SLO 2.1

- Describe the fundamental properties of solutions of ordinary differential equations.
- Give an example of a solution that is both continuous and differentiable.

Linked to SLO 2.2

State the Sturm comparison theorem and explain its significance in oscillation theory.

Apply the Sturm comparison theorem to compare the zeros of $\sin(x)$ and $\sin(2x)$.

Linked to SLO 2.3

Define the Sonin–Polya theorem and explain its implications for oscillatory behaviour.

Analyse how the Sonin–Polya theorem ensures boundedness of oscillatory solutions.

Linked to SLO 2.4

Apply Sturm and Sonin–Polya theorems to a practical example in mechanical vibrations.

Evaluate the importance of these theorems in predicting the behaviour of oscillatory systems.

Answers to Pilot Questions 2

Pilot Questions 2.1 (Properties of Solutions)

- A. Continuous and differentiable
- C. Both continuous and differentiable
- B. Growth rate
- B. Small changes in solutions
- B. They are continuous and differentiable

Pilot Questions 2.2 (Sturm Comparison Theorem)

- B. Zeros of solutions

B. At least one zero of y(x)

B. 2π

B. Oscillation theory

C. More rapidly than those of y''+y=0

Pilot Questions 2.3 (Sonin–Polya Theorem)

A. Bounds on oscillatory behaviour

B. Remain bounded

D. Sonin–Polya theorem

B. Vibrations and wave motion

C. Sturm comparison theorem

Pilot Questions 2.4 (Illustrative Examples and Applications)

B. Interlaced

A. Integer multiples of

B. Within predictable bounds

D. Sonin–Polya theorem

B. Vibrations and wave motion

Series 3: Linear and Non-Linear Systems, Floquet's Theory, and Stability in MTH 401: Theory of Ordinary Differential Equations.

3.1 Linear Systems of Differential Equations

Homogeneous and non-homogeneous systems

Matrix methods and fundamental solutions

3.2 Non-Linear Systems of Differential Equations

Phase plane analysis

Examples of non-linear behaviour

3.3 Floquet's Theory

Periodic coefficients in linear systems

Floquet multipliers and stability

3.4 Stability Analysis

Definitions of stability and instability

Lyapunov methods and applications

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 3.1** formulate linear systems of ordinary differential equations and demonstrate methods for solving homogeneous and non-homogeneous cases,
- **SLO 3.2** apply matrix methods to construct fundamental solutions of linear systems,
- **SLO 3.3** analyse non-linear systems using phase plane techniques and interpret examples of non-linear behaviour,
- **SLO 3.4** explain Floquet's theory for systems with periodic coefficients and evaluate stability using Floquet multipliers, and
- **SLO 3.5** assess stability of systems using Lyapunov methods and illustrate applications in modelling contexts.

3.1 Linear Systems Of Differential Equations

Linear systems of differential equations arise naturally in modelling interconnected processes, such as electrical circuits, mechanical vibrations, and population dynamics. A **linear system** is one in which the dependent variables and their derivatives appear linearly, without products or powers of the variables.

3.1.1 Homogeneous and non-homogeneous systems

A **homogeneous linear system** has the form:

$$y' = Ay$$

where Ais a constant matrix and yis a vector of dependent variables.

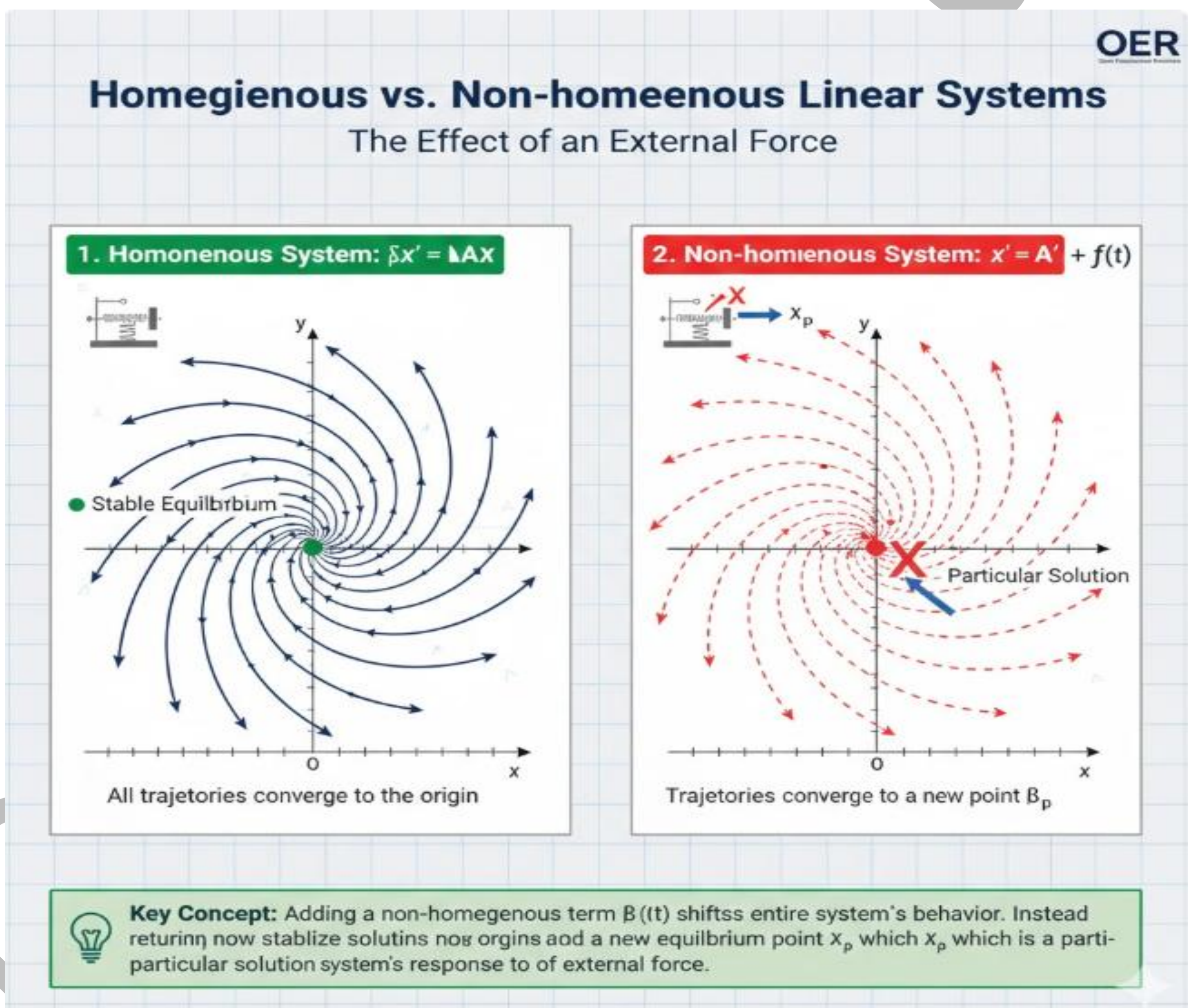
A **non-homogeneous linear system** includes an additional forcing term:

$$y'=Ay+f(x)$$

where f(x)is a vector function representing external inputs or forcing.

Fill in the blank: A linear system with an additional forcing term is called a __________ system.

Comparison of homogeneous and non-homogeneous linear systems.



3.1.2 Matrix methods and fundamental solutions

Matrix methods provide a systematic way to solve linear systems. The key concept is the **fundamental matrix solution**, which is a matrix whose columns are linearly independent solutions of the system.

For the homogeneous system y'=Ay, the solution can be expressed as:

$$y(x)=eAxc$$

where eAxis the matrix exponential and cis a constant vector determined by initial conditions.

Fill in the blank: The solution to a homogeneous linear system can be expressed using the __________ exponential.

: Key points about matrix methods

- ***Homogeneous systems:*** y'=Ay**.**
- ***Non-homogeneous systems:*** y'=Ay+f(x)**.**
- ***Fundamental matrix solution: matrix whose columns are linearly independent solutions.***
- ***General solution:*** y(x)=eAxc**.**

Key Matrix Solution Techniques

Method	Best Used For...	Core Concept
Eigenvalue Method	Constant coefficient matrices ($\mathbf{A}$ is all numbers).	Solutions take the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$, where λ is an eigenvalue and $\mathbf{v}$ is an eigenvector.
Fundamental Matrix ($\mathbf{\Phi}$)	General theory and non-homogeneous cases.	A matrix whose columns are n linearly independent solutions. The general homogeneous solution is $\mathbf{x}_h = \mathbf{\Phi}(t)\mathbf{c}$.
Matrix Exponential ($e^{\mathbf{A}t}$)	Theoretical elegance and initial value problems.	Defined by the power series $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \dots$. The solution is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$.
Variation of Parameters	Finding the particular solution $\mathbf{x}_p$ for non-homogeneous systems.	Uses the fundamental matrix to solve $\mathbf{x}_p = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{f}(t)dt$.

Pilot Questions 3.1

A homogeneous linear system has the form: A. y'=AyB. y'=Ay+f(x)C. y'=f(x)D. y'=Ay2

A non-homogeneous linear system includes: A. No forcing term B. A forcing term f(x)C. Only constant coefficients D. Quadratic terms

The fundamental matrix solution is: A. A matrix whose rows are dependent solutions B. A matrix whose columns are linearly independent solutions C. A determinant of the system D. A scalar exponential

The solution to a homogeneous linear system can be expressed using the: A. Scalar exponential B. Matrix exponential C. Polynomial expansion D. Fourier series

In the system y'=Ay, the vector cis determined by: A. Arbitrary choice B. Initial conditions C. Eigenvalues only D. Forcing terms

3.2 Non-Linear Systems Of Differential Equations

Non-linear systems of differential equations are more complex than linear ones because the dependent variables and their derivatives appear in non-linear combinations. These systems often exhibit rich behaviours such as oscillations, bifurcations, and chaos, making them essential in modelling real-world phenomena like predator–prey dynamics, chemical reactions, and mechanical systems.

3.2.1 Phase plane analysis

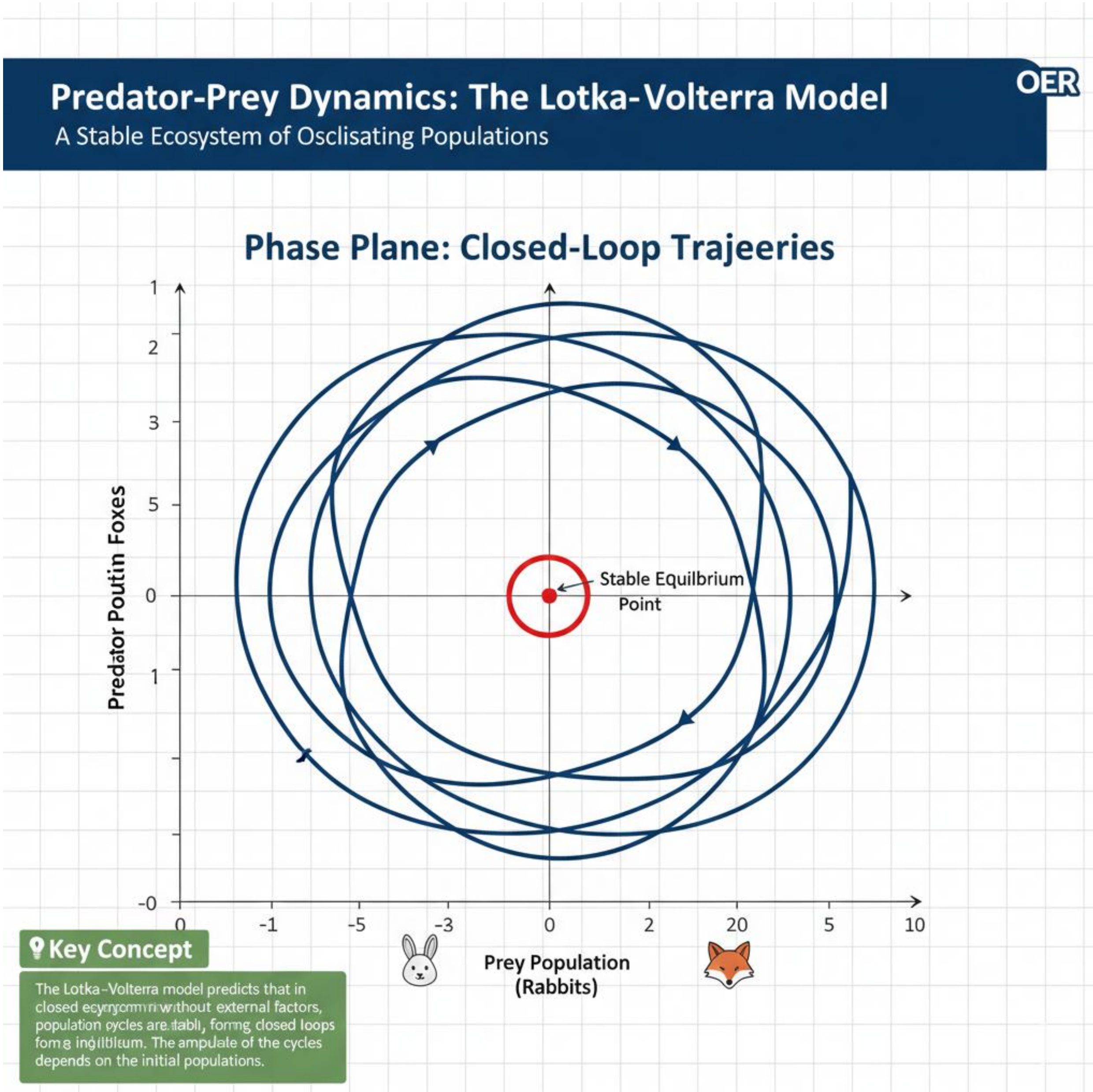
The **phase plane** is a graphical representation of trajectories of a two-dimensional system of differential equations. Each point in the plane represents a state of the system, and the trajectories show how the system evolves over time.

For example, consider the predator–prey model:

$$x'=x(1-y),y'=y(x-1)$$

Here, xand yrepresent populations of prey and predators. Phase plane analysis reveals closed trajectories, indicating oscillatory behaviour of the populations.

Fill in the blank: The phase plane is a graphical representation of __________ of a two-dimensional system.



Phase plane analysis of predator–prey model.

3.2.2 Examples of non-linear behaviour

Non-linear systems can exhibit behaviours not seen in linear systems:

- **Limit cycles**: Closed trajectories in the phase plane that represent sustained oscillations.
- **Bifurcations**: Sudden qualitative changes in system behaviour as parameters vary.

Chaos: Highly sensitive dependence on initial conditions, leading to unpredictable behaviour.

For example, the Van der Pol oscillator:

$$x''-\mu(1-x2)x'+x=0$$

exhibits limit cycles for positive values of .

Fill in the blank: A closed trajectory in the phase plane representing sustained oscillations is called a __________.

Key features of non-linear systems

Limit cycles: sustained oscillations.

Bifurcations: qualitative changes in behaviour.

Chaos: sensitive dependence on initial conditions.

Feature	Description	Physical Example
Multiple Equilibria	**Unlike linear systems (which usually have one origin), non-linear systems can have many steady states.**	**A ball on a hilly landscape (multiple peaks and valleys).**
Limit Cycles	**Isolated closed trajectories in the phase plane that "attract" or "repel" nearby paths.**	**The rhythmic beating of a heart or the cycle of a predator-prey population.**
Bifurcations	**A sudden qualitative change in the system's behavior as**	**A bridge collapsing once a load exceeds a**

Feature	Description	Physical Example
	a parameter is varied.	critical threshold.
Chaos	Extreme sensitivity to initial conditions where behavior looks random but is deterministic.	Long-term weather patterns (The Butterfly Effect).

Pilot Questions 3.2

The phase plane is a graphical representation of: A. Solutions of a two-dimensional system B. Solutions of a one-dimensional system C. Eigenvalues of a matrix D. Fourier coefficients

In the predator–prey model, phase plane analysis reveals: A. Constant solutions B. Closed trajectories C. Divergent solutions D. Linear behaviour

A closed trajectory in the phase plane representing sustained oscillations is called a: A. Bifurcation B. Limit cycle C. Equilibrium point D. Chaos

Sudden qualitative changes in system behaviour as parameters vary are called: A. Limit cycles B. Bifurcations C. Chaos D. Stability

The Van der Pol oscillator exhibits: A. Linear behaviour B. Limit cycles C. Constant solutions D. Divergent trajectories

3.3 Floquet's Theory

Many systems in physics and engineering involve coefficients that vary periodically with time. **Floquet's theory** provides a framework for analysing the stability of such systems. It is particularly useful in studying vibrations, electrical circuits, and control systems where periodic forcing or coefficients occur.

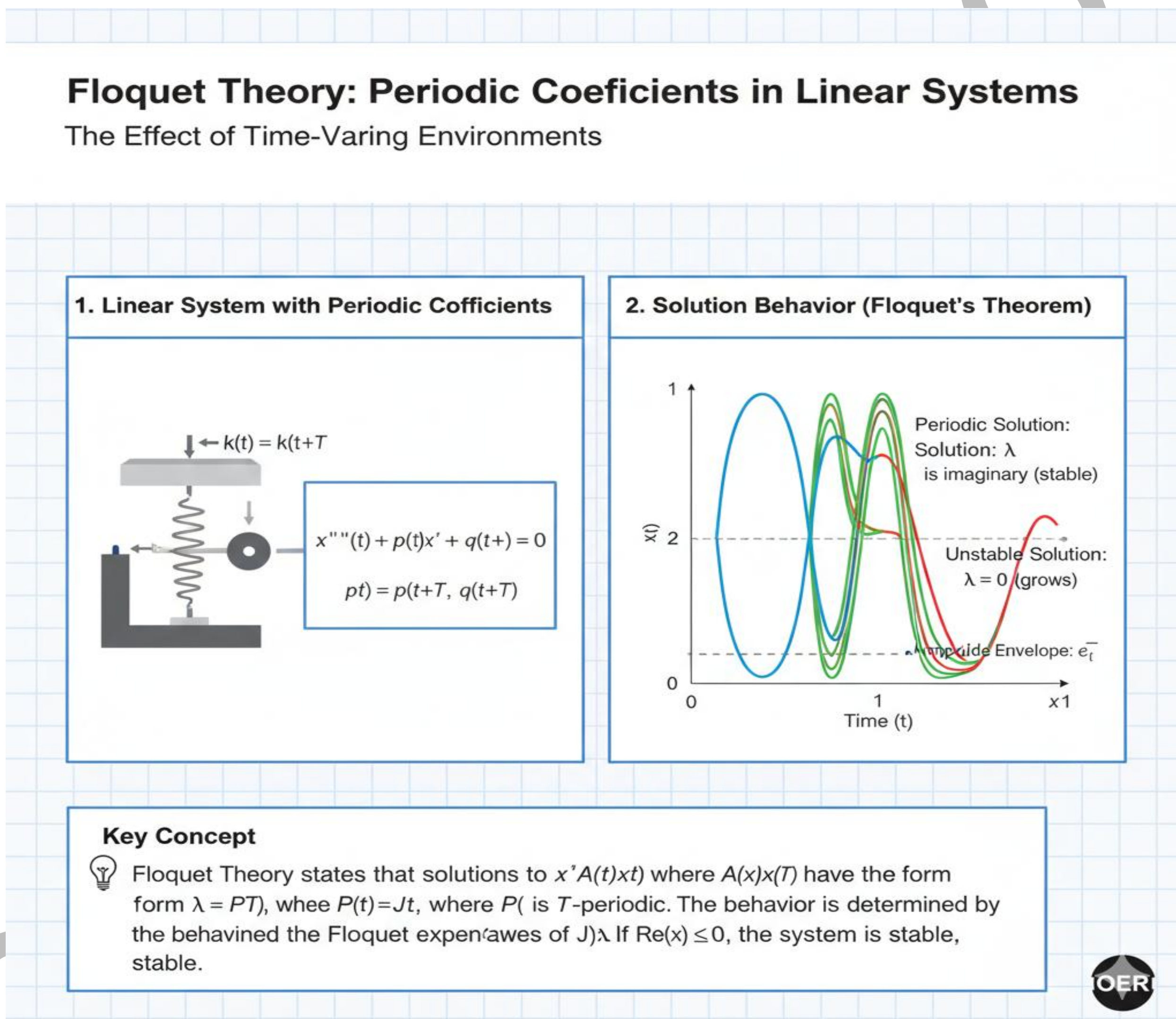
3.3.1 Periodic coefficients in linear systems

Consider a linear system of the form:

$$y'=A(t)y$$

where $A(t)$ is a matrix with **periodic coefficients**, meaning $A(t+T)=A(t)$ for some period T. Floquet's theory states that solutions of such systems can be expressed in terms of a product of a periodic function and an exponential function.

Fill in the blank: Floquet's theory applies to linear systems with __________ coefficients.



Linear system with periodic coefficients analysed using Floquet's theory.

3.3.2 Floquet multipliers and stability

The key concept in Floquet's theory is the **Floquet multiplier**, which is obtained from the eigenvalues of the monodromy matrix (the fundamental matrix solution evaluated after one period).

If all Floquet multipliers have absolute value less than one, the solution is stable.

If any multiplier has absolute value greater than one, the solution is unstable.

If multipliers have absolute value equal to one, the system is marginally stable.

Fill in the blank: The stability of a system in Floquet's theory is determined by the __________ multipliers.

Key points about Floquet's theory

Applies to systems with periodic coefficients.

Solutions expressed as product of periodic and exponential functions.

Stability determined by Floquet multipliers.

Key Concepts & Definitions

Concept	Definition	Significance
Monodromy Matrix ($\mathbf{C}$)	The fundamental matrix evaluated at the end of one period: $\mathbf{\Phi}(T)$ where $\mathbf{\Phi}(0) = \mathbf{I}$.	Captures the total "transformation" the system undergoes over exactly one cycle.
Floquet Multipliers (ρ)	The eigenvalues of the Monodromy Matrix $\mathbf{C}$.	They act as "growth factors." They tell you by what factor the solution is multiplied after one period.
Floquet Exponents (μ)	Related to multipliers by $\rho = e^{\mu T}$.	These are analogous to eigenvalues in constant-coefficient systems ($\mathbf{x} = e^{\mu t}\mathbf{p}(t)$).

Pilot Questions 3.3

Floquet's theory applies to systems with: A. Constant coefficients B. Periodic coefficients C. Random coefficients D. Non-linear terms only

Solutions in Floquet's theory are expressed as a product of: A. Polynomial and exponential functions B. Periodic and exponential functions C. Trigonometric and polynomial functions D. Constant and exponential functions

The stability of a system in Floquet's theory is determined by: A. Eigenvectors B. Floquet multipliers C. Initial conditions D. Forcing terms

If all Floquet multipliers have absolute value less than one, the system is: A. Stable B. Unstable C. Marginally stable D. Divergent

The monodromy matrix is obtained by evaluating the fundamental matrix solution after: A. One derivative B. One period C. One eigenvalue D. One forcing term

3.4 Stability Analysis

The concept of **stability** is central to the study of dynamical systems. Stability determines whether solutions remain close to an equilibrium point or diverge away when subjected to small disturbances. This analysis is crucial in engineering, physics, and applied mathematics, where systems must remain predictable and controllable.

3.4.1 Definitions of stability and instability

Stable equilibrium: An equilibrium point is stable if solutions that start close to it remain close for all time.

Asymptotically stable equilibrium: An equilibrium point is asymptotically stable if solutions not only remain close but also converge to the equilibrium as time progresses.

Unstable equilibrium: An equilibrium point is unstable if solutions starting

near it eventually move away.

For example, in the system $y'=-y$, the equilibrium at $y=0$is asymptotically stable, since all solutions converge to zero.

Fill in the blank: An equilibrium point where solutions converge to it as time progresses is called __________ stable.

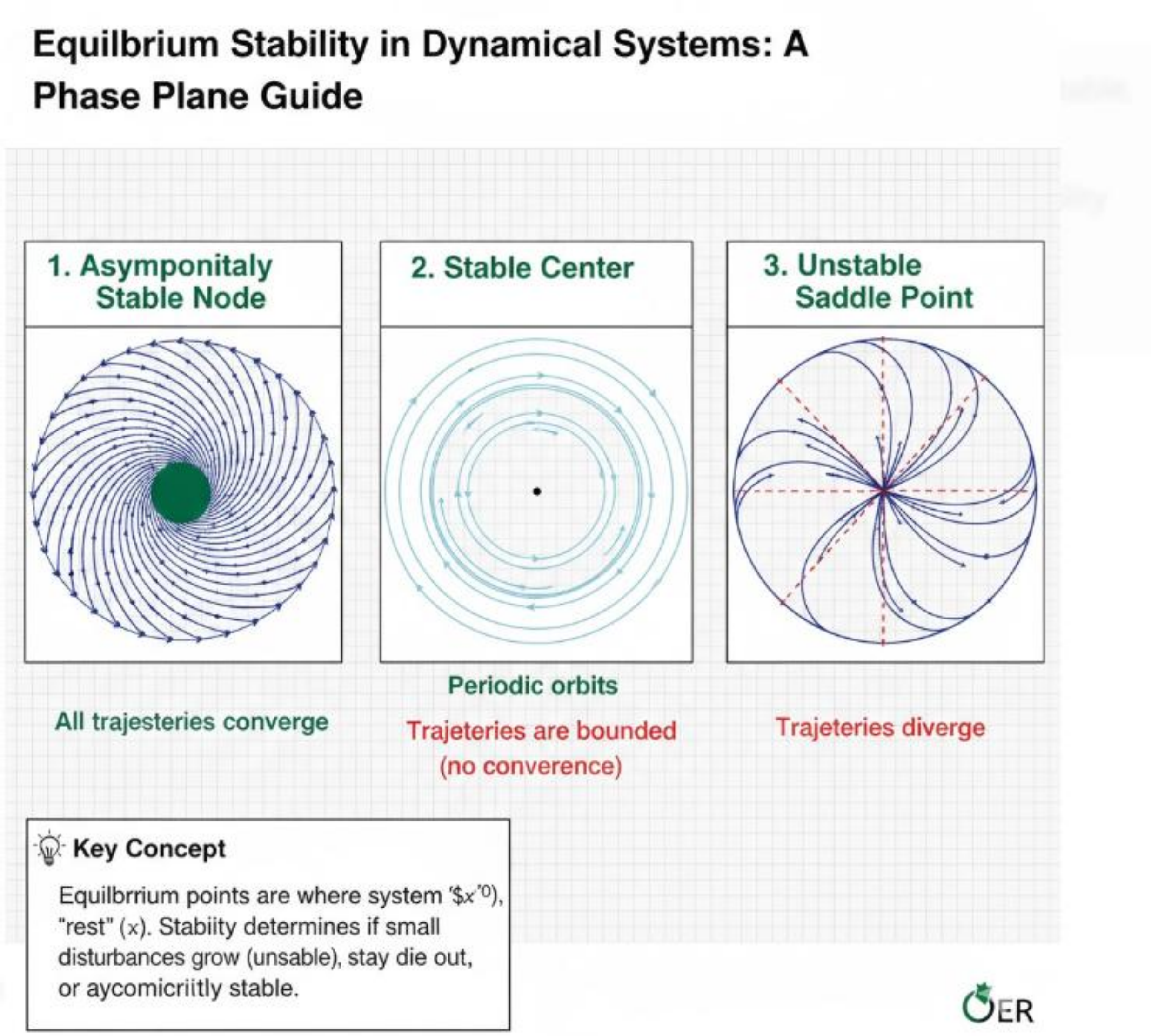


Illustration of stable and unstable equilibria.

3.4.2 Lyapunov methods and applications

The **Lyapunov method** provides a systematic way to assess stability without solving the system explicitly.

A **Lyapunov function** is a scalar function V(x)that decreases along trajectories of the system.

If V(x)is positive definite and its derivative along solutions is negative definite, the equilibrium is asymptotically stable.

Applications of Lyapunov methods include control systems, robotics, and mechanical vibrations, where stability must be guaranteed.

Fill in the blank: A scalar function used to assess stability without solving the system explicitly is called a __________ function.

Key points about stability analysis

- ***Stability determines whether solutions remain close to equilibrium.***
- ***Asymptotic stability means solutions converge to equilibrium.***
- ***Lyapunov functions provide a tool for assessing stability.***

Types of Stability

Status	Mathematical Definition	Physical Intuition
Stable (Lyapunov)	Trajectories starting near x_e stay near x_e for all $t > 0$.	A marble at the bottom of a wide, flat bowl. It moves, but stays close.
Asymptotically Stable	Trajectories not only stay near x_e but eventually converge to x_e.	A marble in a bowl with friction; it eventually settles at the center.
Unstable	Small disturbances grow over time, moving the system away from x_e.	A marble balanced on top of a hill; a tiny tap sends it rolling away.

Pilot Questions 3.4

1. An equilibrium point is stable if: A. Solutions starting near it remain close B. Solutions starting near it diverge away C. Solutions converge to infinity D. Solutions oscillate indefinitely

An equilibrium point is asymptotically stable if: A. Solutions remain close but do not converge B. Solutions converge to the equilibrium C. Solutions diverge away D. Solutions oscillate without bound

An equilibrium point is unstable if: A. Solutions remain close B. Solutions converge to equilibrium C. Solutions move away D. Solutions oscillate with small amplitude

A scalar function used to assess stability without solving the system explicitly is called a: A. Stability function B. Lyapunov function C. Equilibrium function D. Divergence function

If a Lyapunov function is positive definite and its derivative is negative definite, the equilibrium is: A. Stable B. Asymptotically stable C. Unstable D. Marginally stable

Series Summary

In this Series, we studied systems of ordinary differential equations, both linear and non-linear, and explored methods for analysing their behaviour and stability. We began with **linear systems**, distinguishing between homogeneous and non-homogeneous cases and solving them using matrix methods and fundamental solutions. We then examined **non-linear systems**, using phase plane analysis to reveal behaviours such as limit cycles, bifurcations, and chaos. Next, we introduced **Floquet's theory**, which applies to systems with periodic coefficients and uses Floquet multipliers to determine stability. Finally, we explored **stability analysis**, defining stability and instability, and applying **Lyapunov methods** to assess stability without solving the system explicitly. Together, these topics provide powerful tools for understanding and predicting the behaviour of dynamic systems.

Glossary of Terms

Linear system: A system of differential equations where variables and derivatives appear linearly.

Non-linear system: A system where variables and derivatives appear in non-linear combinations, often leading to complex behaviours.

Phase plane: A graphical representation of trajectories of a two-dimensional system.

Limit cycle: A closed trajectory in the phase plane representing sustained oscillations.

Bifurcation: A qualitative change in system behaviour as parameters vary.

Chaos: Sensitive dependence on initial conditions leading to unpredictable behaviour.

Floquet's theory: A framework for analysing systems with periodic coefficients.

Floquet multipliers: Eigenvalues of the monodromy matrix used to determine stability in Floquet's theory.

Lyapunov function: A scalar function used to assess stability without solving the system explicitly.

Stability: The property of a system where solutions remain close to equilibrium under small disturbances.

Concept Check Questions 3

Linked to SLO 3.1

Formulate a homogeneous linear system and demonstrate how to solve it using matrix methods.

Explain the difference between homogeneous and non-homogeneous linear systems.

Linked to SLO 3.2

Construct a fundamental matrix solution for a given linear system.

Apply matrix exponential methods to solve a homogeneous system.

Linked to SLO 3.3

Analyse a non-linear system using phase plane techniques.

Identify examples of limit cycles, bifurcations, and chaos in non-linear systems.

Linked to SLO 3.4

Explain Floquet's theory and its application to systems with periodic coefficients.

Evaluate stability using Floquet multipliers.

Linked to SLO 3.5

Define stability and asymptotic stability.

Assess stability using Lyapunov functions and illustrate with an example.

Answers to Pilot Questions 3

Pilot Questions 3.1 (Linear Systems)

A. y'=Ay

B. A forcing term f(x)

B. A matrix whose columns are linearly independent solutions

B. Matrix exponential

B. Initial conditions

Pilot Questions 3.2 (Non-Linear Systems)

A. Solutions of a two-dimensional system

B. Closed trajectories

B. Limit cycle

B. Bifurcations

B. Limit cycles

Pilot Questions 3.3 (Floquet's Theory)

B. Periodic coefficients

B. Periodic and exponential functions

B. Floquet multipliers

A. Stable

B. One period

Pilot Questions 3.4 (Stability Analysis)

A. Solutions starting near it remain close

B. Solutions converge to the equilibrium

C. Solutions move away

B. Lyapunov function

B. Asymptotically stable

Series 4: Boundary Value Problems And Green's Functions

Series Outline

4.1 Boundary Value Problems (BVPs)

Definition and distinction from initial value problems

Examples of boundary conditions

4.2 Existence and Uniqueness of Solutions to BVPs

Theoretical results

Illustrative examples

4.3 Green's Functions

Definition and construction

Properties and applications

4.4 Applications of Green's Functions

Solving boundary value problems

Physical applications (heat conduction, vibrations, electrostatics)

.

Introduction

Boundary value problems (BVPs) arise when conditions are specified at more than one point, unlike initial value problems where conditions are given at a single starting point. BVPs are fundamental in mathematical physics and engineering, appearing in contexts such as heat conduction, wave propagation, and electrostatics.

To solve BVPs, we often use **Green's functions**, which provide a powerful method for representing solutions. Green's functions act like "building blocks" that capture the influence of boundary conditions and external forcing.

The aim of this Series is to introduce boundary value problems, explain the existence and uniqueness of their solutions, define and construct Green's functions, and apply them to practical problems. By the end of this Series, you should be able to distinguish BVPs from IVPs, understand the role of Green's functions, and apply them to solve real-world problems.

Series Learning Outcomes

Upon completing this Series, you should be able to:

- **SLO 4.1** define boundary value problems and distinguish them from initial value problems, providing examples of boundary conditions,
- **SLO 4.2** explain the existence and uniqueness of solutions to boundary value problems and illustrate with worked examples,
- **SLO 4.3** construct Green's functions for simple boundary value problems and analyse their properties,
- **SLO 4.4** apply Green's functions to solve boundary value problems and evaluate their significance in physical applications such as heat conduction, vibrations, and electrostatics.

4.1 Boundary Value Problems (BVPs)

Boundary value problems (BVPs) are a class of ordinary differential equations where conditions are specified at more than one point in the domain. This contrasts with **initial value problems (IVPs)**, where conditions are given only at a single starting point. BVPs are fundamental in physics and engineering, especially in problems involving equilibrium states, vibrations, and steady-state heat conduction.

4.1.1 Definition and distinction from initial value problems

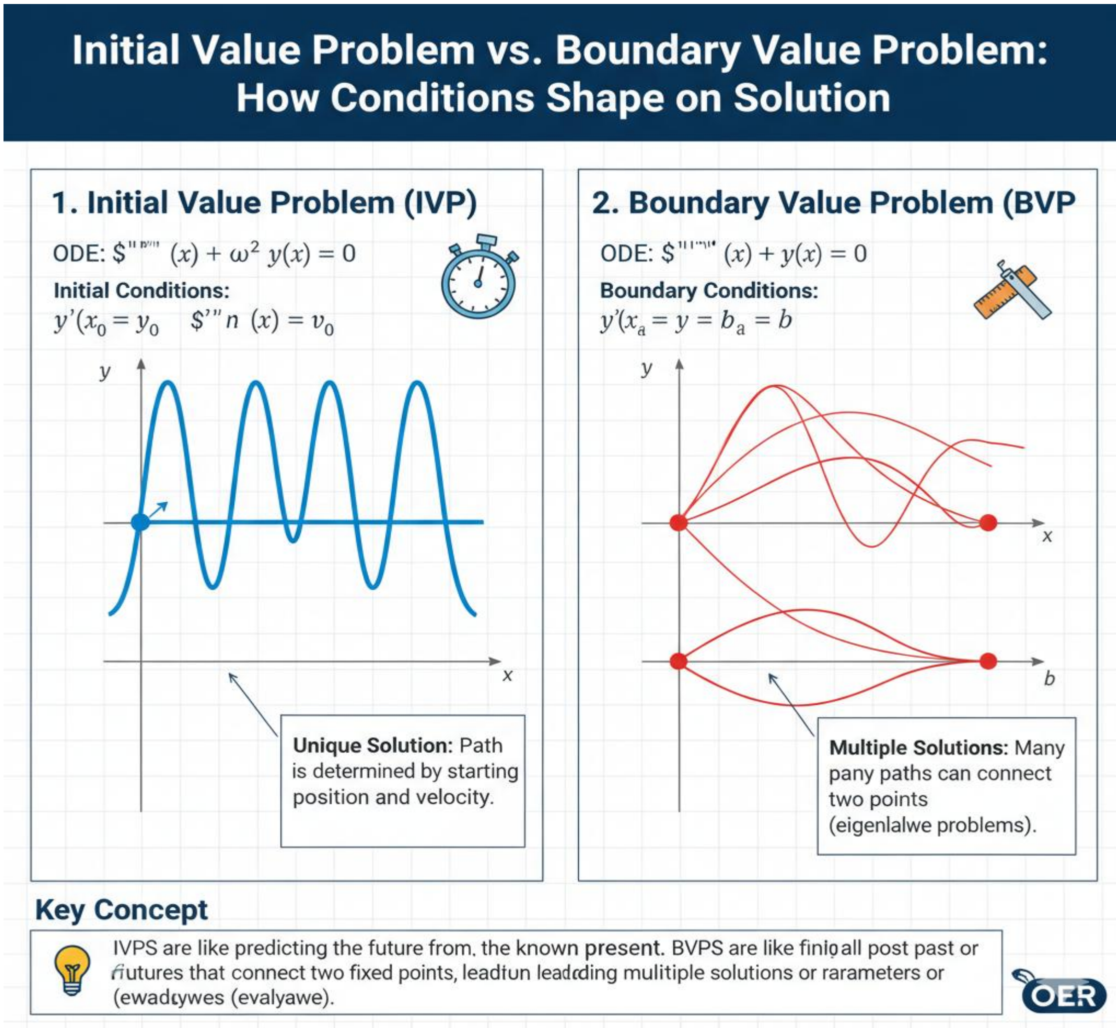
- **Initial Value Problem (IVP):** Conditions specified at a single point, usually at the start of the interval (e.g., $y(0)=1$).
- **Boundary Value Problem (BVP):** Conditions specified at two or more points, often at the endpoints of the interval (e.g., $y(0)=0, y(1)=1$).

Example:

$$y''+y=0, y(0)=0, y(\pi)=0$$

This is a boundary value problem because conditions are given at two points.

Fill in the blank: A boundary value problem specifies conditions at __________ points in the domain.



Distinction between initial value problems and boundary value problems.

4.1.2 Examples of boundary conditions

Boundary conditions can take different forms:

- **Dirichlet boundary condition:** Specifies the value of the function at the boundary (e.g., $y(0)=0$).
- **Neumann boundary condition:** Specifies the value of the derivative at the boundary (e.g., $y'(1)=2$).
- **Mixed boundary condition:** Combines both function and derivative conditions.

Example: Heat conduction in a rod:

$$u''(x)=-q(x),u(0)=0,u(L)=T$$

Here, $u(0)=0$ and $u(L)=T$ are Dirichlet boundary conditions.

Fill in the blank: A boundary condition that specifies the value of the derivative at the boundary is called a __________ condition.

Types of boundary conditions

Dirichlet: value of the function.

Neumann: value of the derivative.

Mixed: combination of both.

Pilot Questions 4.1

A boundary value problem specifies conditions at: A. One point B. Two or more points C. No points D. Random points

An initial value problem specifies conditions at: A. Two points B. One point C. Infinite points D. No points

The condition $y(0)=0, y(\pi)=0$ is an example of: A. Initial value problem B. Boundary value problem C. Mixed condition D. Neumann condition

A boundary condition that specifies the value of the function at the boundary is called: A. Dirichlet condition B. Neumann condition C. Mixed condition D. Periodic condition

A boundary condition that specifies the value of the derivative at the boundary is called: A. Dirichlet condition B. Neumann condition C. Mixed condition D. Periodic condition

4.2 Existence And Uniqueness Of Solutions To BVPs

Boundary value problems (BVPs) differ from initial value problems (IVPs) in that their solutions are not always guaranteed to exist or be unique. This makes the study of existence and uniqueness central to understanding when BVPs can be applied reliably in mathematics, physics, and engineering.

4.2.1 Theoretical results

For linear BVPs, existence and uniqueness depend on the associated differential operator and the boundary conditions.

If the homogeneous problem (with zero forcing term) has only the trivial solution, then the non-homogeneous problem will have a unique solution.

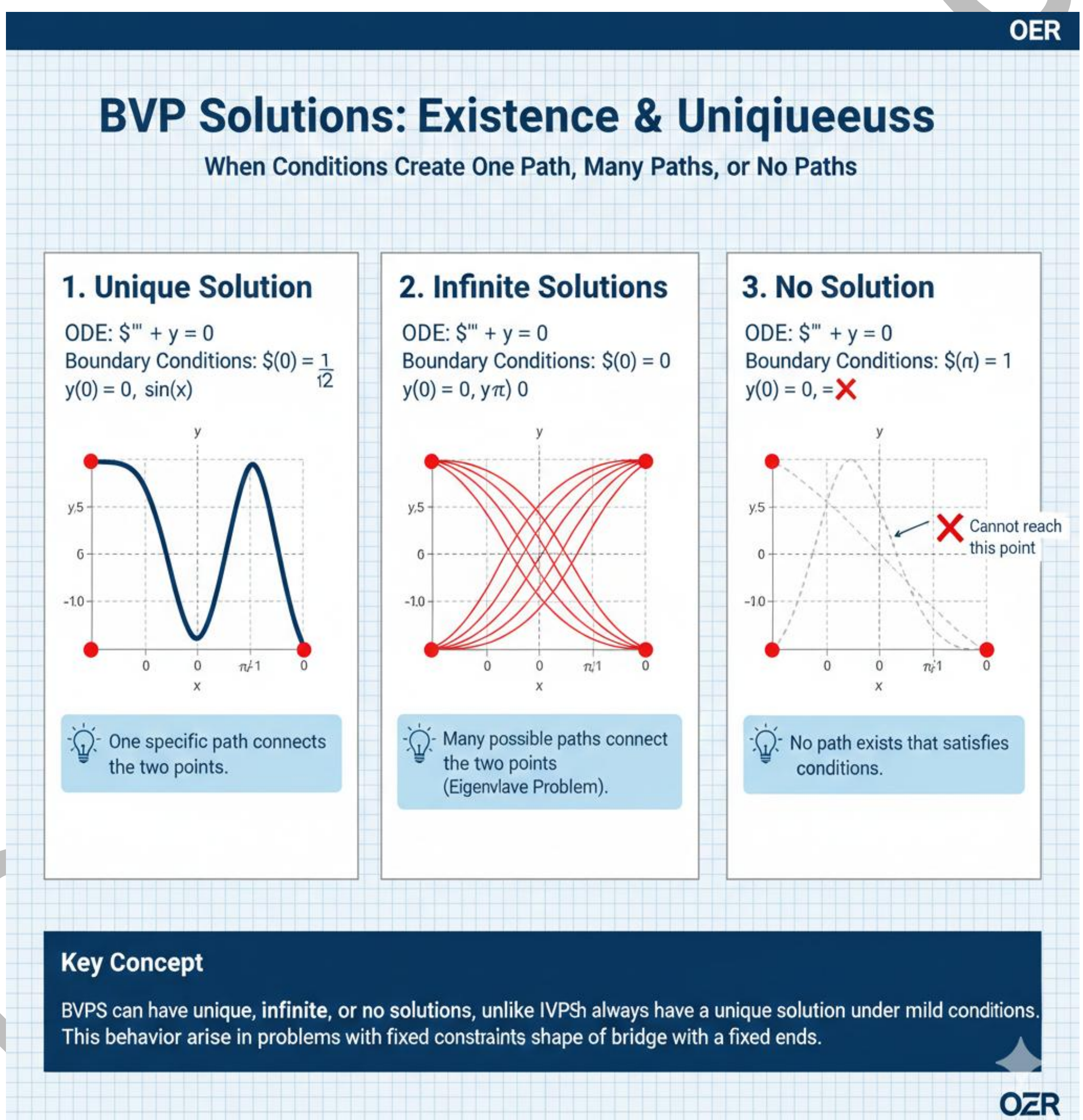
If the homogeneous problem admits non-trivial solutions, then the non-homogeneous problem may have no solution or infinitely many solutions.

Example:

$$y''+y=0, y(0)=0, y(\pi)=0$$

This problem has non-trivial solutions such as $y(x)=\sin(x)$. Hence, uniqueness is not guaranteed.

Fill in the blank: A boundary value problem has a unique solution if the associated homogeneous problem has only the __________ solution.



Existence and uniqueness scenarios in boundary value problems.

4.2.2 Illustrative examples

Example 1 (Unique solution):

$$y''=-1, y(0)=0, y(1)=0$$

Solution: y(x)=-12x(x-1). This is unique.

Example 2 (Non-unique solution):

$$y''+y=0, y(0)=0, y(\pi)=0$$

Solutions include y(x)=sin(x), showing non-uniqueness.

Fill in the blank: The solution y(x)=-12x(x-1)is an example of a __________ solution to a BVP.

Key points about existence and uniqueness of BVPs

Existence and uniqueness depend on operator and boundary conditions.

Homogeneous problem with only trivial solution → unique solution.

Homogeneous problem with non-trivial solutions → no solution or infinitely many solutions.

Outcome	***Mathematical Condition (Linear Case)***	***Physical Example***
Unique Solution	***The corresponding homogeneous BVP has only the trivial solution ($y=0$).***	***A taut guitar string held fixed at both ends with a specific weight hanging from it.***
Infinite Solutions	***The homogeneous BVP has non-trivial solutions (eigenfunctions).***	***A bridge or beam vibrating at one of its natural frequencies (resonance).***

Outcome	***Mathematical Condition (Linear Case)***	***Physical Example***
No Solution	***The boundary conditions contradict the internal "physics" of the ODE.***	***Attempting to stretch a rigid rod between two points that are further apart than the rod's maximum length.***

Pilot Questions 4.2

Existence and uniqueness of BVPs depend on: A. The operator and boundary conditions B. Initial conditions only C. Random choice of parameters D. The forcing term alone

If the homogeneous problem has only the trivial solution, the BVP has: A. No solution B. A unique solution C. Infinitely many solutions D. Random solutions

If the homogeneous problem has non-trivial solutions, the BVP may have: A. A unique solution B. No solution or infinitely many solutions C. Exactly one solution D. No dependence on boundary conditions

The problem $y''=-1, y(0)=0, y(1)=0$has: A. No solution B. A unique solution C. Infinitely many solutions D. Non-trivial homogeneous solutions

The problem $y''+y=0, y(0)=0, y(\pi)=0$has: A. A unique solution B. Non-unique solutions C. No solution D. Random solutions

4.3 Green's Functions

Green's functions are powerful tools for solving boundary value problems. They act as kernels that capture the influence of boundary conditions and external forcing, allowing us to represent solutions in an integral form.

4.3.1 Definition and construction

A Green's function $G(x,\xi)$is defined for a differential operator Land boundary conditions such that:

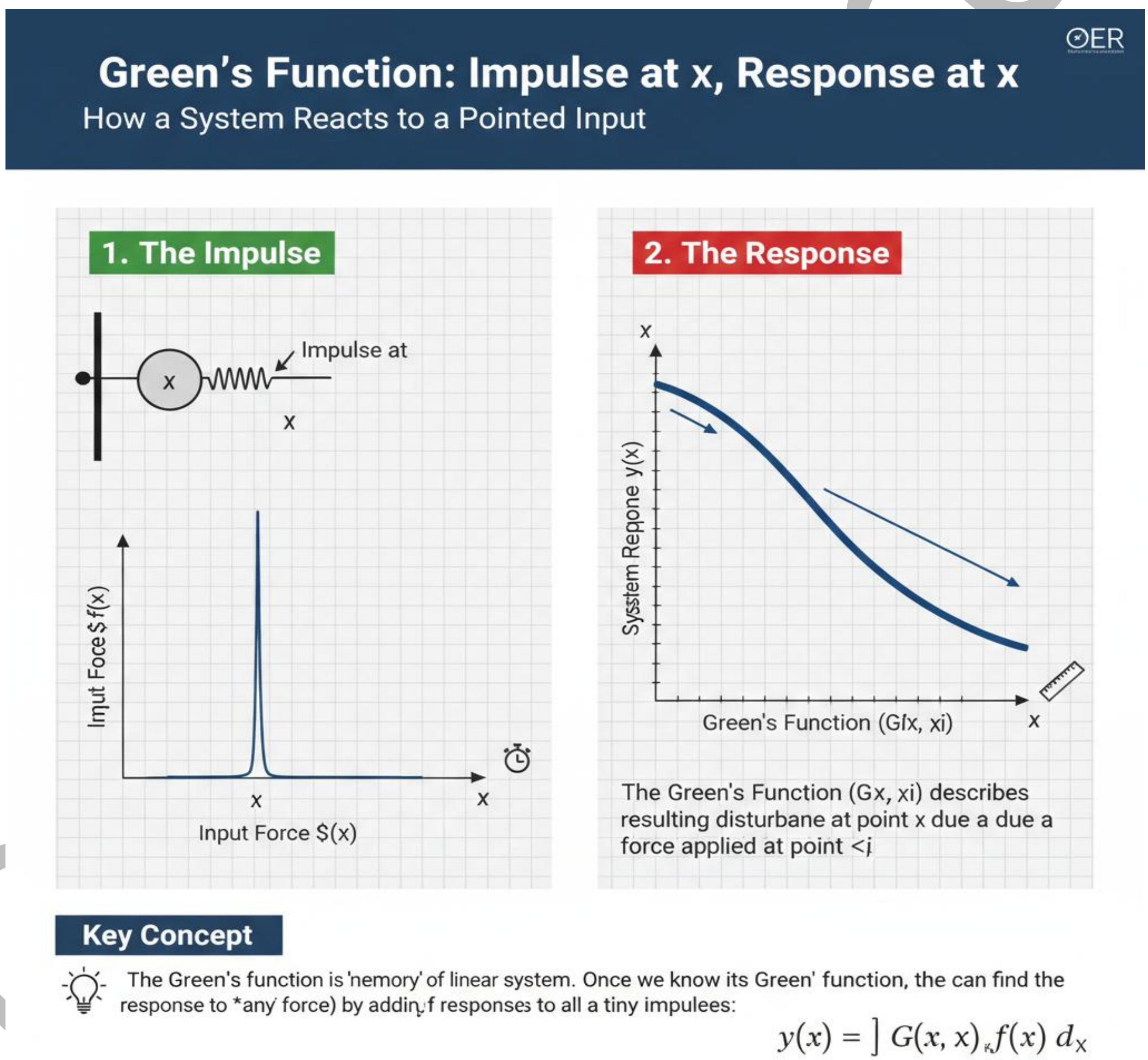
$$LG(x,\xi)=\delta(x-\xi)$$

where $\delta(x-\xi)$is the Dirac delta function.

The Green’s function represents the response of the system at point xdue to a unit impulse applied at point . Once constructed, the solution to a boundary value problem can be expressed as:

$$y(x)=abG(x,\xi)f(\xi)\ d\xi$$

Fill in the blank: A Green’s function represents the response of the system at point xdue to a unit __________ applied at point .



Green’s function as system response to a unit impulse.

4.3.2 Properties and applications

Key properties of Green's functions:

Linearity: Solutions can be superposed.

Symmetry: In many cases, $G(x,\xi)=G(\xi,x)$.

Dependence on boundary conditions: The form of $G(x,\xi)$changes with boundary conditions.

Applications:

Solving boundary value problems in physics and engineering.

Analysing heat conduction, vibrations, and electrostatics.

Providing integral representations of solutions.

Fill in the blank: In many cases, Green's functions satisfy the property of __________, meaning $G(x,\xi)=G(\xi,x)$.

Key points about Green's functions

Defined via $LG(x,\xi)=\delta(x-\xi)$**.**

Represent system response to a unit impulse.

Properties: linearity, symmetry, dependence on boundary conditions.

Applications: heat conduction, vibrations, electrostatics.

Key Properties

Property	Description	Meaning
Continuity	$G(x, \xi)$ is continuous everywhere in the interval.	The physical displacement doesn't "break" at the point of impact.
Slope Jump	The derivative $G'(x, \xi)$ has a specific jump discontinuity at $x = \xi$.	The force (impulse) creates a sharp "kink" in the shape of the solution.
Symmetry	Often $G(x, \xi) = G(\xi, x)$ (for self-adjoint operators).	**Maxwell's Reciprocity:** The effect at x from a force at ξ is the same as the effect at ξ from a force at x.
Boundary Satisfaction	$G(x, \xi)$ must satisfy the boundary conditions of the problem.	The function "knows" where the edges of the system are fixed.

Pilot Questions 4.3

A Green's function is defined such that: A. LG(x,ξ)=0B. LG(x,ξ)=δ(x-ξ)C. LG(x,ξ)=f(x)D. LG(x,ξ)=y(x)

A Green's function represents the response of the system at point xdue to: A. A constant input at B. A unit impulse at C. A boundary condition at xD. A periodic forcing term

The solution to a boundary value problem using Green's function is expressed as: A. A polynomial expansion B. A Fourier series C. An integral representation D. A matrix exponential

In many cases, Green's functions satisfy the property of: A. Linearity B. Symmetry C. Periodicity D. Divergence

Applications of Green's functions include: A. Population growth models B. Heat conduction, vibrations, electrostatics C. Random processes D. Algebraic equations

4.4 Applications Of Green's Functions

Green's functions are not just theoretical constructs; they are widely applied in physics, engineering, and applied mathematics. Their power lies in transforming boundary value problems into integral equations, making complex problems more tractable.

4.4.1 Solving boundary value problems

Once a Green's function $G(x,\xi)$ is constructed for a given operator and boundary conditions, the solution to the boundary value problem:

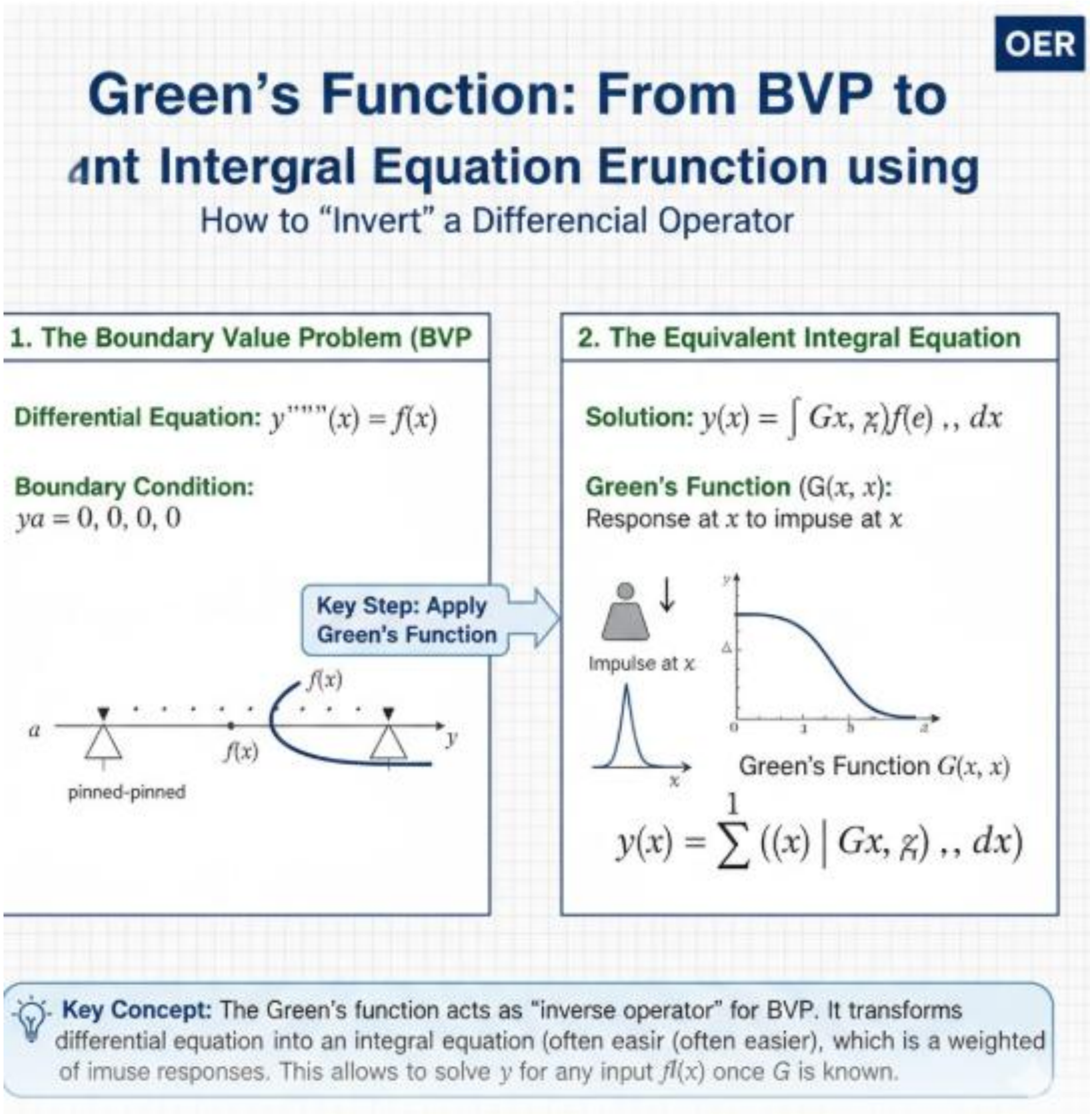
$$Ly(x)=f(x), \text{with boundary conditions}$$

can be expressed as:

$$y(x)=abG(x,\xi)f(\xi)\ d\xi$$

This integral representation allows us to compute solutions directly, even when the forcing term $f(x)$ is complicated.

Fill in the blank: Green's functions transform boundary value problems into __________ equations.



Solving boundary value problems using Green's functions.

4.4.2 Physical applications

Green's functions appear in many physical contexts:

- Heat conduction: Representing temperature distribution in a rod subject to boundary conditions.
- Vibrations: Analysing oscillations in mechanical systems such as strings and beams.
- Electrostatics: Computing electric potential due to charge distributions with boundary conditions.

Quantum mechanics: Describing propagation of particles and fields.

Fill in the blank: In electrostatics, Green's functions are used to compute the __________ potential due to charge distributions.

Box 4.4: Applications of Green's functions

Heat conduction: temperature distribution.

Vibrations: oscillations in mechanical systems.

Electrostatics: electric potential due to charges.

Quantum mechanics: particle and field propagation.

Core Applications Across Physics

Field	Physical Interpretation of $G(x, \xi)$	Governing Equation
Electrostatics	The electric potential at x due to a **unit point charge** at ξ.	Poisson's Equation: $\nabla^2\phi = -\rho/\epsilon_0$
Structural Mechanics	The deflection of a beam at x due to a **concentrated load** at ξ.	Euler-Bernoulli Beam Equation
Heat Conduction	The temperature at x at time t resulting from an **instantaneous heat pulse** at ξ.	Heat Equation: $u_t = \alpha\nabla^2 u$
Acoustics/Vibrations	The sound pressure or displacement at x caused by a **point source vibration** at ξ.	Wave Equation: $u_{tt} = c^2\nabla^2 u$
Quantum Mechanics	The probability amplitude (propagator) for a particle to move from ξ to x.	Schrödinger Equation

Pilot Questions 4.4

Green's functions transform boundary value problems into: A. Differential equations B. Integral equations C. Algebraic equations D. Polynomial expansions

The solution to a BVP using Green's function is expressed as: A. A Fourier series B. An integral representation C. A matrix exponential D. A polynomial expansion

In heat conduction, Green's functions represent: A. Charge distribution B. Temperature distribution C. Oscillations in beams D. Particle propagation

In electrostatics, Green's functions are used to compute: A. Electric potential B. Magnetic field C. Temperature gradient D. Vibrational frequency

Applications of Green's functions include: A. Population growth models B. Heat conduction, vibrations, electrostatics, quantum mechanics C. Random processes D. Algebraic equations

Series Summary

In this Series, we explored the theory and applications of boundary value problems (BVPs) and Green's functions. We began by defining BVPs and distinguishing them from initial value problems, highlighting the role of boundary conditions such as Dirichlet and Neumann. We then examined the existence and uniqueness of solutions, noting that uniqueness is guaranteed when the homogeneous problem has only the trivial solution. Next, we introduced Green's functions, defining them as kernels that represent the system's response to a unit impulse and explaining their properties of linearity, symmetry, and dependence on boundary conditions. Finally, we applied Green's functions to solve BVPs and explored their wide-ranging applications in physics and engineering, including heat conduction, vibrations, electrostatics, and quantum mechanics.

Glossary of Terms

Boundary value problem (BVP): A differential equation with conditions specified at more than one point.

Initial value problem (IVP): A differential equation with conditions specified at a single starting point.

Dirichlet boundary condition: Specifies the value of the function at the boundary.

Neumann boundary condition: Specifies the value of the derivative at the boundary.

Existence and uniqueness: Conditions under which a BVP has solutions and whether those solutions are unique.

Green’s function: A kernel representing the system’s response to a unit impulse, used to solve BVPs.

Integral representation: Expression of a solution as an integral involving the Green’s function.

Symmetry property: In many cases, $G(x,\xi)=G(\xi,x)$.

Concept Check Questions 4

Linked to SLO 4.1

Define a boundary value problem and distinguish it from an initial value problem.

Give examples of Dirichlet and Neumann boundary conditions.

Linked to SLO 4.2

State the conditions under which a BVP has a unique solution.

Provide an example of a BVP with non-unique solutions.

Linked to SLO 4.3

Define a Green’s function and explain its construction.

List the key properties of Green’s functions.

Linked to SLO 4.4

Express the solution of a BVP using Green’s function in integral form.

Discuss applications of Green’s functions in physics and engineering.

Answers to Pilot Questions 4

Pilot Questions 4.1 (BVPs)

B. Two or more points

B. One point

B. Boundary value problem

A. Dirichlet condition

B. Neumann condition

Pilot Questions 4.2 (Existence and Uniqueness)

A. The operator and boundary conditions

B. A unique solution

B. No solution or infinitely many solutions

B. A unique solution

B. Non-unique solutions

Pilot Questions 4.3 (Green's Functions)

B. LG(x,ξ)=δ(x-ξ)

B. A unit impulse at

C. An integral representation

B. Symmetry

B. Heat conduction, vibrations, electrostatics

Pilot Questions 4.4 (Applications)

B. Integral equations

B. An integral representation

B. Temperature distribution

A. Electric potential

B. Heat conduction, vibrations, electrostatics, quantum mechanics

Series 5: Sturm–Liouville Theory and Orthogonal Functions

5.1 Sturm–Liouville Problems

Definition and formulation

Examples of Sturm–Liouville systems

5.2 Properties of Eigenvalues and Eigenfunctions

Real eigenvalues

Orthogonality of eigenfunctions

5.3 Orthogonal Functions and Expansions

Concept of orthogonality

Fourier series and generalised expansions

5.4 Applications of Sturm–Liouville Theory

Vibrations of strings and membranes

Heat conduction and wave equations

Introduction

The Sturm–Liouville theory is a cornerstone of applied mathematics and mathematical physics. It provides a systematic framework for solving boundary value problems involving second-order linear differential equations. The theory introduces eigenvalues and eigenfunctions, which play a crucial role in expanding functions into orthogonal series.

Orthogonal functions, such as those arising from Sturm–Liouville problems, form the basis for Fourier series and other expansions used in physics and engineering. These expansions allow us to represent complicated functions in terms of simpler building blocks, making analysis and computation more manageable.

The aim of this Series is to introduce Sturm–Liouville problems, explain the properties of eigenvalues and eigenfunctions, explore orthogonal functions and expansions, and apply the theory to practical problems such as vibrations, heat conduction, and wave propagation.

Series Learning Outcomes

Upon completing this Series, you should be able to:

SLO 5.1 formulate Sturm–Liouville problems and demonstrate their role in boundary value problems,

SLO 5.2 explain the properties of eigenvalues and eigenfunctions, and verify orthogonality in Sturm–Liouville systems,

SLO 5.3 apply the concept of orthogonality to construct function expansions, including Fourier series and generalised orthogonal expansions,

SLO 5.4 evaluate applications of Sturm–Liouville theory in physical contexts such as vibrations, heat conduction, and wave equations.

5.1 Sturm–Liouville Problems

The **Sturm–Liouville problem** is a special type of boundary value problem involving a second-order linear differential equation with variable coefficients. It plays a central role in mathematical physics, especially in the study of vibrations, heat conduction, and wave propagation.

5.1.1 Definition and formulation

A Sturm–Liouville problem has the general form:

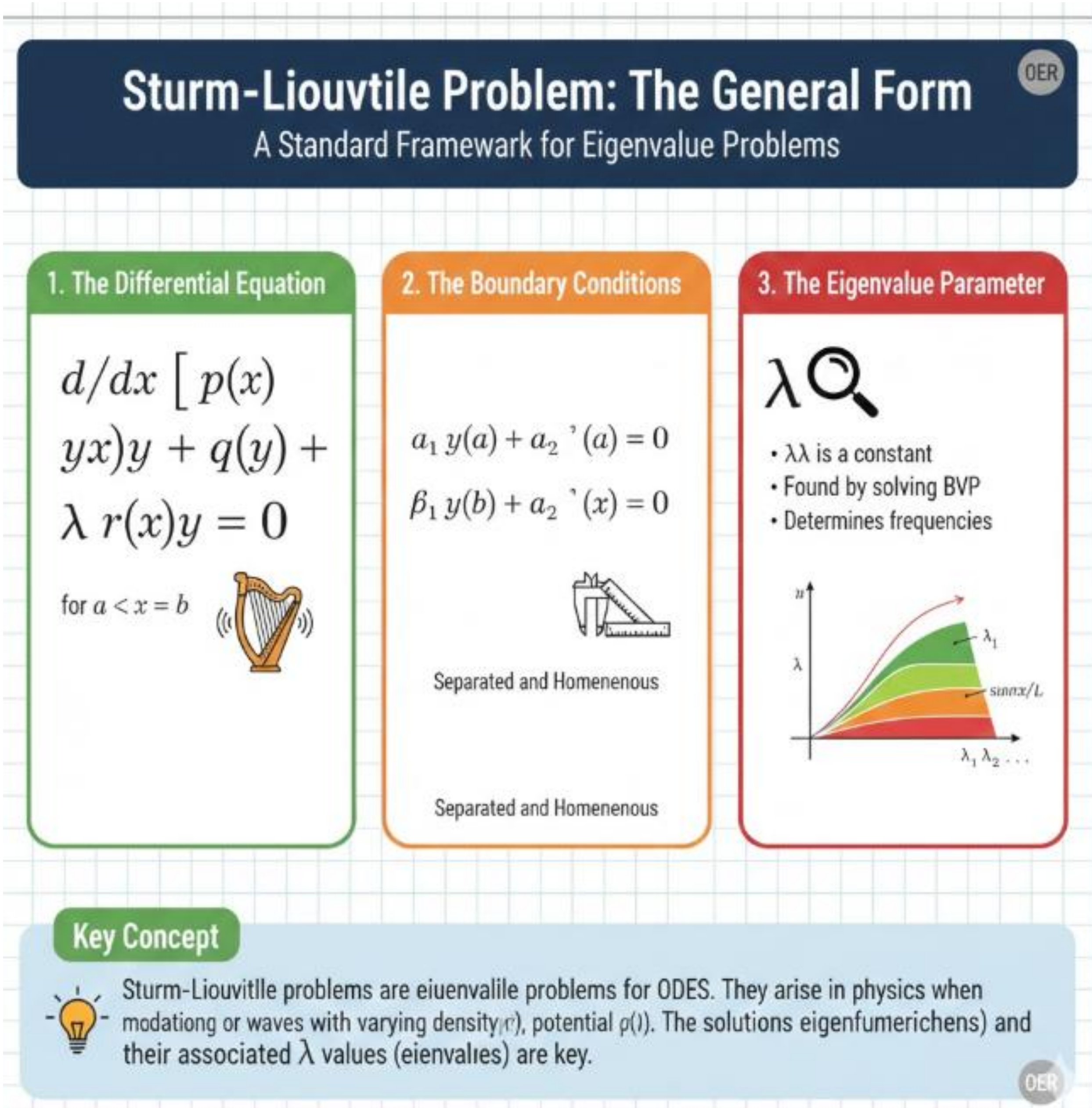
$$ddx[p(x)dydx]+[\lambda w(x)-q(x)]y=0,a<x<b$$

with boundary conditions at $x=a$and $x=b$.

Here:

- $p(x)$, $q(x)$, and $w(x)$are given functions with $p(x)>0$and $w(x)>0$.
- is the **eigenvalue**.
- $y(x)$is the **eigenfunction**.

Fill in the blank: In a Sturm–Liouville problem, is called the __________ and $y(x)$is the __________.



General form of a Sturm–Liouville problem.

5.1.2 Examples of Sturm–Liouville systems

Example 1 (Legendre equation):

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\lambda y=0, -1<x<1$$

This is a Sturm–Liouville problem with weight function $w(x)=1$.

Example 2 (Bessel equation):

$$ddx[xdydx]+(\lambda x\text{-}2x)y=0,0<x<a$$

This is a Sturm–Liouville problem with weight function $w(x)=x$.

Fill in the blank: The Legendre equation is an example of a __________ problem with weight function $w(x)=1$.

Key points about Sturm–Liouville problems

General form: $ddx[p(x)y']+[\lambda w(x)\text{-}q(x)]y=0$**.**

Eigenvalues and eigenfunctions $y(x)$**.**

Examples: Legendre equation, Bessel equation.

Property	***Significance***
Real Eigenvalues	***All possible values of λ are real numbers (no imaginary frequencies).***
Orthogonality	***Solutions (eigenfunctions) for different λ are orthogonal. This allows us to use them like "building blocks" (similar to Fourier series).***
Completeness	***Any "well-behaved" function can be represented as a sum of these eigenfunctions.***
Ordered Zeros	***The n-th eigenfunction has exactly n-1 zeros in the interval (a, b).***

Pilot Questions 5.1

A Sturm–Liouville problem involves a: A. First-order linear equation B. Second-order linear equation with variable coefficients C. Non-linear equation D. Algebraic equation

In a Sturm–Liouville problem, is called the: A. Eigenfunction B. Eigenvalue C. Weight function D. Boundary condition

In a Sturm–Liouville problem, $y(x)$is called the: A. Eigenvalue B. Eigenfunction C. Weight function D. Operator

The Legendre equation is an example of a: A. Sturm–Liouville problem B. Initial value problem C. Non-linear system D. Algebraic equation

The weight function in the Bessel equation is: A. $w(x)=1$B. $w(x)=x$C. $w(x)=x2$D. $w(x)=0$

5.2.2 Orthogonality of eigenfunctions

Another key property is the **orthogonality of eigenfunctions**.

If $ym(x)$and $yn(x)$are eigenfunctions corresponding to distinct eigenvalues mand n, then:

$$abym(x)yn(x)w(x)\ dx=0$$

where $w(x)$is the weight function.

This orthogonality allows us to expand arbitrary functions in terms of eigenfunctions, similar to Fourier series.

Fill in the blank: Eigenfunctions corresponding to distinct eigenvalues are __________ with respect to the weight function.

[Insert Box here] Caption: Box 5.2: Key properties of eigenvalues and eigenfunctions

Eigenvalues are real due to self-adjointness.

Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Orthogonality enables function expansions.

Summary: Properties of Eigenvalues & Eigenfunctions

In **Sturm–Liouville (S-L) Theory**, the relationship between the operator and its solutions mirrors the behavior of symmetric matrices in linear algebra. When we solve $(p(x)y')' + q(x)y + \lambda r(x)y = 0$ subject to self-adjoint boundary conditions, the resulting eigenvalues and eigenfunctions possess several "gold-standard" mathematical properties.

---

1. Reality of Eigenvalues

Even if the coefficients of the differential equation are complex, for a regular S-L problem, all eigenvalues λ_n are **guaranteed to be real numbers**.

- **Physical Meaning:** This ensures that frequencies or energy levels in physical models are observable, real-world quantities rather than imaginary abstractions.

2. Discrete and Ordered Spectrum

The eigenvalues can be arranged in a strictly increasing infinite sequence:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \to \infty$$

- There is a **smallest eigenvalue** (λ_1), often called the ground state or fundamental frequency.
- There is no "largest" eigenvalue; the frequencies increase without bound.

3. Orthogonality

This is the most powerful tool for solving non-homogeneous problems. Two eigenfunctions $y_n(x)$ and $y_m(x)$ corresponding to distinct eigenvalues ($\lambda_n \neq \lambda_m$) are **orthogonal** with respect to the weight function $r(x)$:

$$\int_a^b y_n(x) y_m(x) r(x)\, dx = 0$$

4. Simple Eigenvalues

For each eigenvalue λ_n, there is exactly **one** linearly independent eigenfunction $y_n(x)$ (up to a constant multiple). In mathematical terms, every eigenvalue has a geometric multiplicity of one.

5. The Oscillation Theorem (Nodal Properties)

The n-th eigenfunction $y_n(x)$ (corresponding to λ_n) has exactly $n-1$ **zeros** (nodes) in the open interval (a, b).

- **Example:** The first eigenfunction ($n = 1$) never crosses the x-axis. The second ($n = 2$) crosses it exactly once.

6. Completeness (Eigenfunction Expansion)

Any piecewise smooth function $f(x)$ satisfying the same boundary conditions can be represented as an infinite sum of these eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

This is the generalized version of a **Fourier Series**. We calculate the coefficients c_n using the orthogonality property:

This is the generalized version of a **Fourier Series**. We calculate the coefficients c_n using the orthogonality property:

$$c_n = \frac{\int_a^b f(x)y_n(x)r(x)dx}{\int_a^b y_n^2(x)r(x)dx}$$

Pilot Questions 5.2

In Sturm–Liouville problems, eigenvalues are always: A. Complex B. Real C. Imaginary D. Random

Eigenvalues are real because the operator is: A. Linear B. Self-adjoint C. Non-linear D. Arbitrary

Eigenfunctions corresponding to distinct eigenvalues are: A. Parallel B. Orthogonal C. Identical D. Random

Orthogonality of eigenfunctions is defined with respect to the: A. Boundary conditions B. Weight function C. Eigenvalues D. Initial conditions

Orthogonality of eigenfunctions allows us to: A. Ignore boundary conditions B. Expand functions in terms of eigenfunctions C. Eliminate eigenvalues D. Solve algebraic equations only

5.3 Orthogonal Functions And Expansions

Orthogonal functions are fundamental in mathematical analysis and physics because they allow us to represent complicated functions as sums of simpler components. Sturm–Liouville theory provides a natural framework for generating such orthogonal functions.

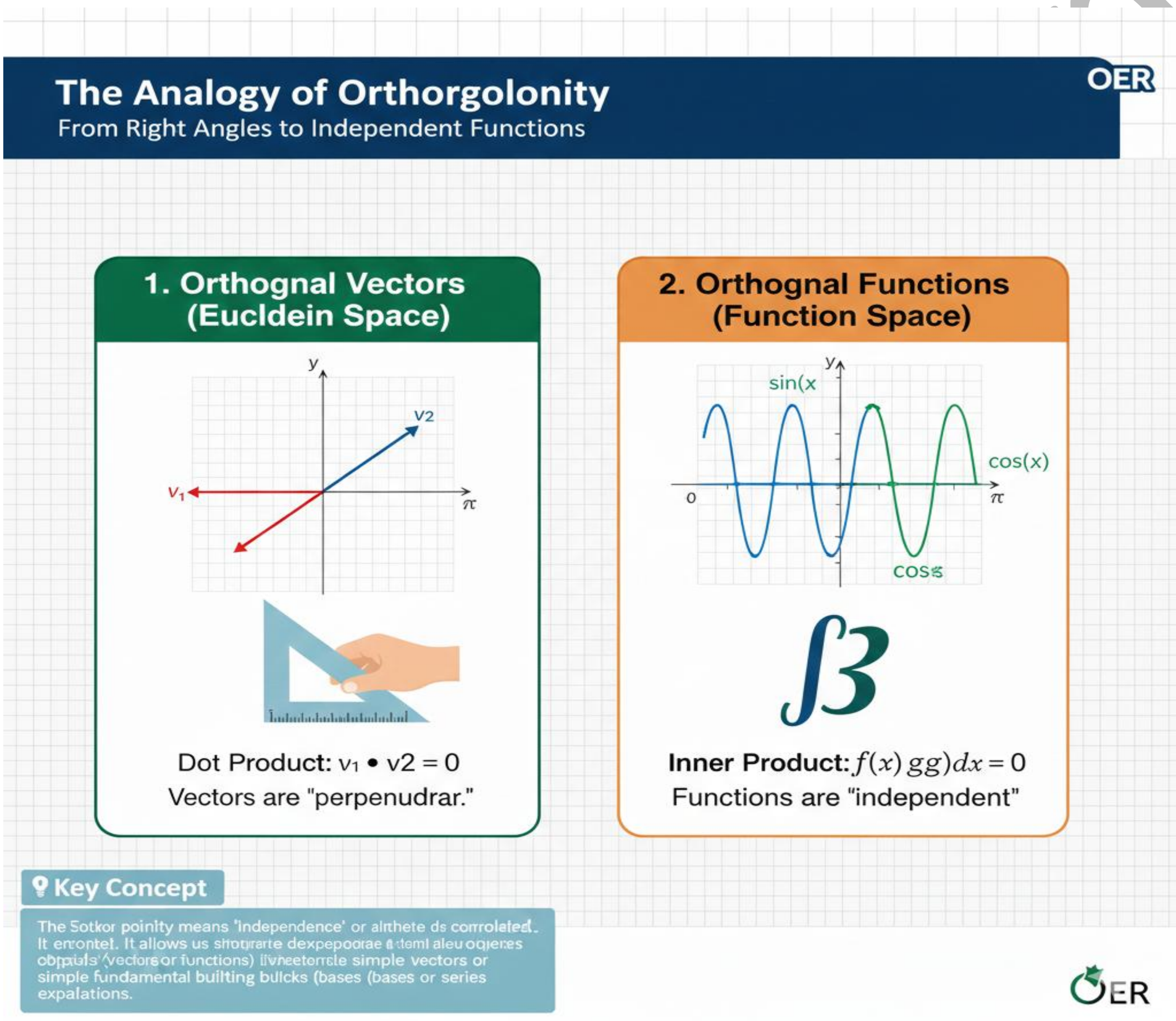
5.3.1 Concept of orthogonality

Two functions f(x)and g(x)are said to be **orthogonal** with respect to a weight function w(x)on the interval [a,b]if:

abf(x)g(x)w(x) dx=0

This concept generalises the idea of perpendicular vectors in Euclidean space to functions in function space.

Fill in the blank: Two functions are orthogonal if their weighted inner product over the interval is equal to __________.



Orthogonality of functions compared to orthogonality of vectors.

5.3.2 Fourier series and generalised expansions

Orthogonal functions form the basis of **Fourier series** and other expansions.

Fourier series: Any periodic function can be expressed as a sum of sines and cosines, which are orthogonal functions.

f(x)=a0+n=1[ancos(nx)+bnsin(nx)]

Generalised expansions: Sturm–Liouville eigenfunctions can be used to expand functions in terms of orthogonal sets beyond sines and cosines.

Fill in the blank: Fourier series expand periodic functions in terms of __________ and __________, which are orthogonal functions.

Key points about orthogonal functions and expansions

3. Key Types of Expansions

Expansion Type	Basis Functions	Interval / Weight	Application
Classical Fourier	$\sin(nx), \cos(nx)$	$[-\pi, \pi]$, $w(x) = 1$	Signal processing, heat in rods.
Fourier–Bessel	$J_m(k_{mn}x)$	$[0, R]$, $w(x) = x$	Vibrations of circular membranes.
Fourier–Legendre	$P_n(x)$	$[-1, 1]$, $w(x) = 1$	Gravitational & Electric potentials.
Hermite	$H_n(x)$	$(-\infty, \infty)$, $w(x) = e^{-x^2}$	Quantum Harmonic Oscillator.

Orthogonality generalises perpendicularity to function space.

Fourier series: expansion in sines and cosines.

Sturm–Liouville eigenfunctions provide generalised orthogonal expansions.

Pilot Questions 5.3

Two functions are orthogonal if: A. Their product is always zero B. Their weighted inner product over the interval is zero C. They are linearly dependent D. They are identical

Orthogonality of functions generalises the concept of: A. Parallel vectors B. Perpendicular vectors C. Random vectors D. Complex numbers

Fourier series expand periodic functions in terms of: A. Polynomials B. Sines and cosines C. Exponentials only D. Random functions

Sturm–Liouville eigenfunctions can be used to: A. Expand functions in terms of orthogonal sets B. Solve algebraic equations only C. Eliminate eigenvalues D. Ignore boundary conditions

Orthogonal expansions are useful because they: A. Complicate function representation B. Simplify function representation into basic building blocks C. Eliminate periodicity D. Remove eigenfunctions

5.4 Applications Of Sturm–Liouville Theory

Sturm–Liouville theory is not only a mathematical framework but also a powerful tool for solving real-world problems in physics and engineering. Its applications arise wherever boundary value problems and orthogonal expansions are needed.

5.4.1 Vibrations of strings and membranes

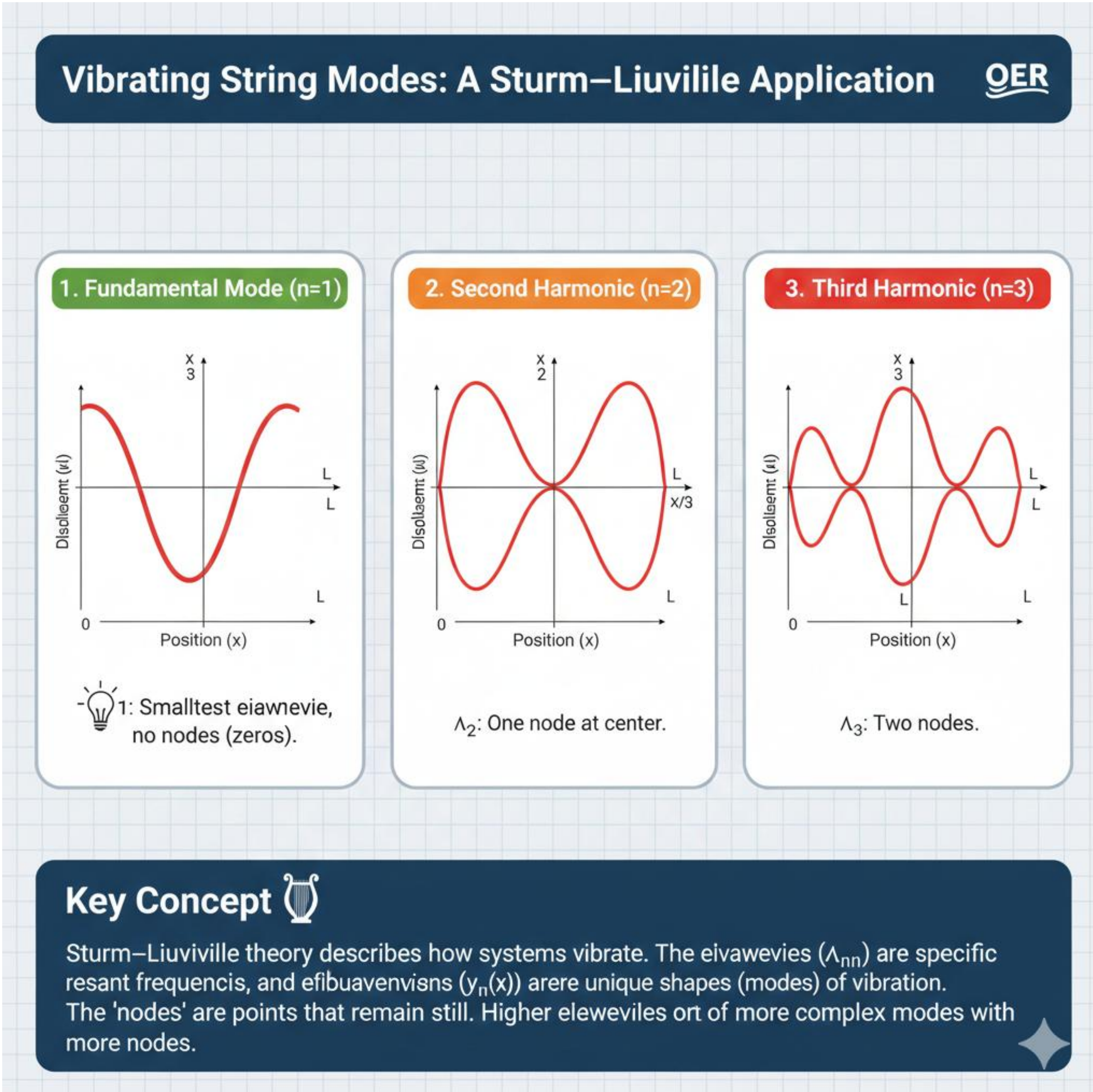
The vibration of a stretched string can be modelled by the wave equation:

2ut2=c22ux2

Separation of variables reduces this to a Sturm–Liouville problem in space.

Eigenfunctions represent the modes of vibration, and eigenvalues correspond to frequencies.

Fill in the blank: In string vibration problems, eigenvalues correspond to __________ of vibration.



Modes of vibration of a string as eigenfunctions.

5.4.2 Heat conduction and wave equations

Heat conduction in a rod is governed by the heat equation:

ut=k2ux2

Using separation of variables, the spatial part becomes a Sturm–Liouville problem.

Eigenfunctions describe temperature distributions, while eigenvalues determine decay rates.

Fill in the blank: In heat conduction problems, eigenfunctions describe __________ distributions in the rod.

5.4.3 Other physical applications

Electromagnetic theory: Sturm–Liouville systems arise in solving Maxwell's equations in bounded regions.

Quantum mechanics: Schrödinger's equation often reduces to a Sturm–Liouville form, with eigenvalues representing energy levels.

Acoustics: Vibrations of membranes and air columns can be analysed using Sturm–Liouville theory.

Fill in the blank: In quantum mechanics, eigenvalues of Sturm–Liouville problems represent __________ levels of particles.

Applications of Sturm–Liouville theory

Vibrations: eigenfunctions as modes, eigenvalues as frequencies.

Heat conduction: eigenfunctions as temperature distributions.

Quantum mechanics: eigenvalues as energy levels.

Electromagnetism and acoustics: boundary value problems solved via Sturm–Liouville systems.

Application	***Role of S-L Theory***	***Resulting Physical Insights***
Acoustics & Music	***Solving the wave equation for strings, air columns, and membranes.***	***Determines the harmonics and timbre of musical instruments.***

Application	*Role of S-L Theory*	*Resulting Physical Insights*
Structural Engineering	***Analyzing the stability and vibration of beams, bridges, and skyscrapers.***	***Identifies critical buckling loads and resonance frequencies to prevent collapse.***
Quantum Mechanics	***Solving the time-independent Schrödinger equation for various potentials.***	***Explains quantized energy levels in atoms (e.g., electron shells).***
Heat Transfer	***Modeling temperature distribution in non-uniform rods or cooling fins.***	***Predicts how quickly a component will cool based on its thermal conductivity.***
Electromagnetics	***Determining the modes of propagation in waveguides and fiber optics.***	***Optimizes the bandwidth and signal integrity of communication cables.***

Pilot Questions 5.4

In string vibration problems, eigenvalues correspond to: A. Amplitudes B. Frequencies C. Temperatures D. Energy levels

In heat conduction problems, eigenfunctions describe: A. Frequencies of vibration B. Temperature distributions C. Energy levels D. Magnetic fields

In quantum mechanics, eigenvalues of Sturm–Liouville problems represent: A. Frequencies B. Energy levels C. Temperatures D. Boundary conditions

Applications of Sturm–Liouville theory include: A. Vibrations, heat conduction, quantum mechanics, electromagnetism B. Algebraic equations only C. Random processes D. Population growth models

The wave equation for a vibrating string reduces to a Sturm–Liouville problem through: A. Fourier transform B. Separation of variables C. Matrix methods D. Random substitution

Series Summary

In this Series, we studied the Sturm–Liouville theory, which provides a systematic framework for solving boundary value problems involving second-order linear differential equations. We began with the definition and formulation of Sturm–Liouville problems, highlighting the role of eigenvalues and eigenfunctions. We then examined the properties of eigenvalues and eigenfunctions, noting that eigenvalues are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal. Next, we explored orthogonal functions and expansions, showing how Fourier series and generalised expansions arise naturally from Sturm–Liouville systems. Finally, we applied the theory to physical contexts such as vibrations of strings and membranes, heat conduction, wave equations, electromagnetism, and quantum mechanics, demonstrating its wide-ranging importance in applied mathematics and physics.

Glossary of Terms

Sturm–Liouville problem: A boundary value problem of the form ddx[p(x)y']+[λw(x)-q(x)]y=0.

Eigenvalue (): A scalar parameter associated with non-trivial solutions of the Sturm–Liouville problem.

Eigenfunction (y(x)): A non-trivial solution corresponding to an eigenvalue.

Self-adjoint operator: An operator that ensures eigenvalues are real and eigenfunctions are orthogonal.

Orthogonality: Property where the weighted inner product of two functions is zero.

Fourier series: Expansion of periodic functions in terms of sines and cosines.

Generalised expansion: Representation of functions using Sturm–Liouville eigenfunctions.

Applications: Vibrations, heat conduction, wave equations, electromagnetism, quantum mechanics.

Concept Check Questions 5

Linked to SLO 5.1

Define a Sturm–Liouville problem and explain its general form.

Give examples of Sturm–Liouville systems such as Legendre and Bessel equations.

Linked to SLO 5.2

Explain why eigenvalues in Sturm–Liouville problems are real.

Verify the orthogonality of eigenfunctions corresponding to distinct eigenvalues.

Linked to SLO 5.3

Define orthogonality of functions with respect to a weight function.

Expand a periodic function using Fourier series.

Linked to SLO 5.4

Apply Sturm–Liouville theory to the vibration of a string.

Discuss applications of Sturm–Liouville theory in quantum mechanics and heat conduction.

Answers to Pilot Questions 5

Pilot Questions 5.1 (Sturm–Liouville Problems)

B. Second-order linear equation with variable coefficients

B. Eigenvalue

B. Eigenfunction

A. Sturm–Liouville problem

B. $w(x)=x$

Pilot Questions 5.2 (Properties of Eigenvalues and Eigenfunctions)

B. Real

B. Self-adjoint

B. Orthogonal

B. Weight function

B. Expand functions in terms of eigenfunctions

Pilot Questions 5.3 (Orthogonal Functions and Expansions)

B. Their weighted inner product over the interval is zero

B. Perpendicular vectors

B. Sines and cosines

A. Expand functions in terms of orthogonal sets

B. Simplify function representation into basic building blocks

Pilot Questions 5.4 (Applications)

B. Frequencies

B. Temperature distributions

B. Energy levels

A. Vibrations, heat conduction, quantum mechanics, electromagnetism

B. Separation of variables